Probabilistic Computation

Lecture 15

Computing with Less Randomness, or with Imperfect Randomness

Soundness Amplification for BPP

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      - t = $O(n^d/\delta^2)$ enough for Pr[error] ≤ $2^{-n\hat{}d}$

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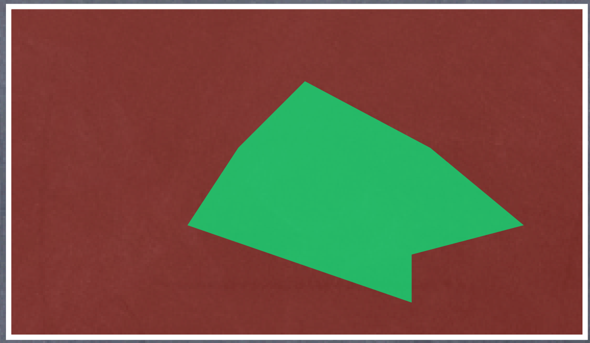

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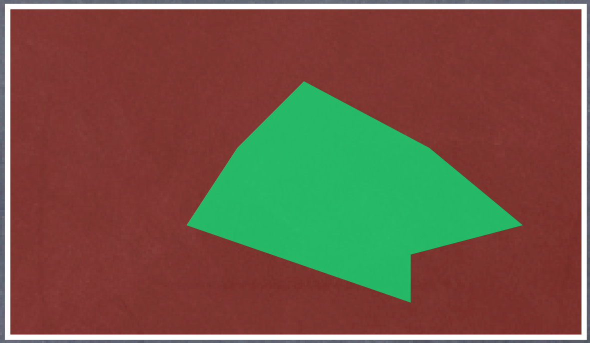

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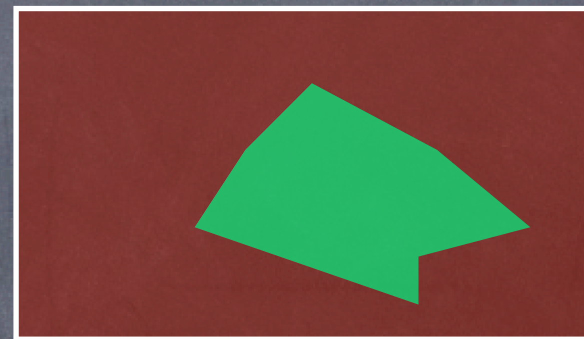

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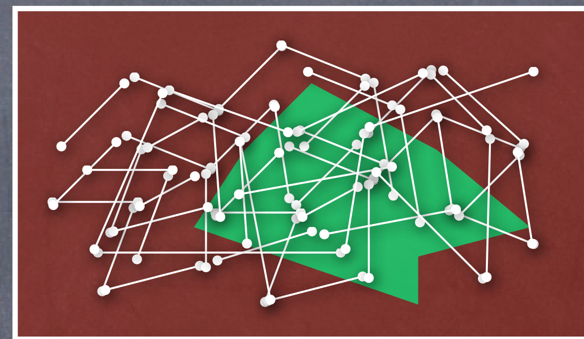

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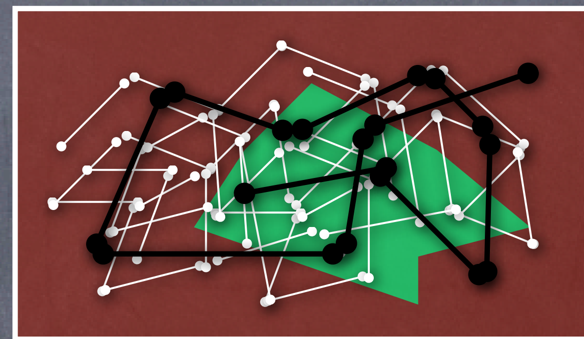

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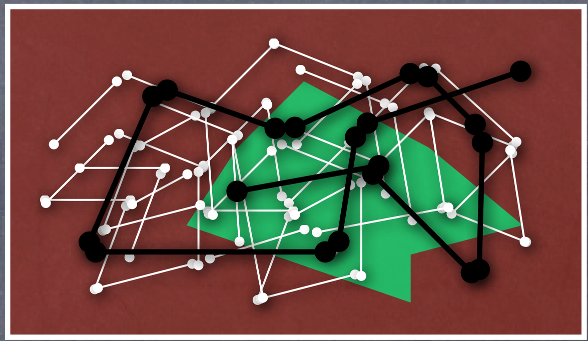

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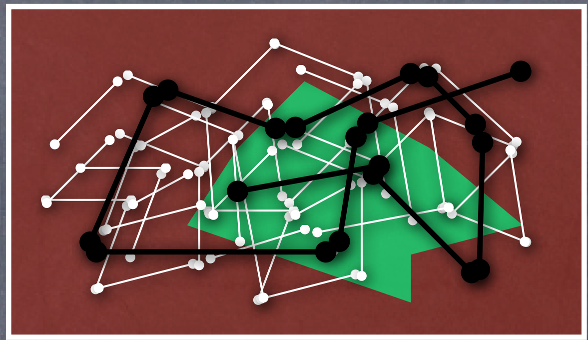

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    - Can we work with imperfect random sources?

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  - Don't know the exact distribution, but belongs to a known class of distributions

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- Weaker guarantees: e.g. Block source

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  - If on perfect randomness, Pr[error] < $1/(e2^n)$, then on imperfect randomness with bias < 1/m, Pr[error] < $1/2^n$

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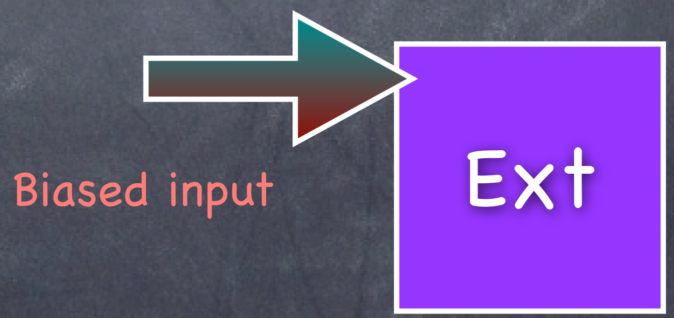

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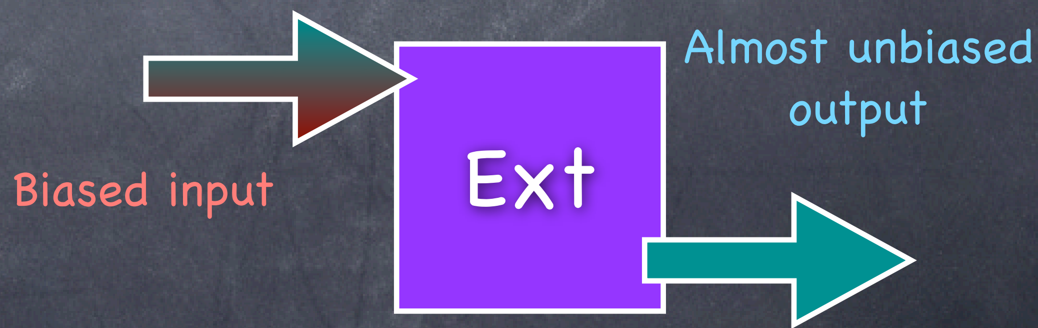

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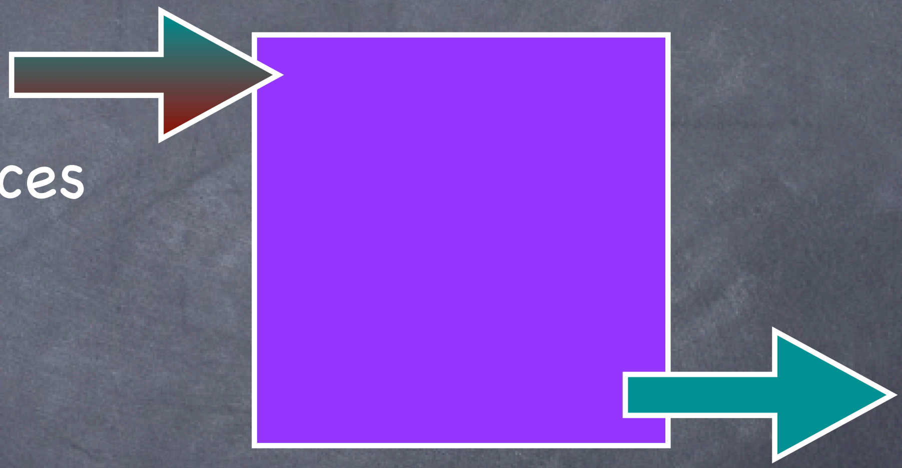

