Interactive Proofs

Lecture 16

What the all-powerful can convince mere mortals of

Recap

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- Non-deterministic Computation

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- Non-deterministic Computation
- Polynomial Hierarchy

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    - Non-determinism on steroids!

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- Today: Interactive Proofs

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- Today: Interactive Proofs
  - Non-determinism and Probabilistic computation on steroids!

Interactive Proofs

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- **Prover** wants to convince **verifier** that x has some property

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  - i.e. x is in language L

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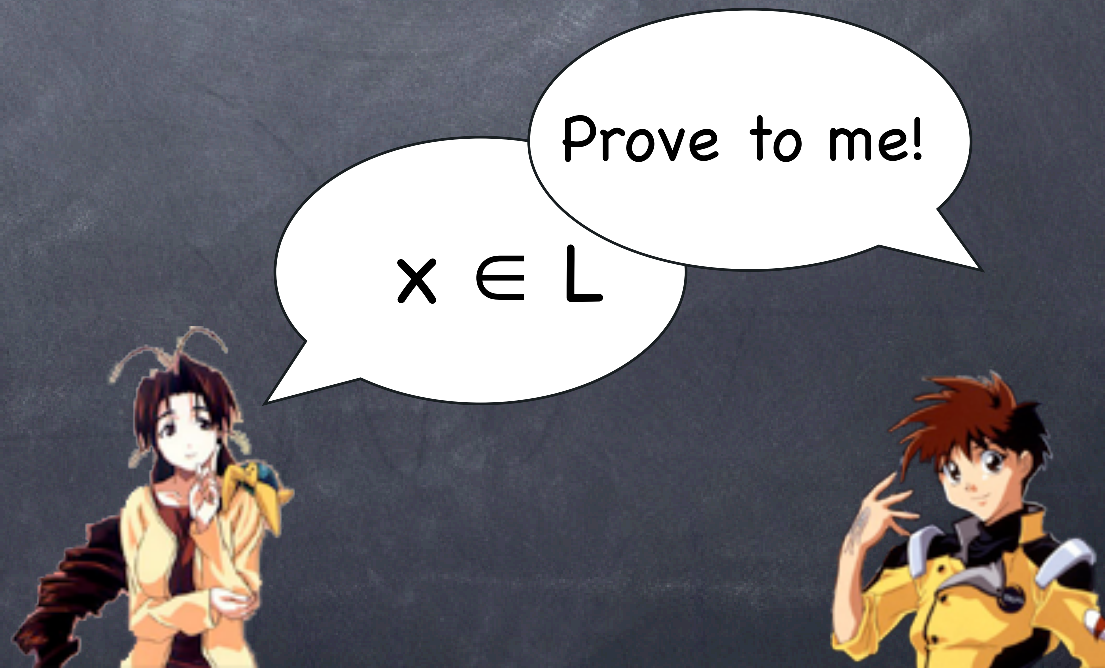

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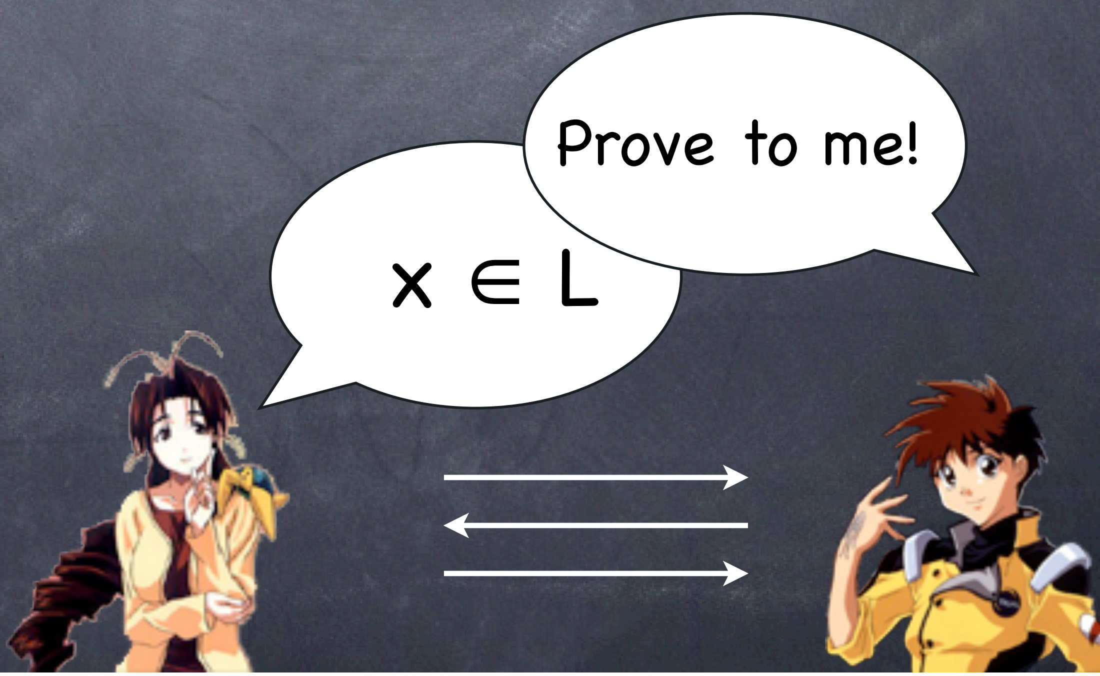