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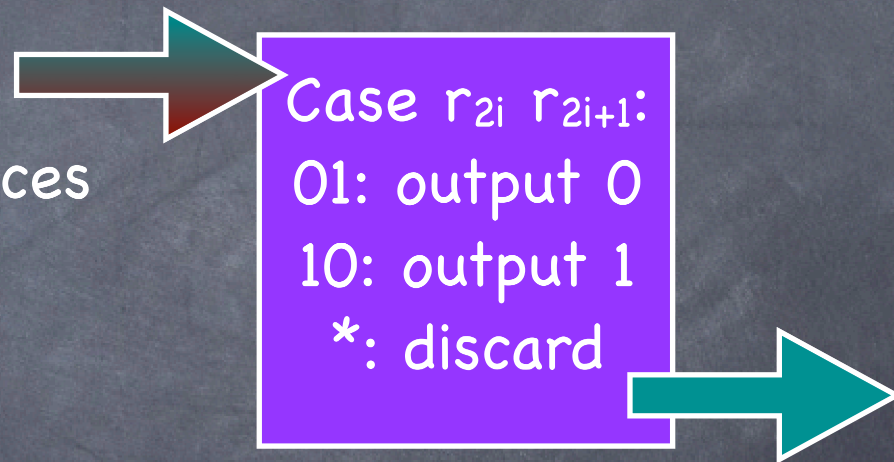

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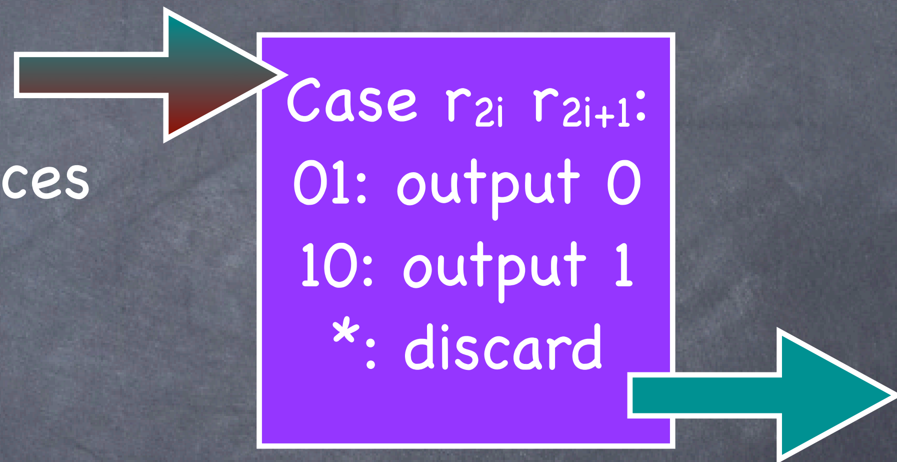

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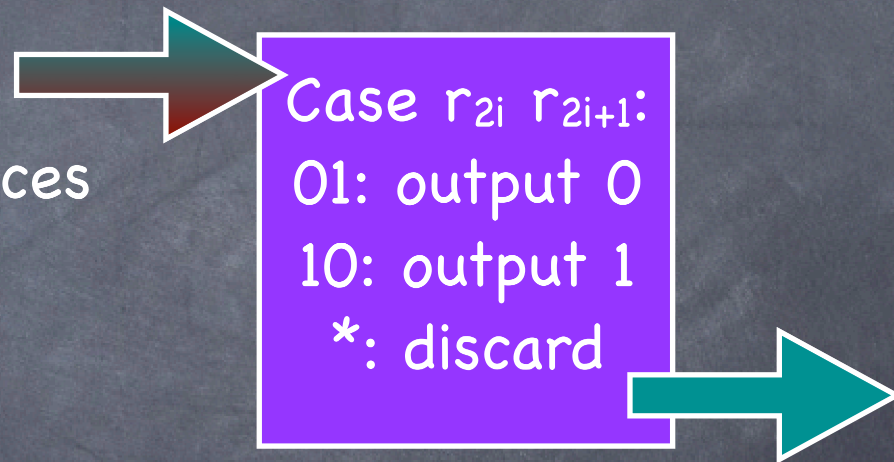

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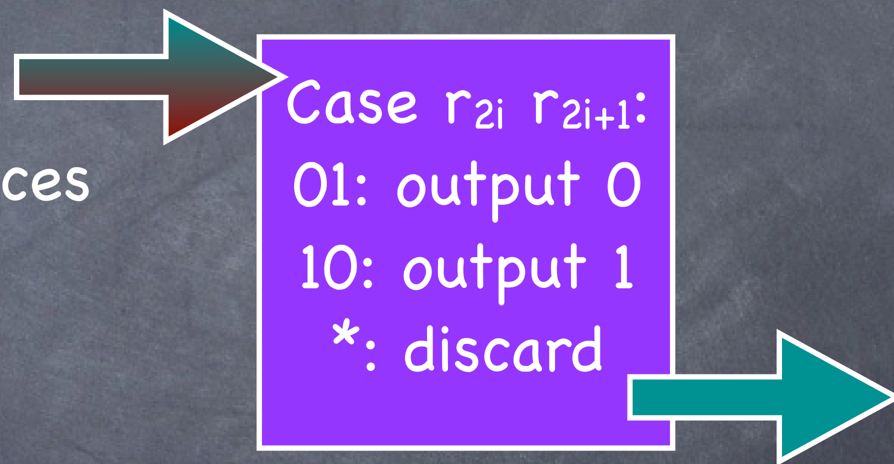

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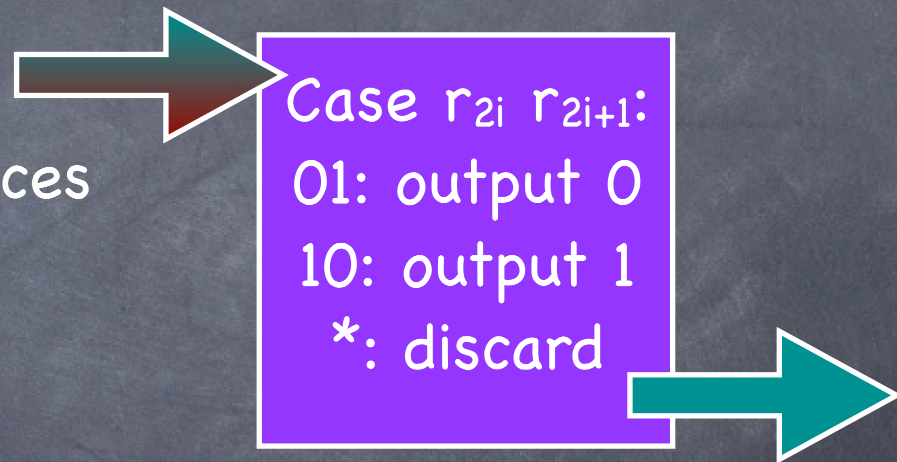

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- Can be generalized to sources which are (hidden) Markov chains

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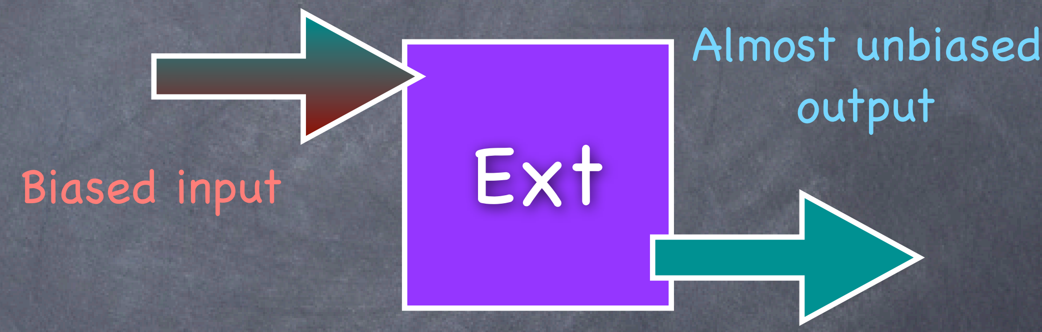