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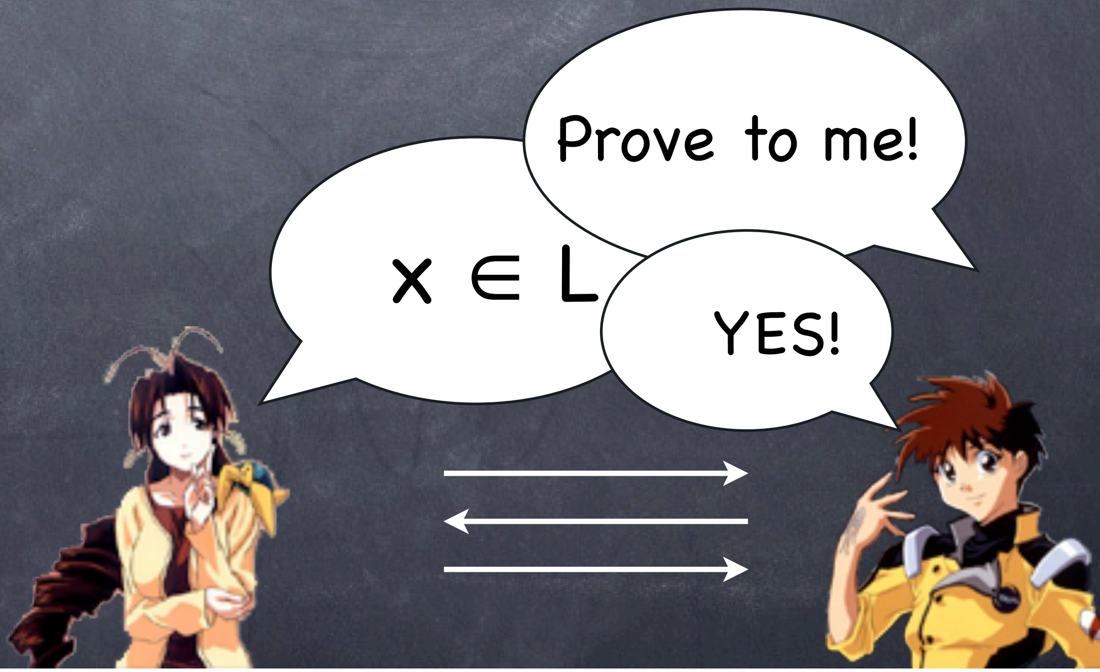

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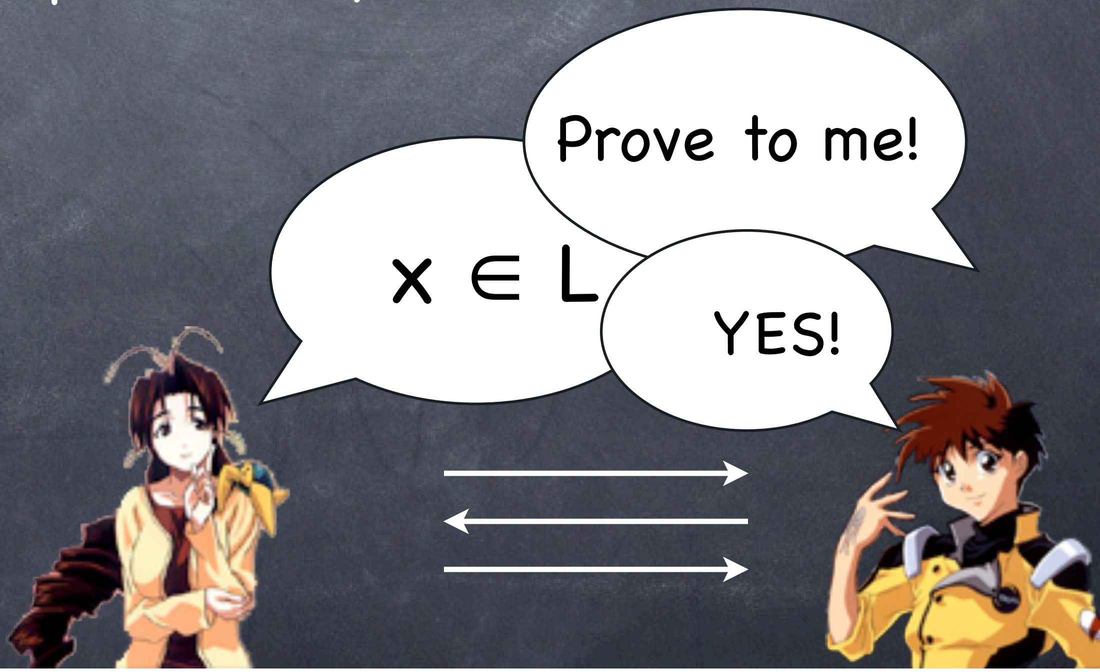

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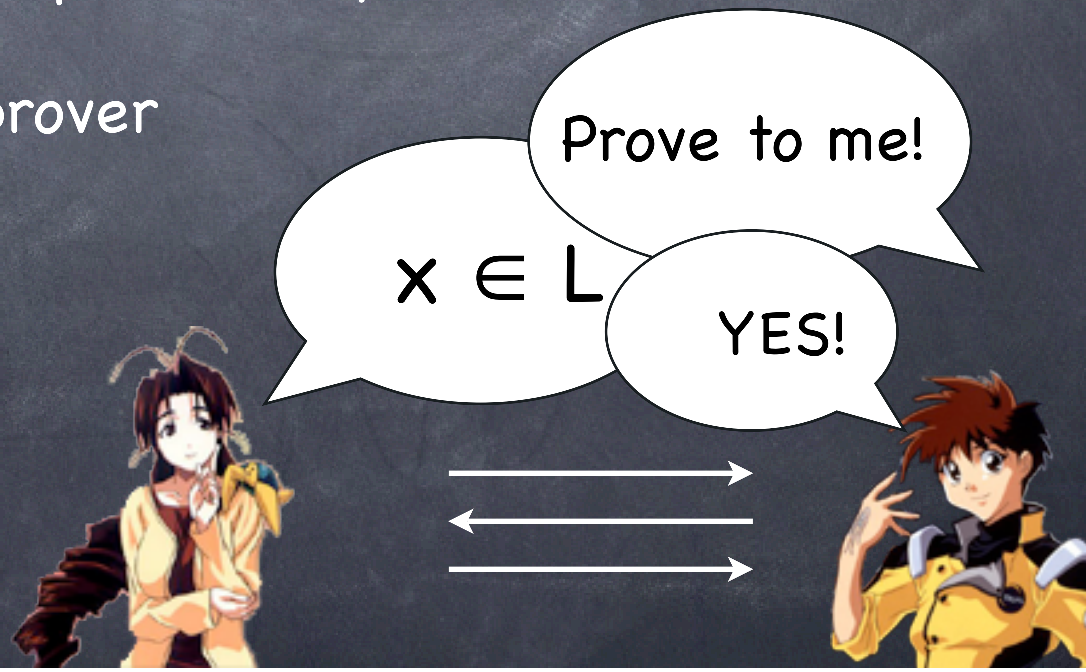

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- All powerful prover, computationally bounded verifier
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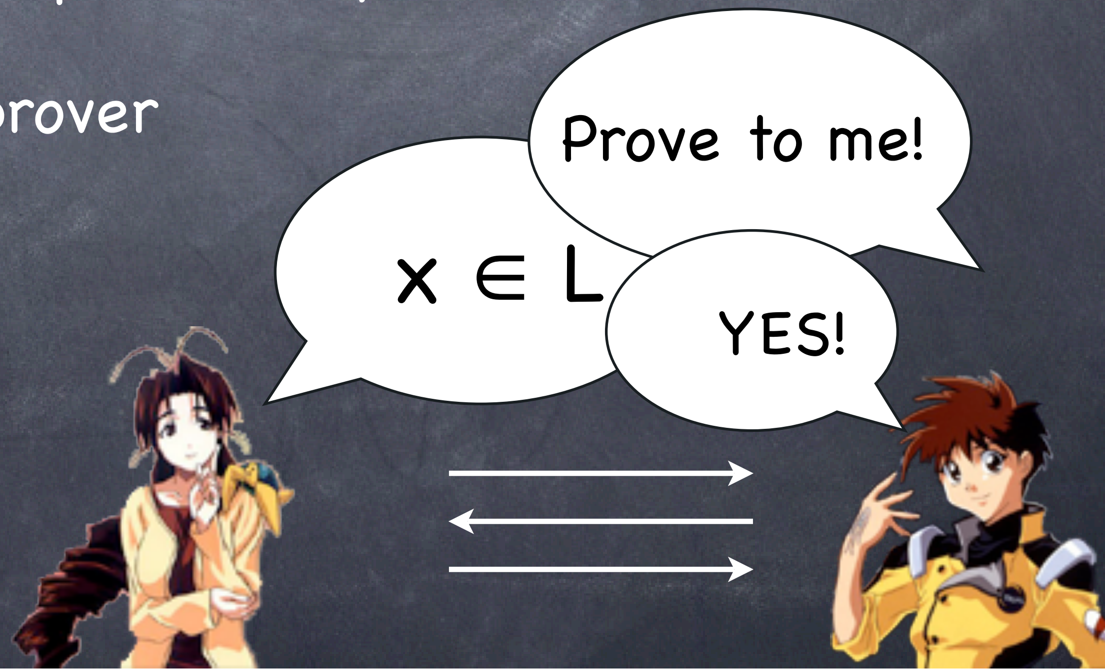

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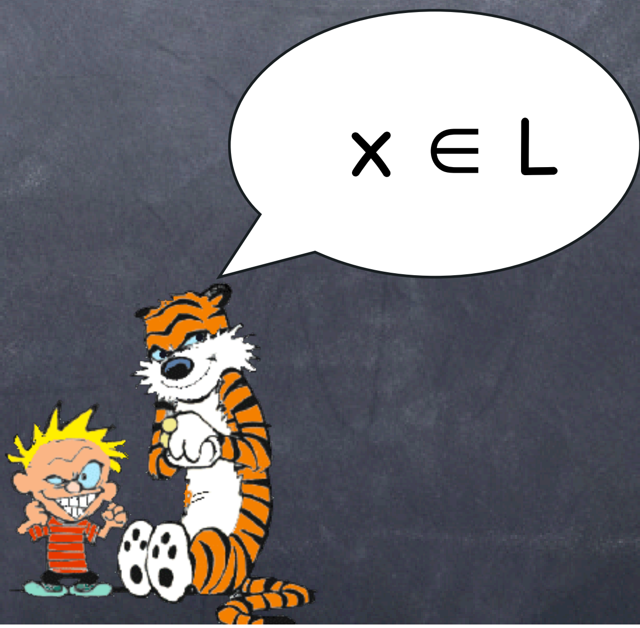

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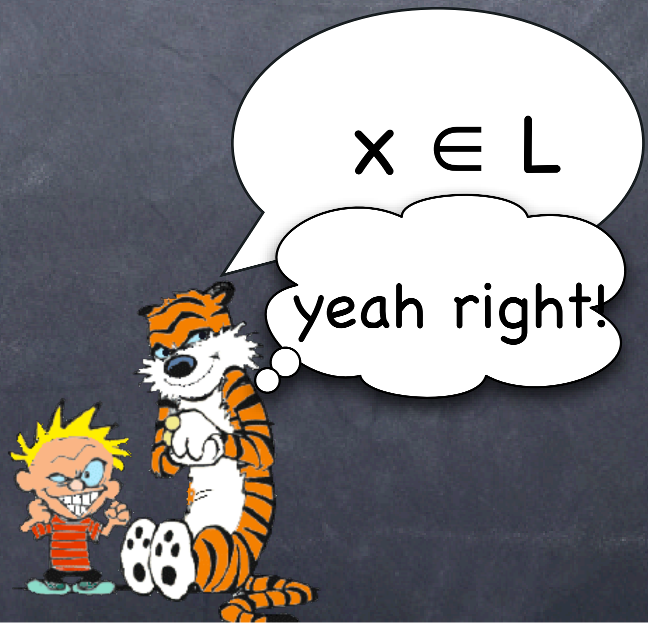