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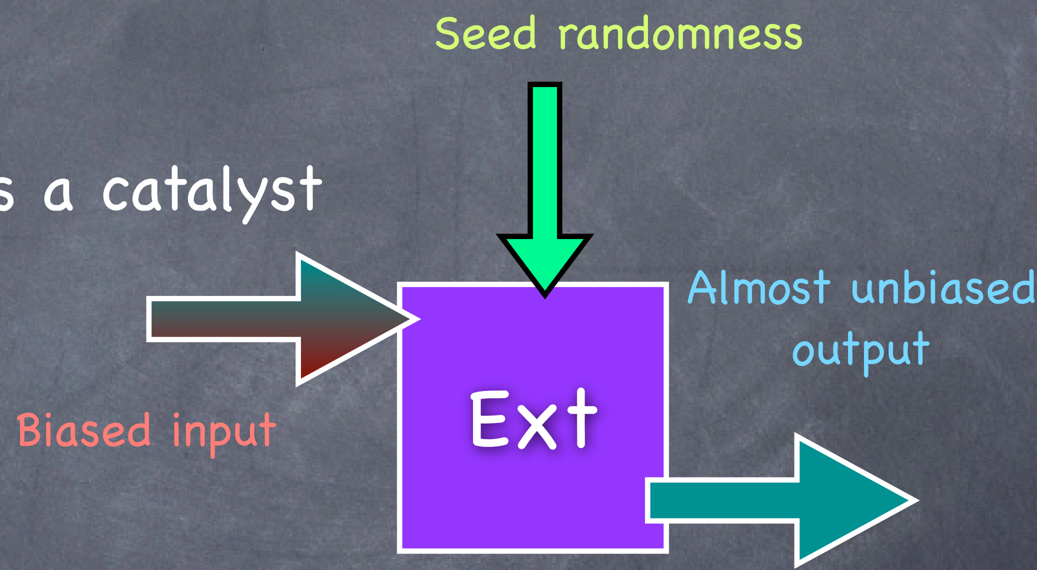

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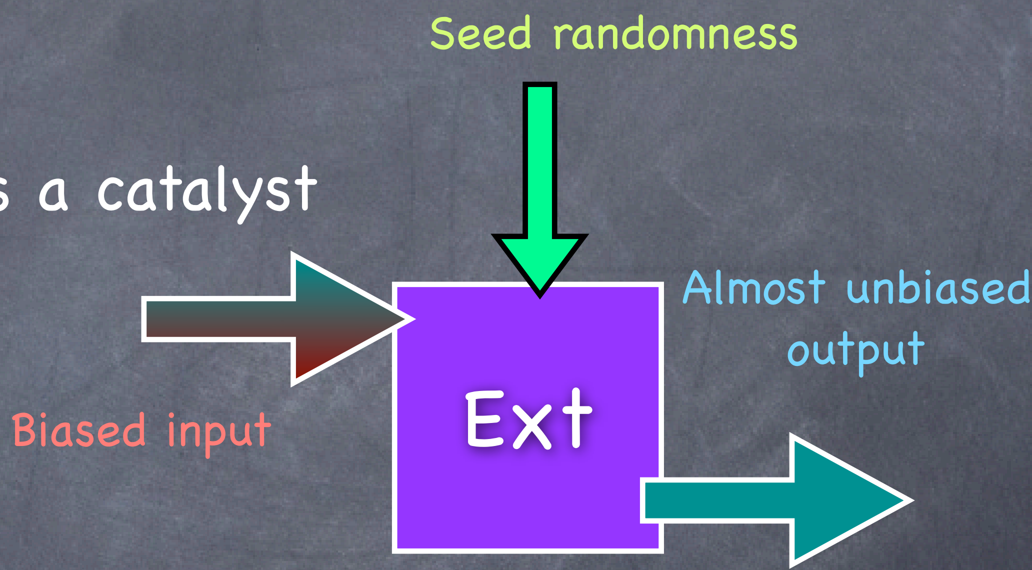

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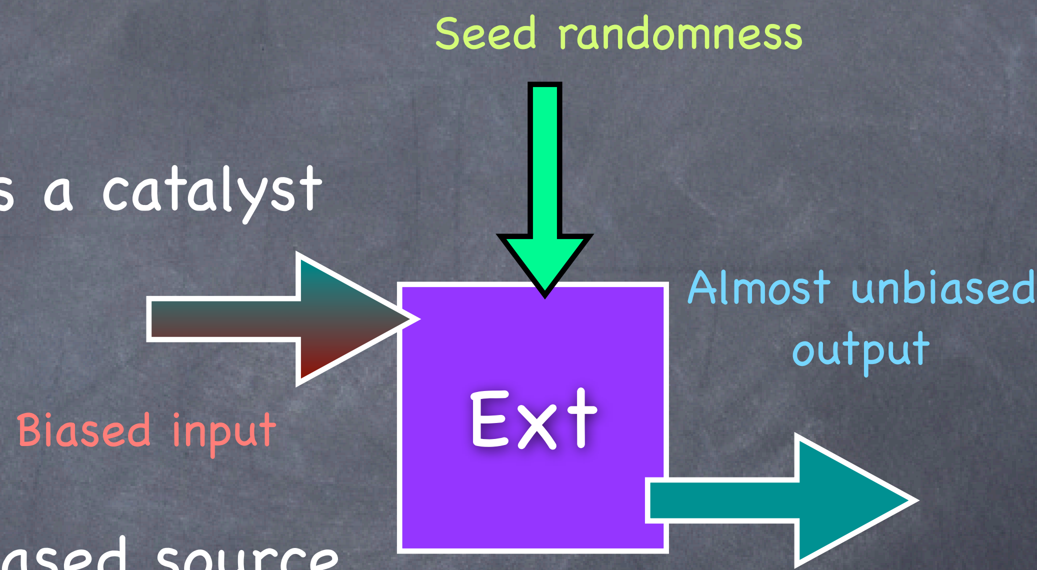

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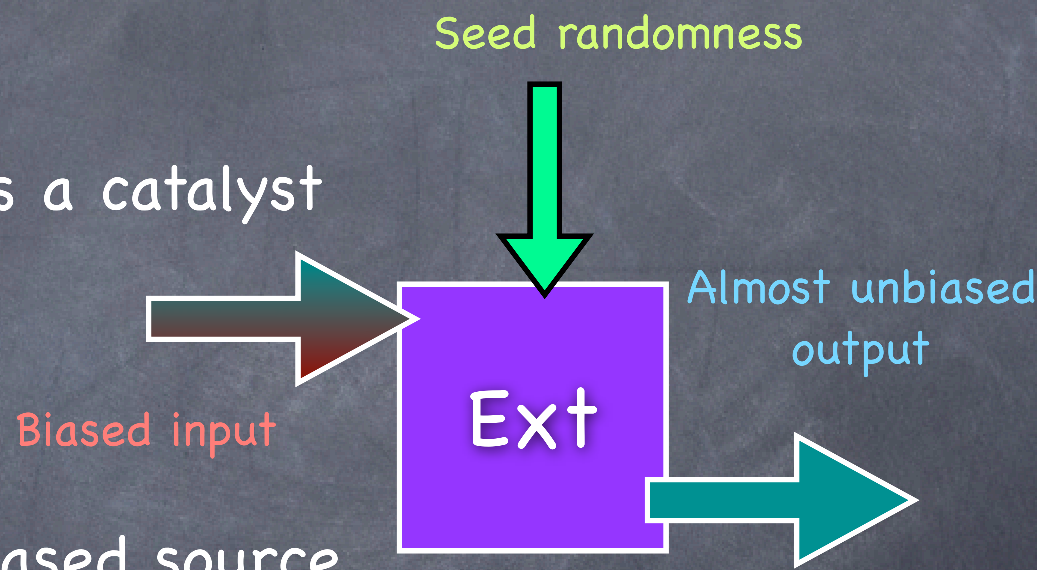

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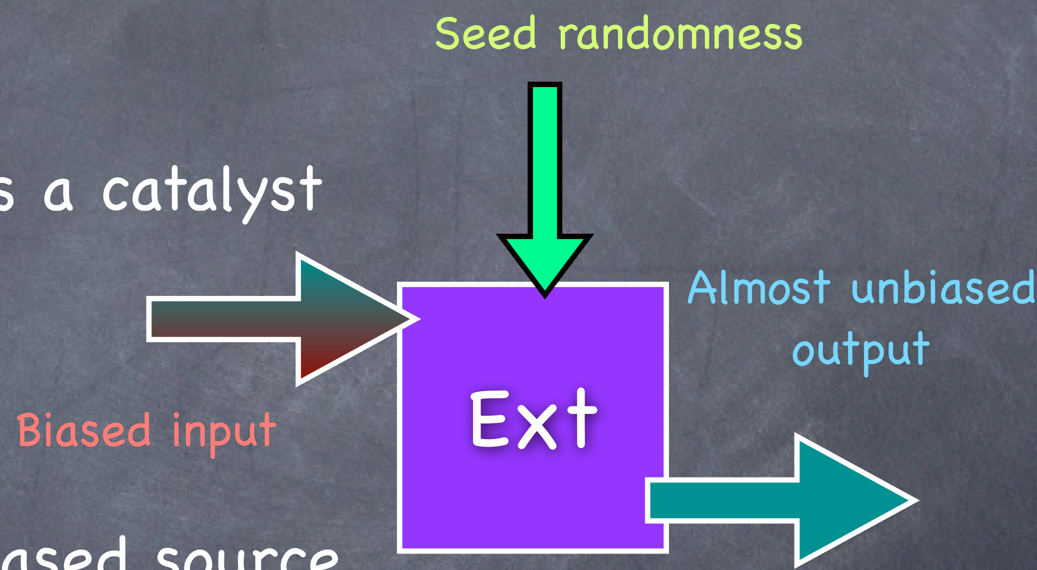
    - Error probability remains bounded [Exercise]



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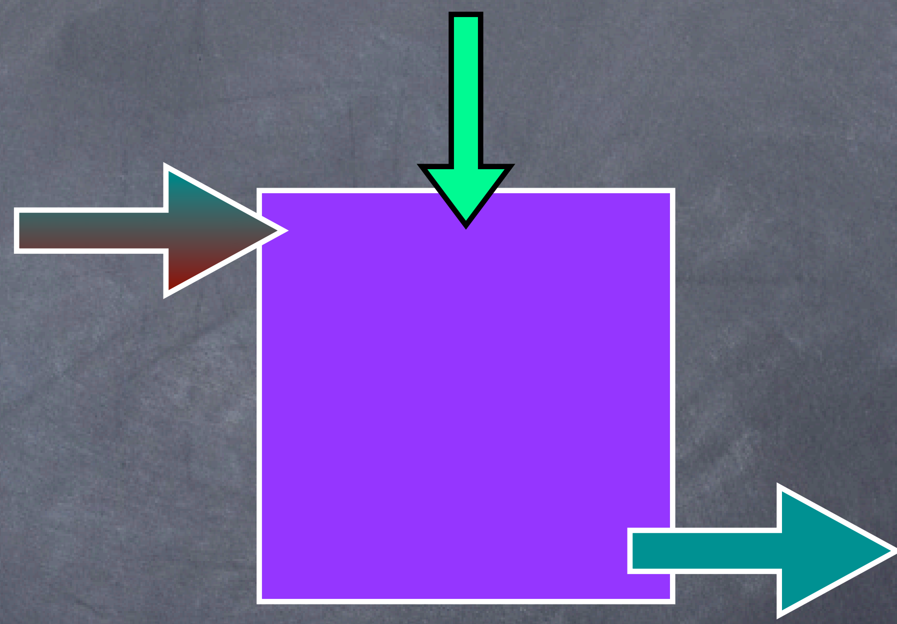

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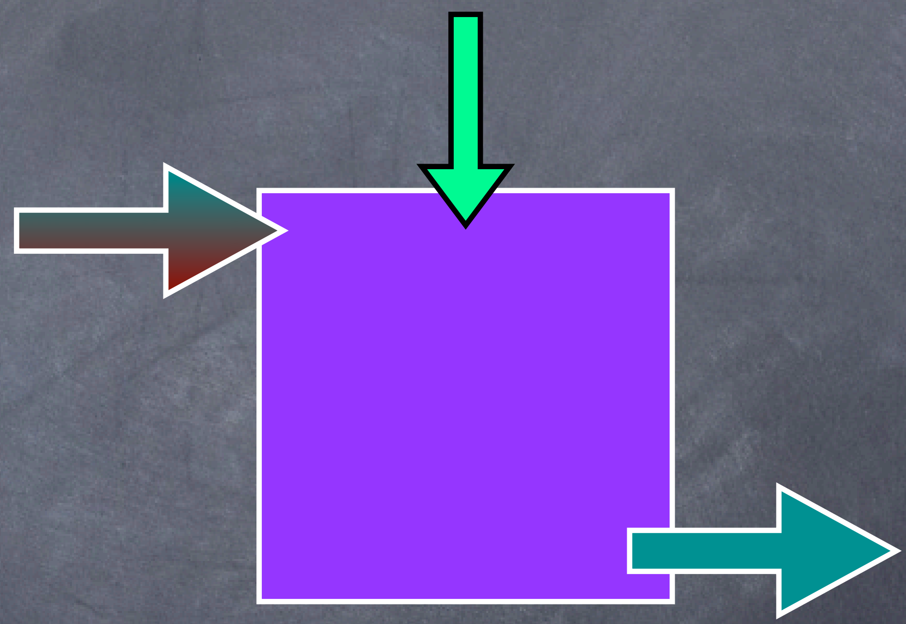

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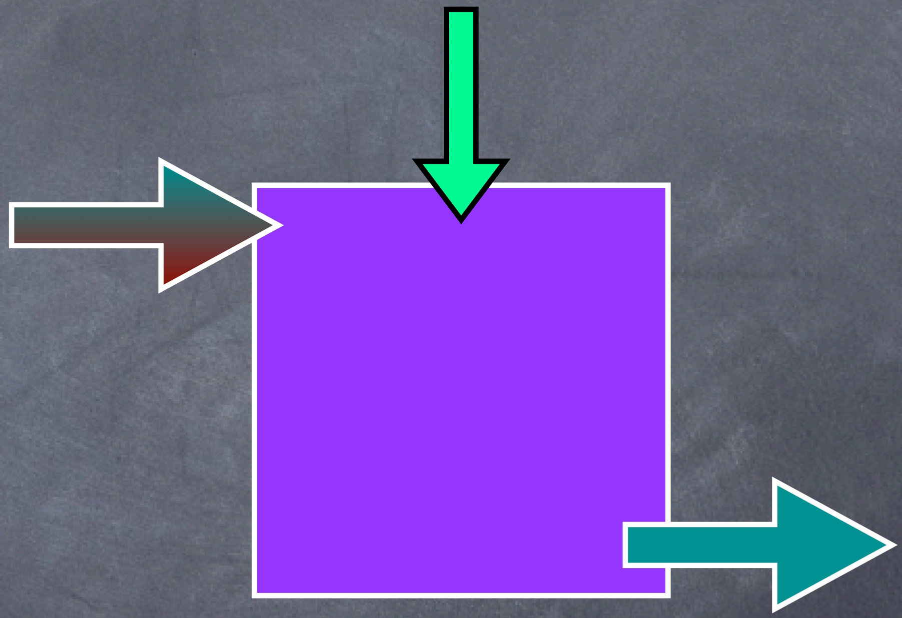