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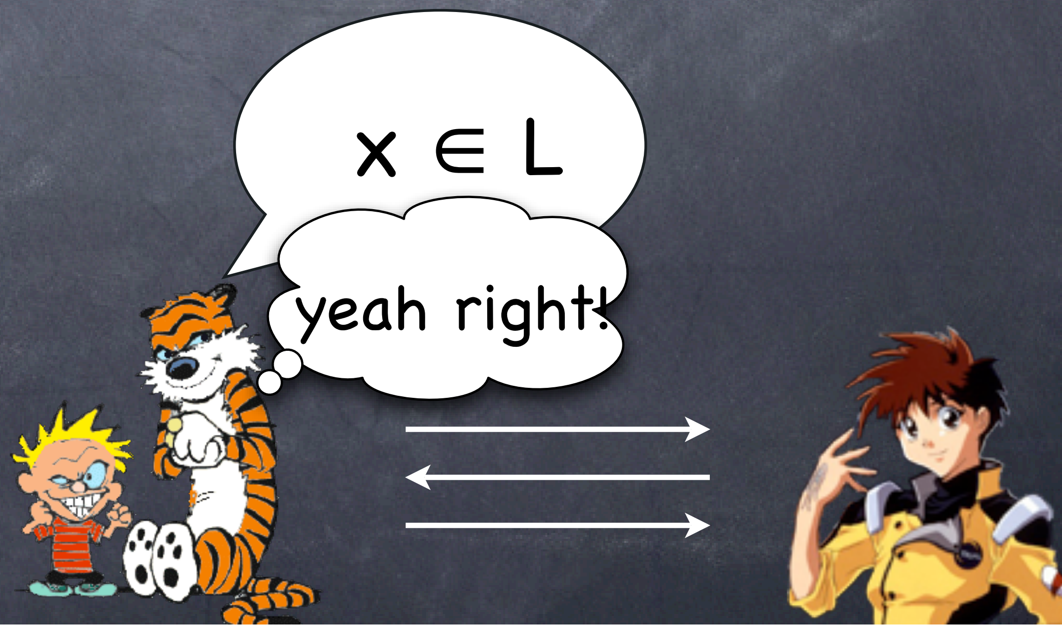

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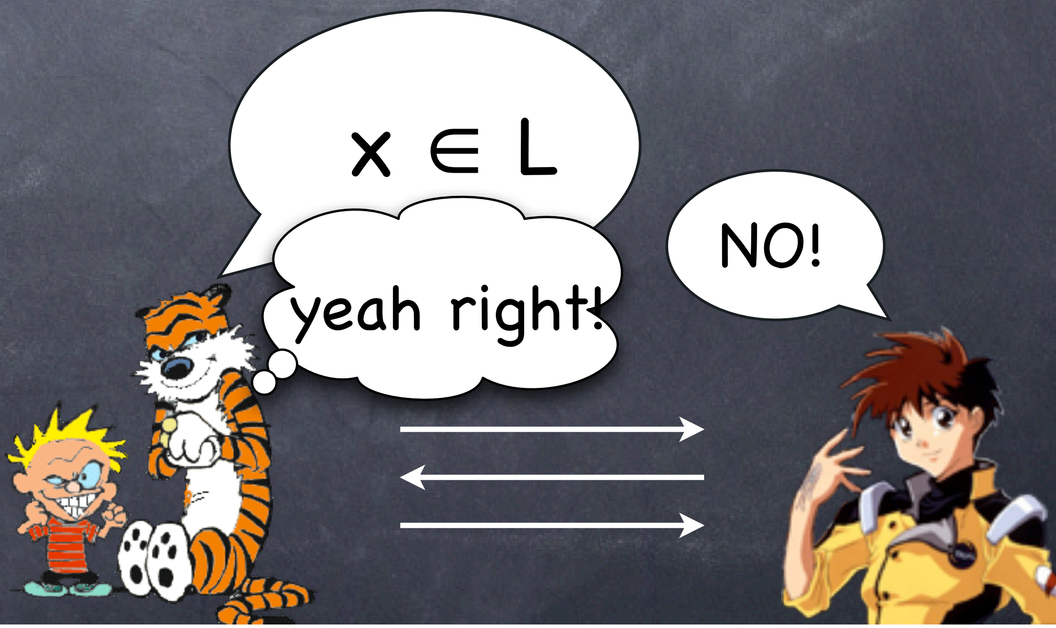

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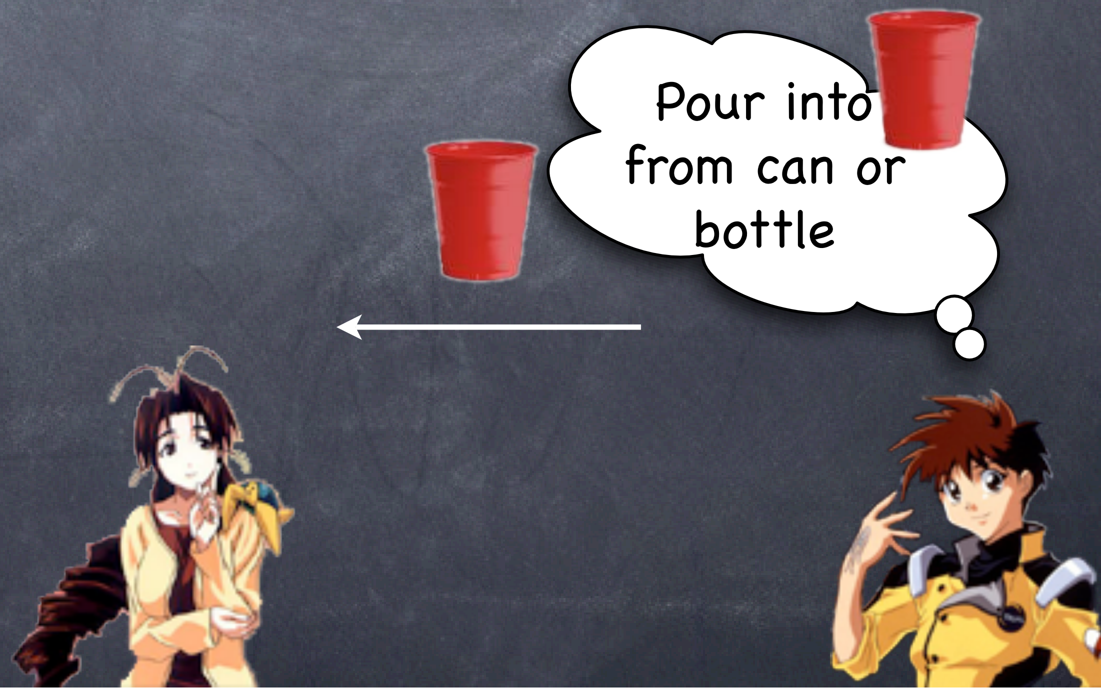