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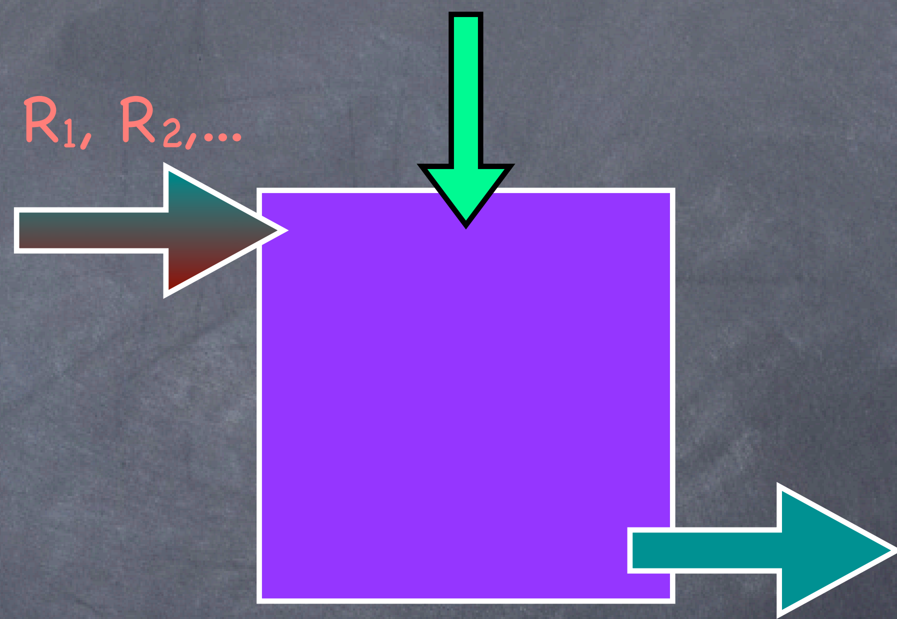

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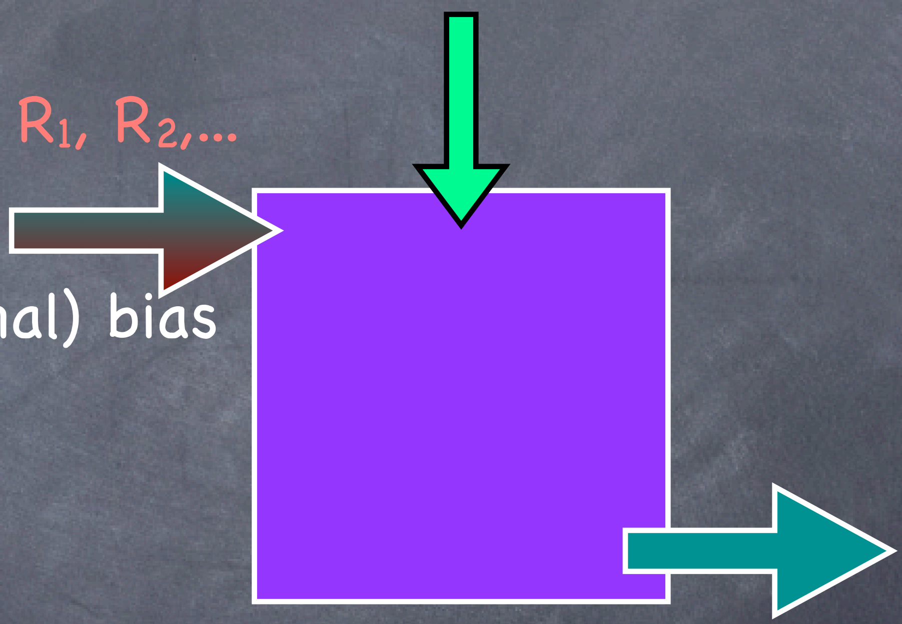

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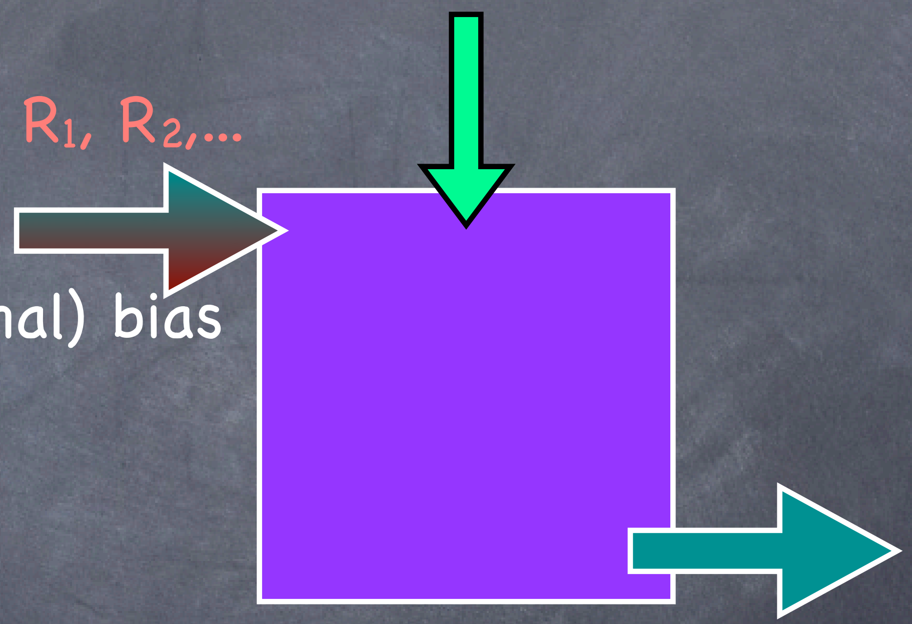

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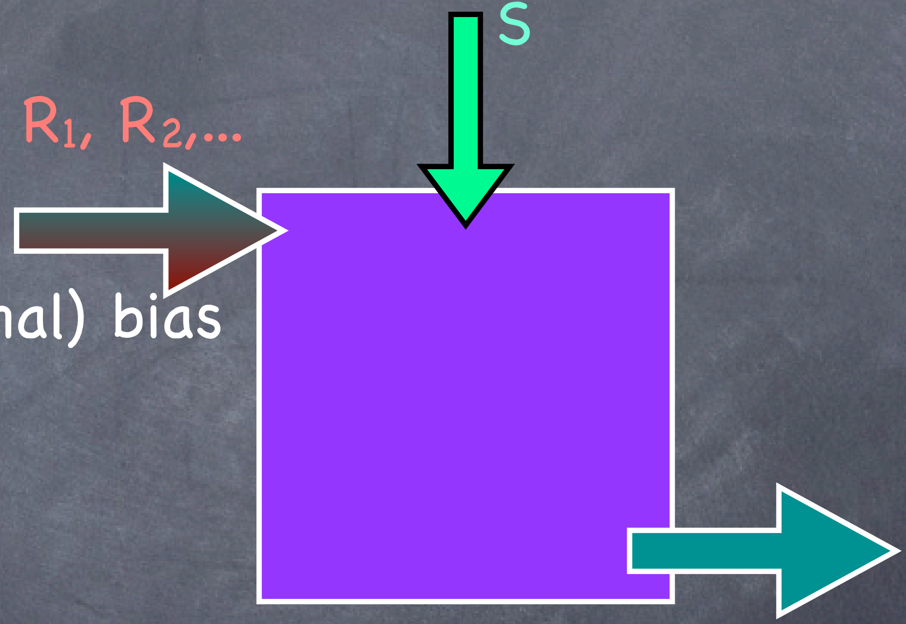

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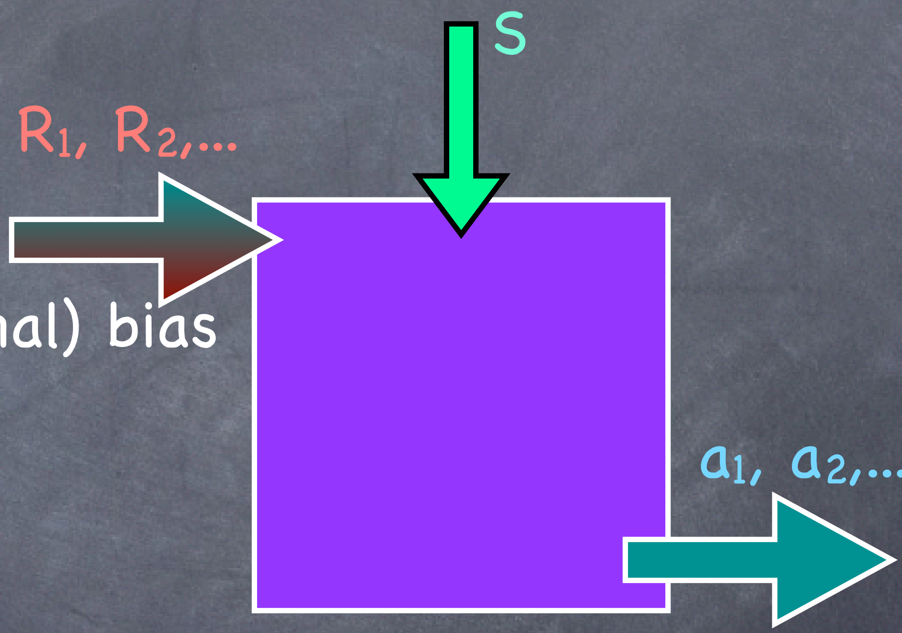