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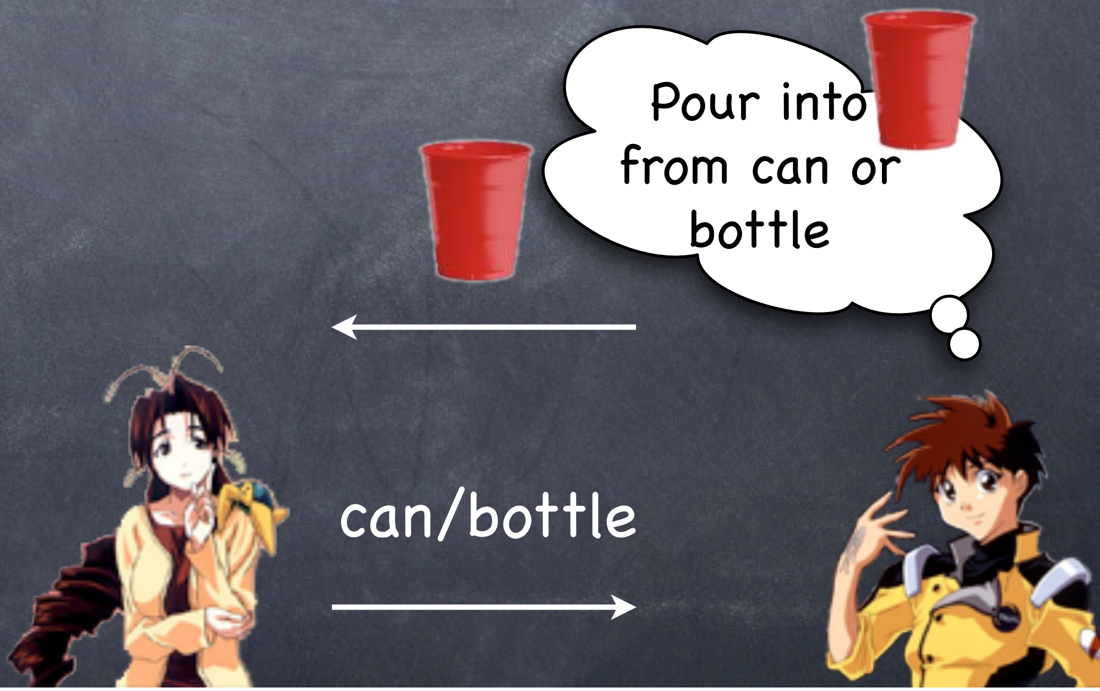

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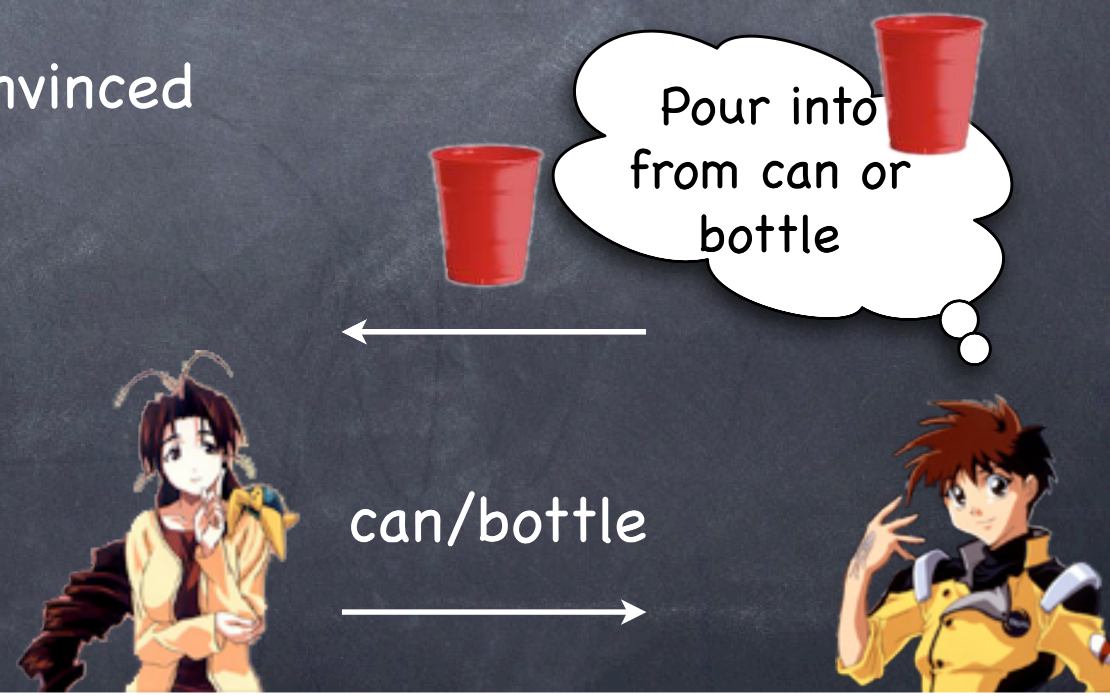