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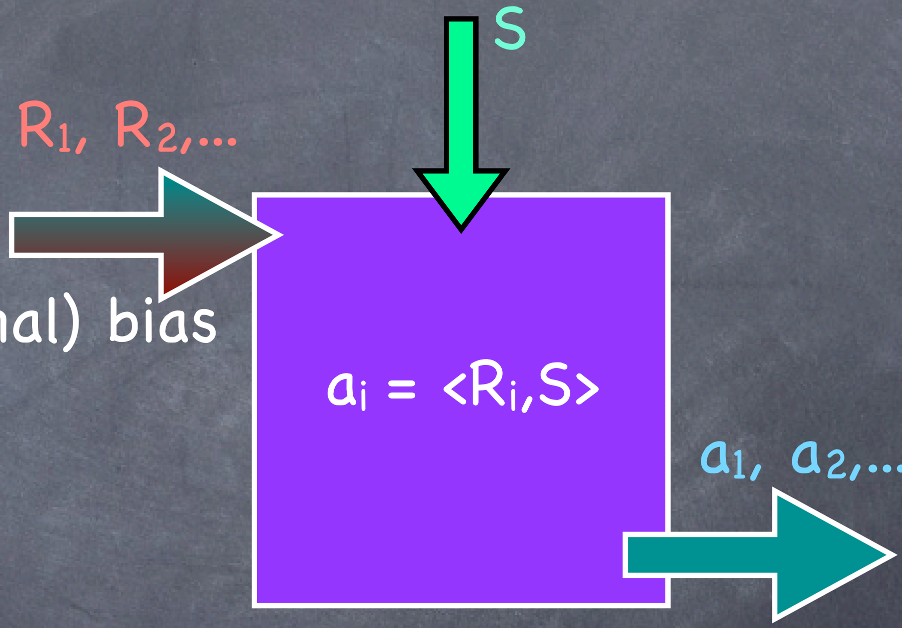
  - Plan: to get to a small (conditional) bias (O(1/m)) for each output bit.
    - Weak extraction



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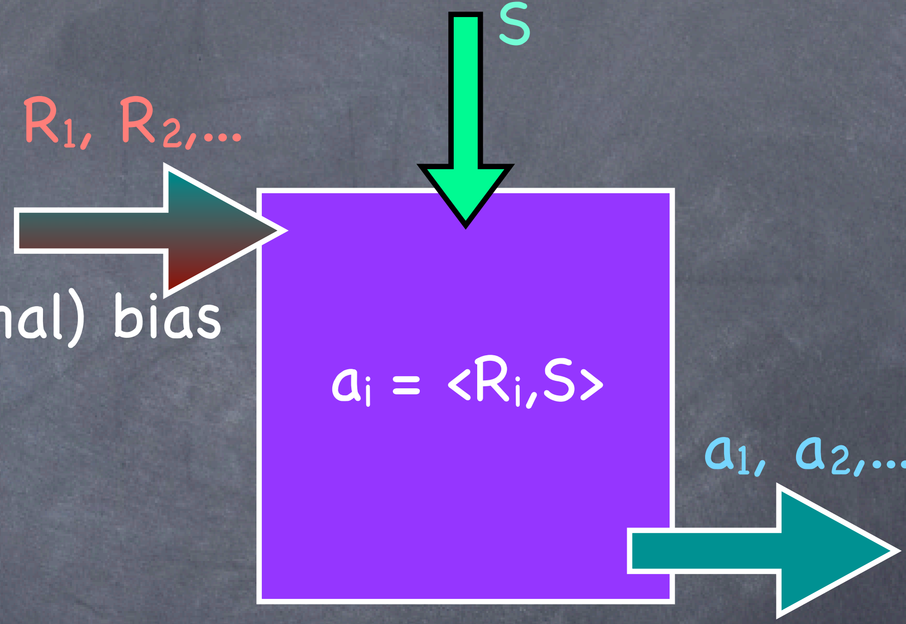

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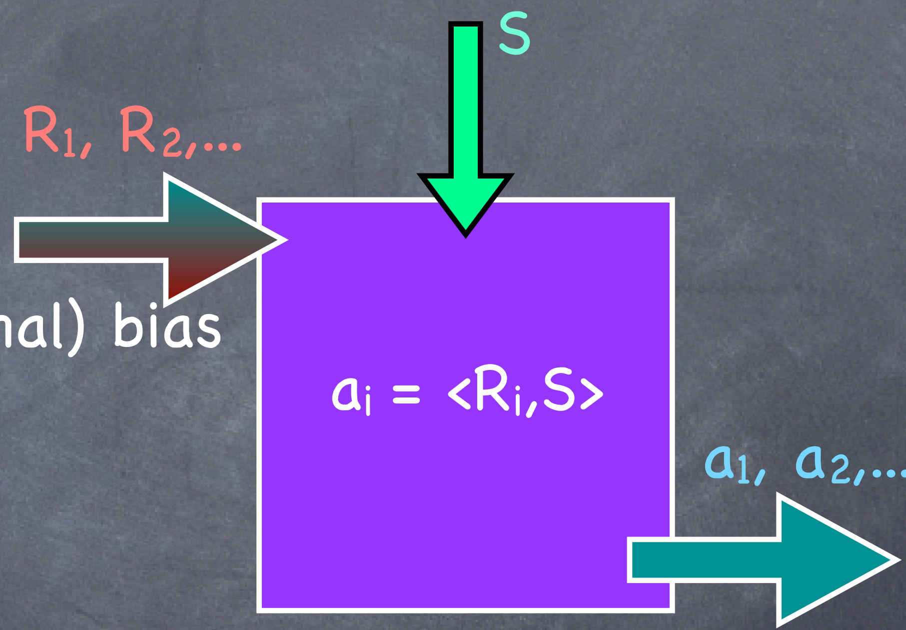

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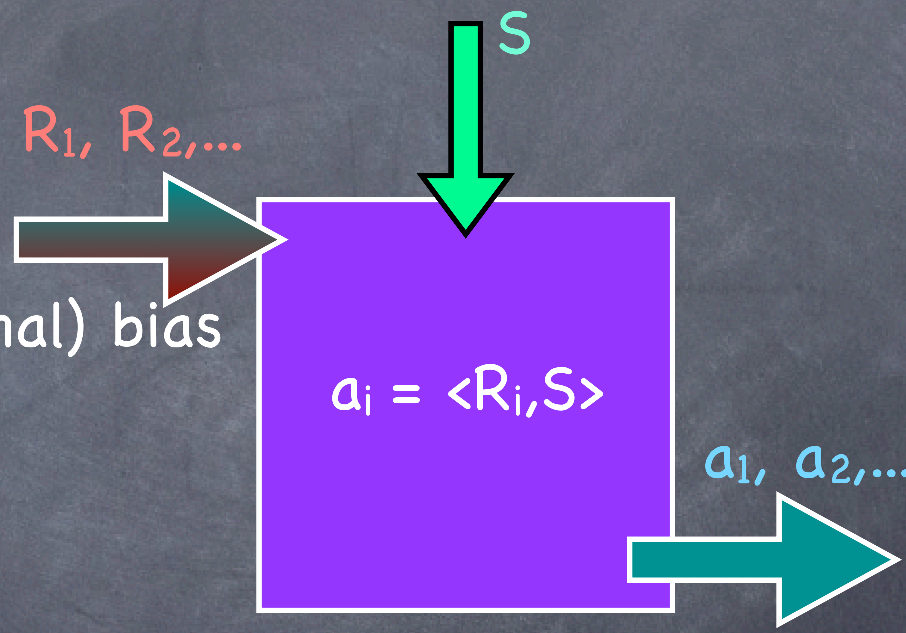
  - Collision prob ≤ max prob ≤ $(1/2 + \delta/2)^d$ = 1/poly(m)