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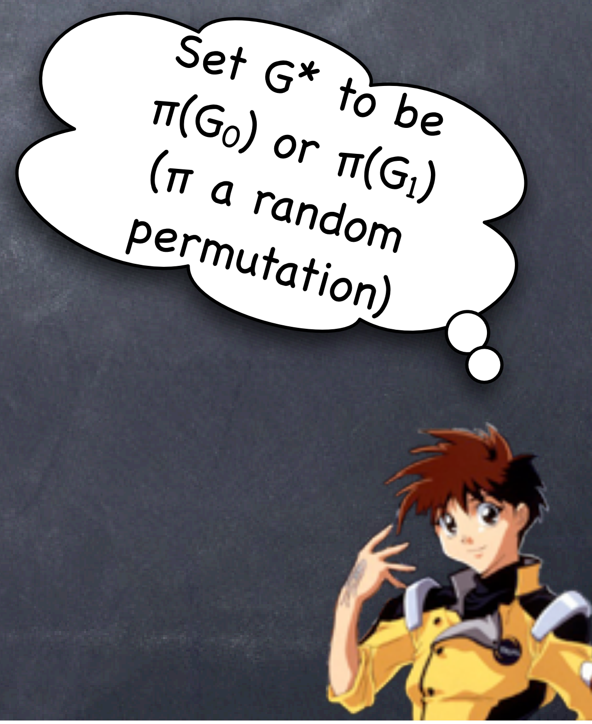

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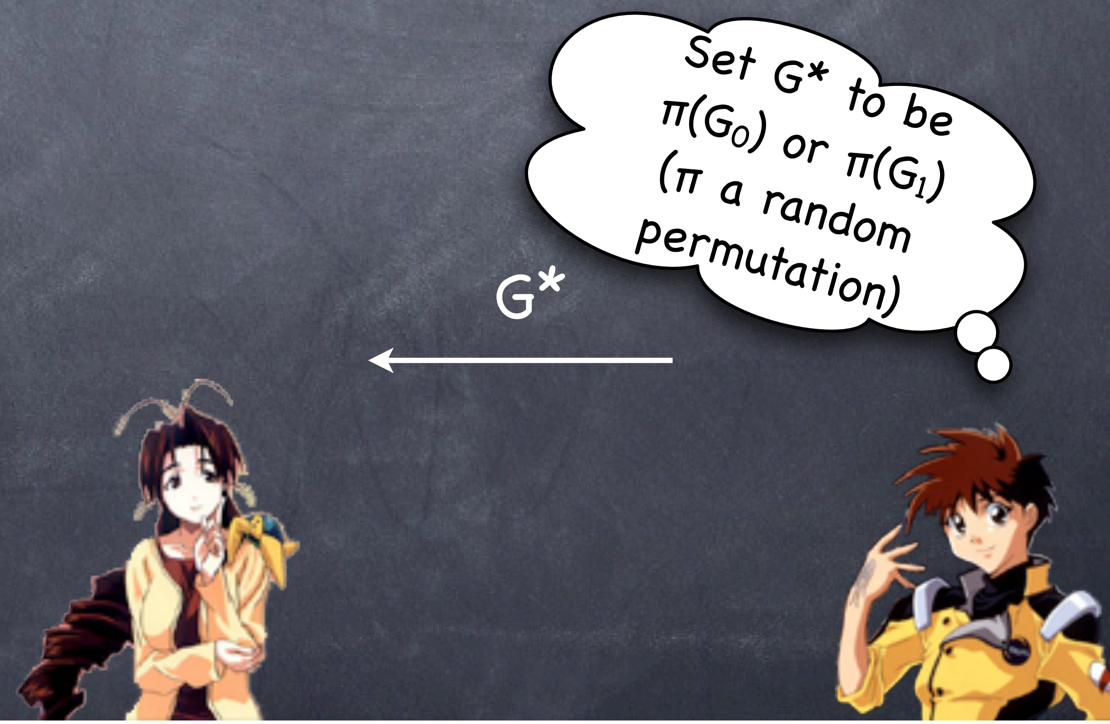

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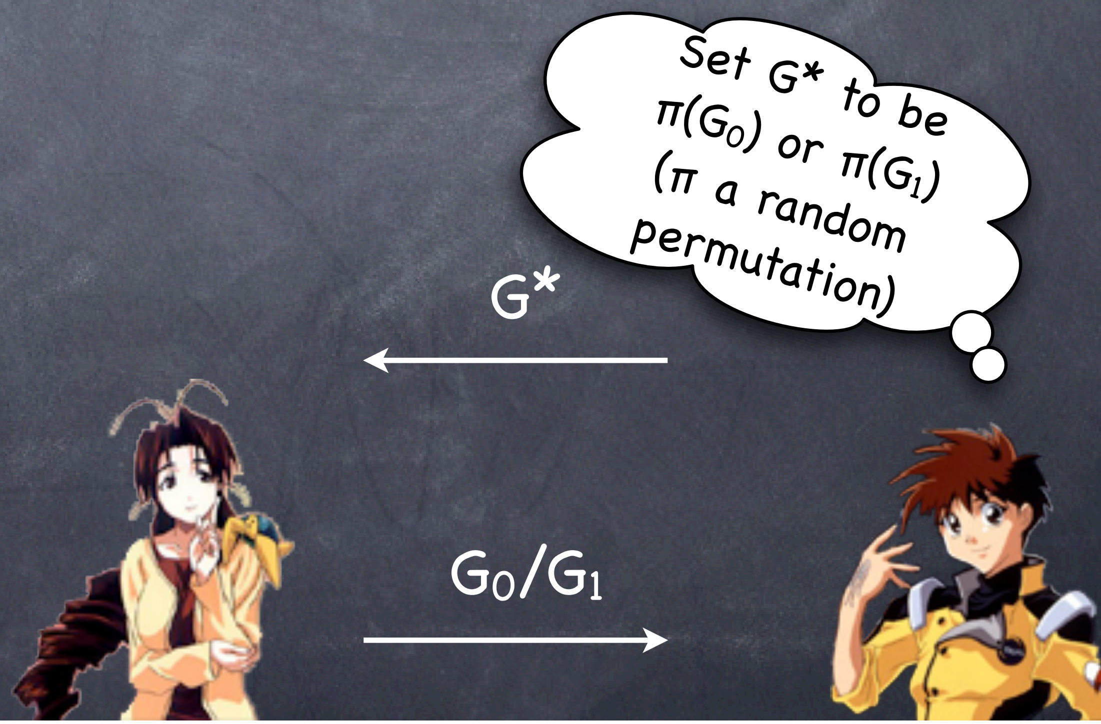