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- Bottom line: Can efficiently run BPP algorithms using very general classes of sources of randomness

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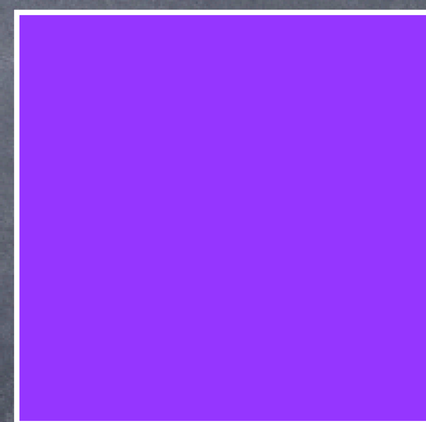

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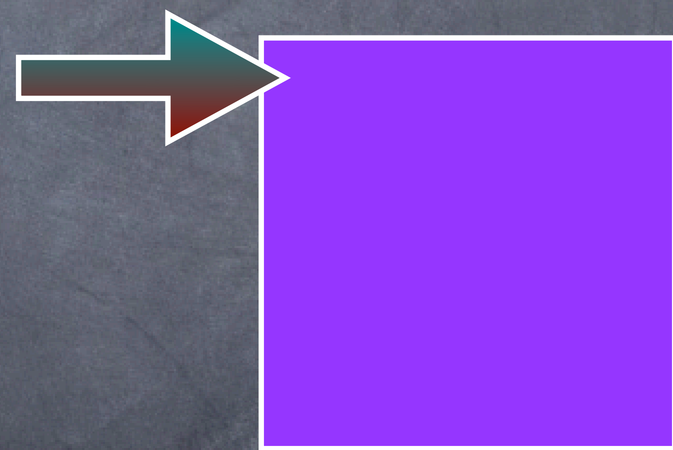

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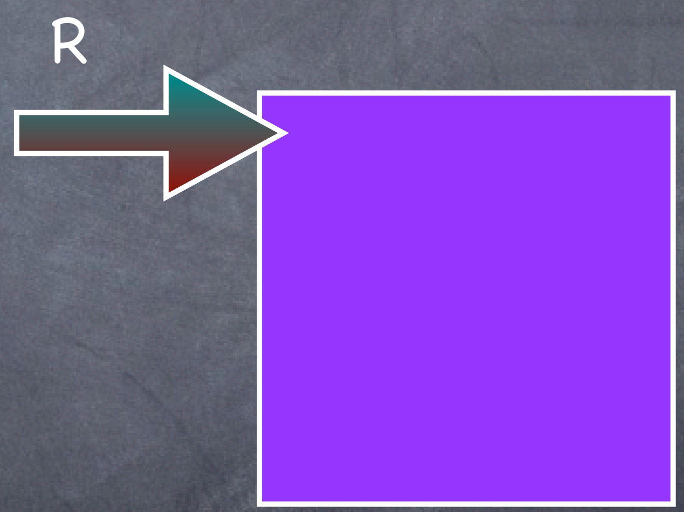

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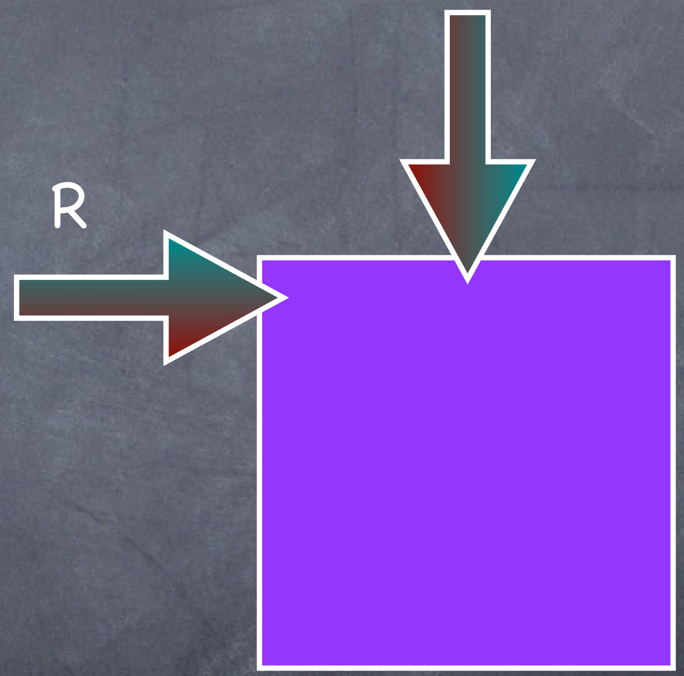

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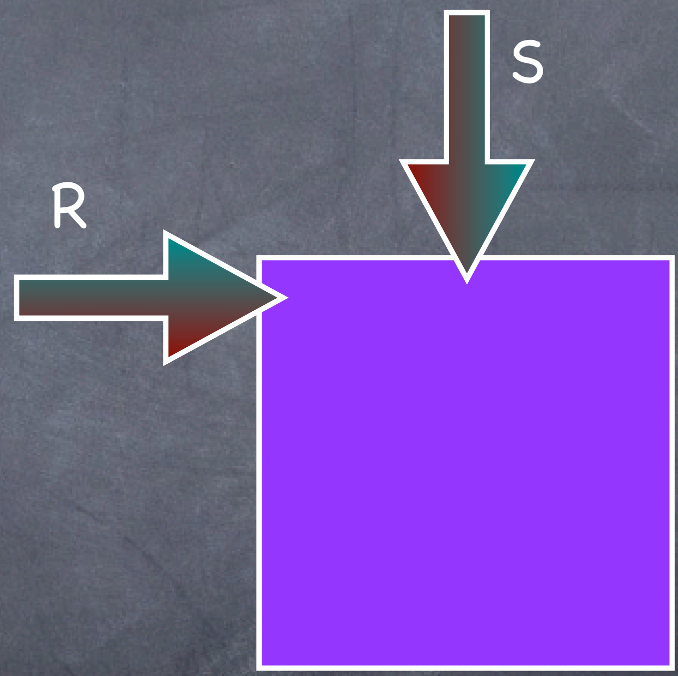

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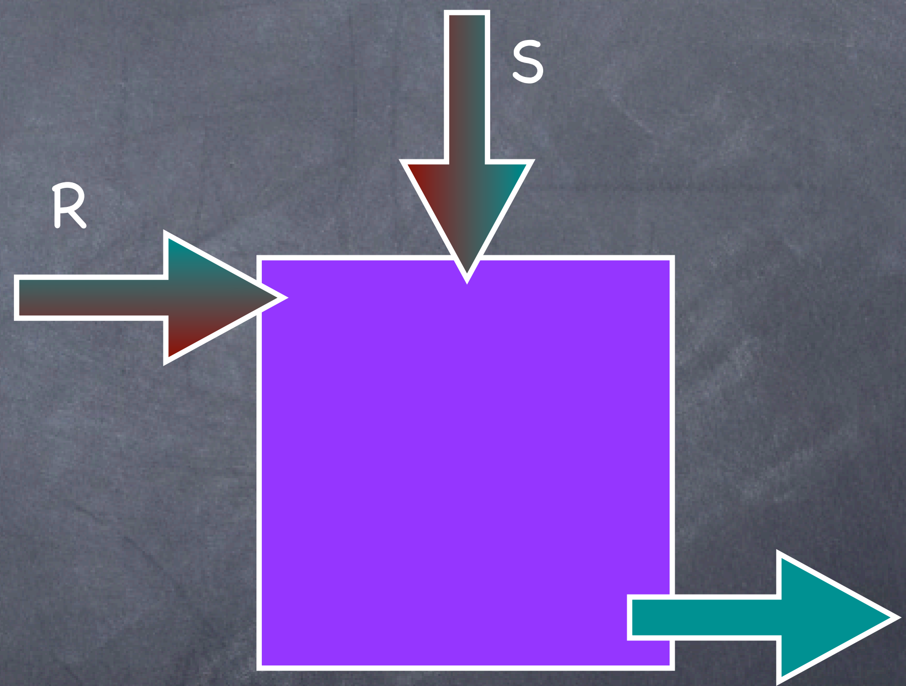

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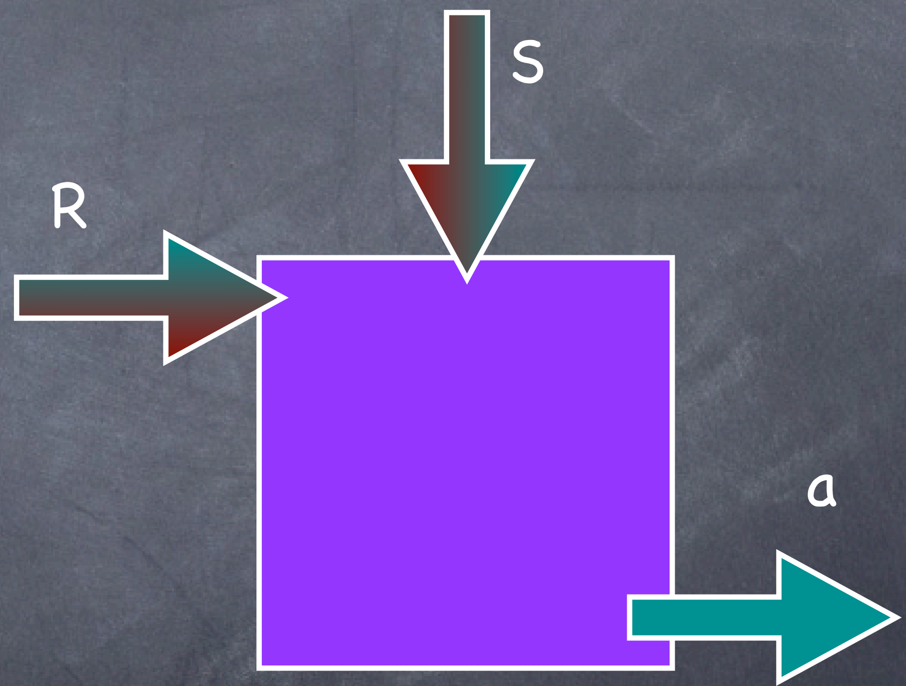

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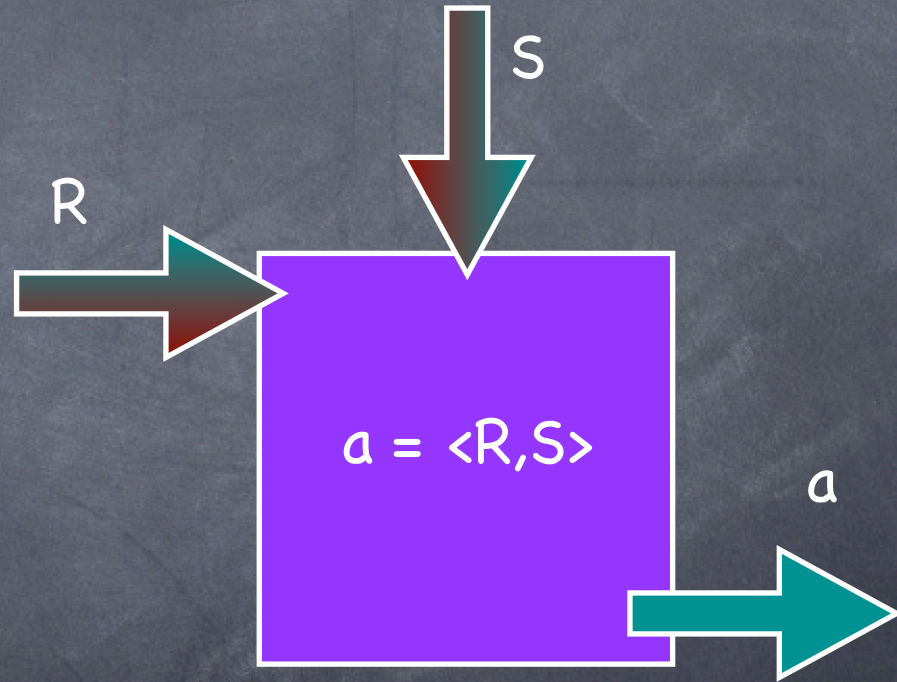

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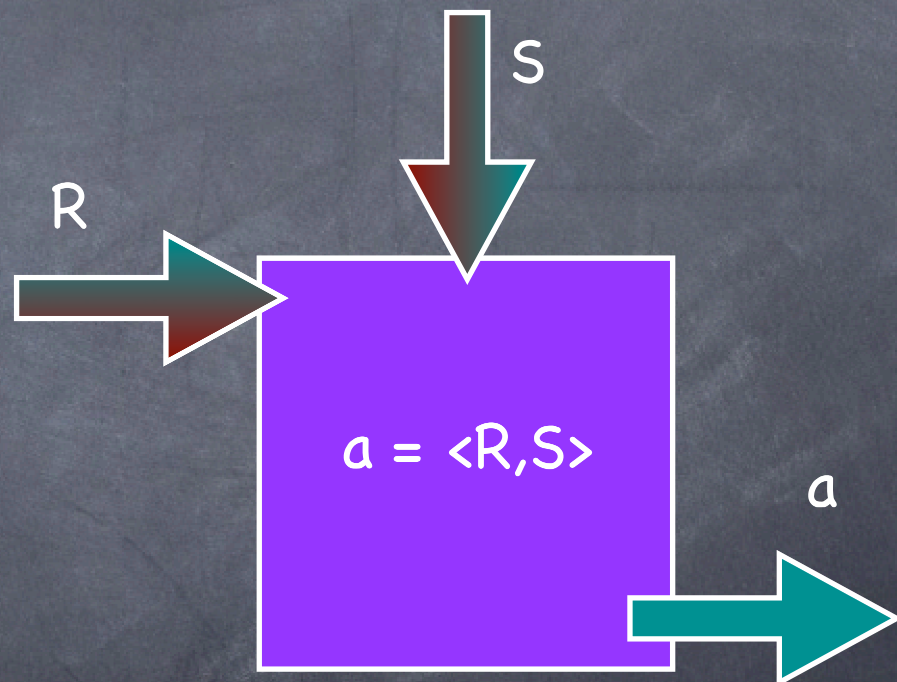

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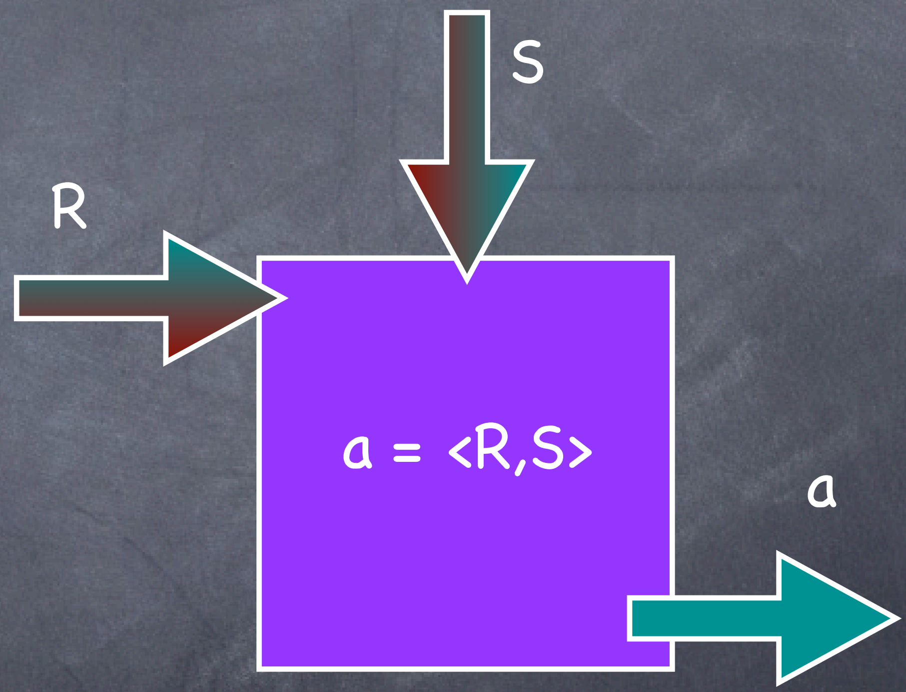
  - Known, with a few more sources



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