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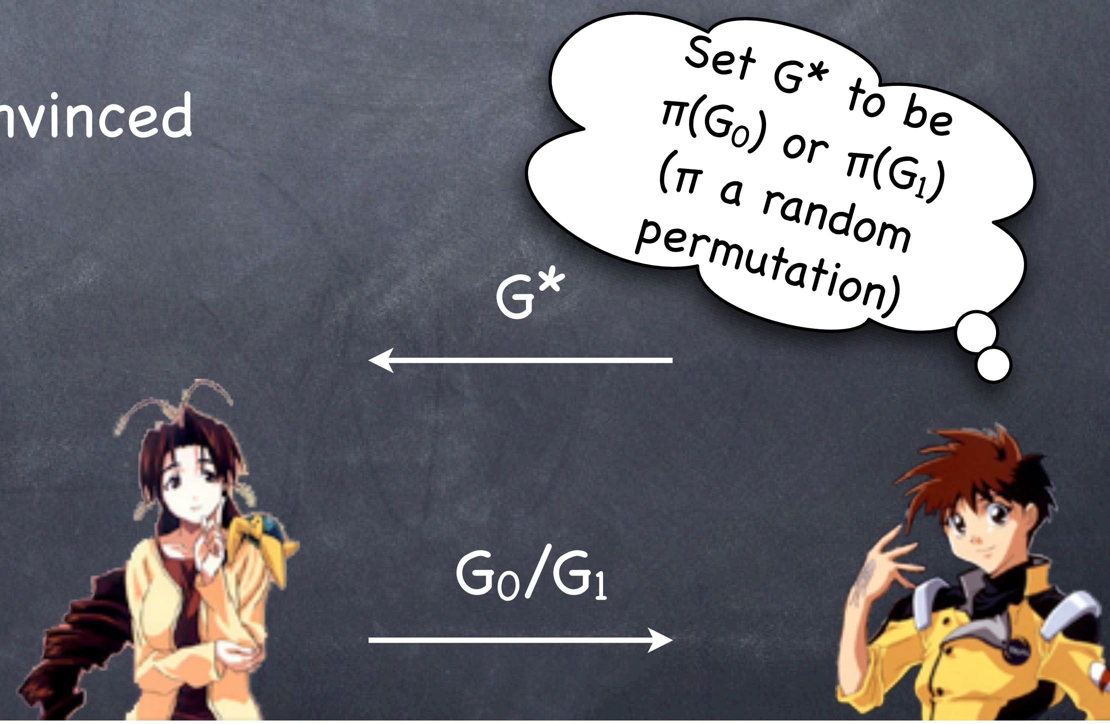

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  - Except with probability at most 1/3

Deterministic IP?

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- Deterministic Prover IP = IP
  - For each input prover can choose the random tape which maximizes Pr[yes] (probability over honest verifier's randomness)

Public and Private Coins

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- Private coins: Verifier does not send everything about the coins
  - e.g. GNI protocol: verifier keeps coin tosses hidden; uses it to create challenge

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- Arthur-Merlin proof-systems

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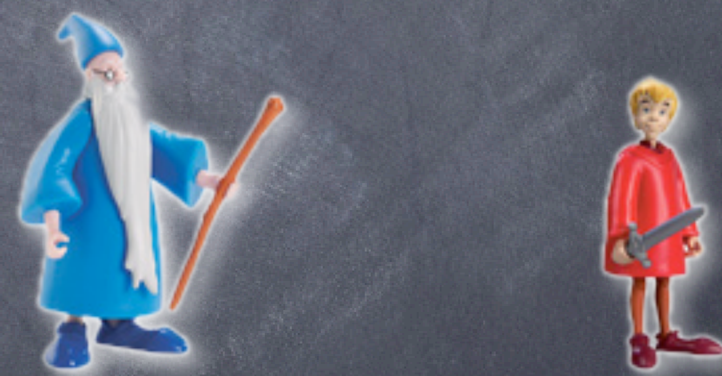

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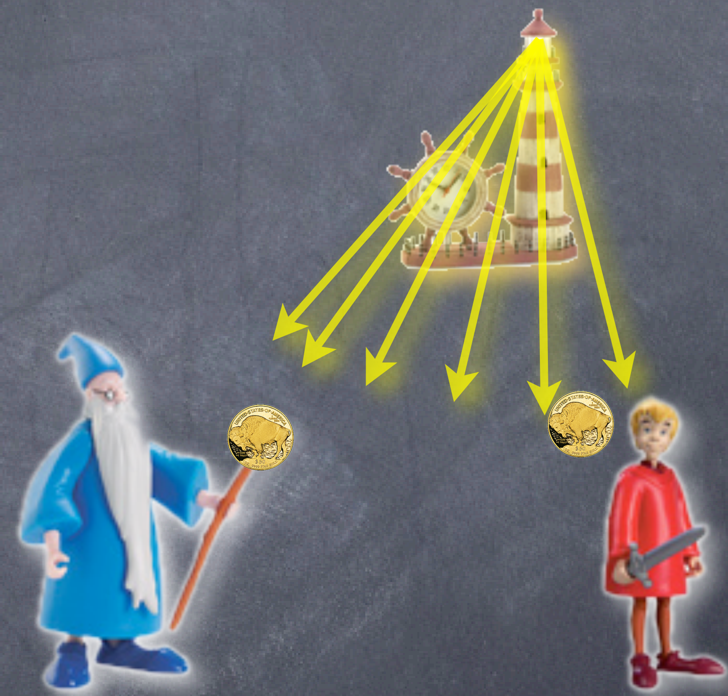

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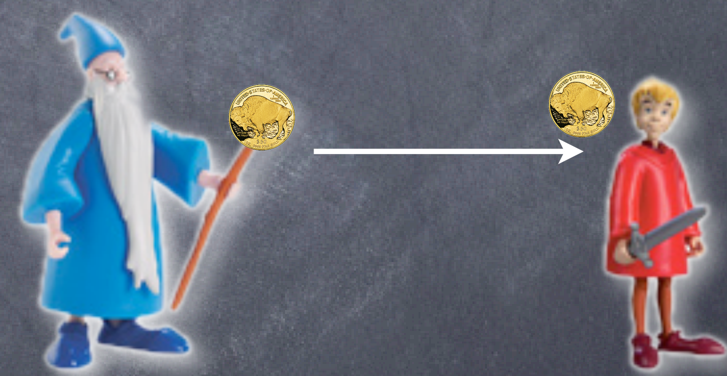

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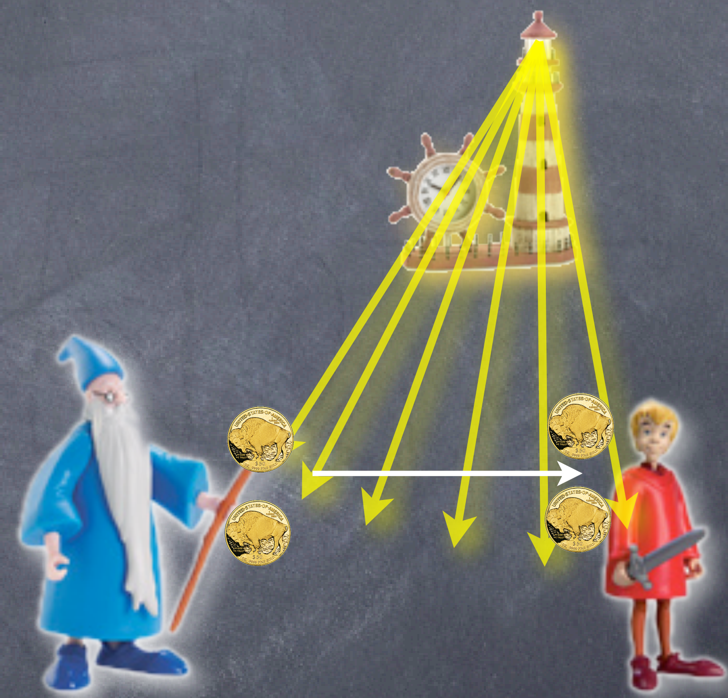

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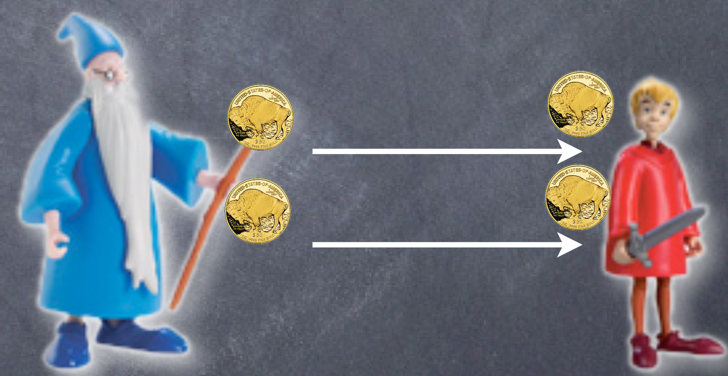

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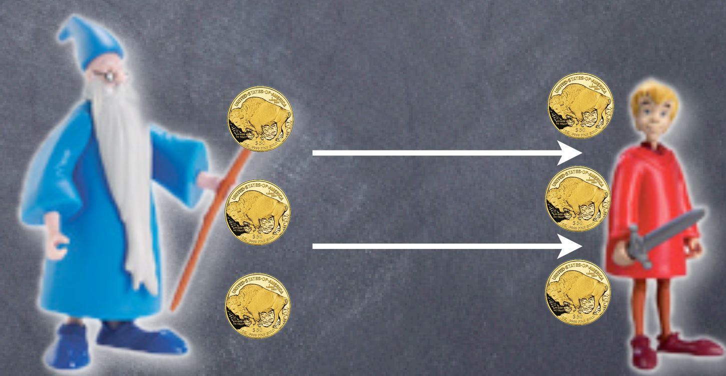

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- Contain NP and BPP

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  - Prover to prove that $|\{H: H \equiv G_0 \text{ or } H \equiv G_1\}| > n!$

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- If $|S| > 2K$, Pr[yes] > 2/3. If $|S| \leq K$, Pr[yes] $\leq$ 1/3
- But what if $K/|U|$ is exponentially small?

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  - Clearly no single function for all S!

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- Hash collision probability $= 1/|R|$

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- Pr[Yes] has a constant gap between |S| > 2K and |S| < K [Exercise]