Probabilistic Computation

Lecture 12

Flipping coins, taking chances

PP, BPP

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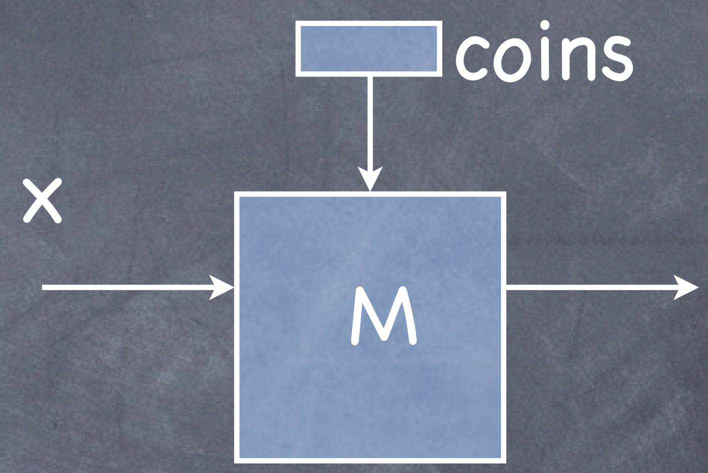
Probabilistic Computation



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    - Last two: If on any x neither, M doesn't decide a language!
      - When M does decide, much better than random guess

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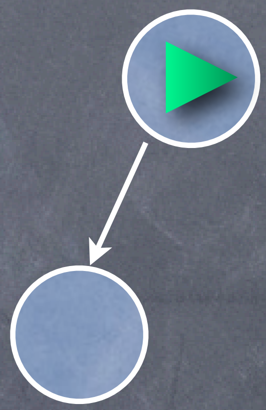

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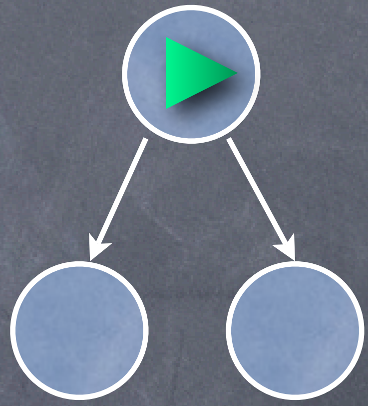

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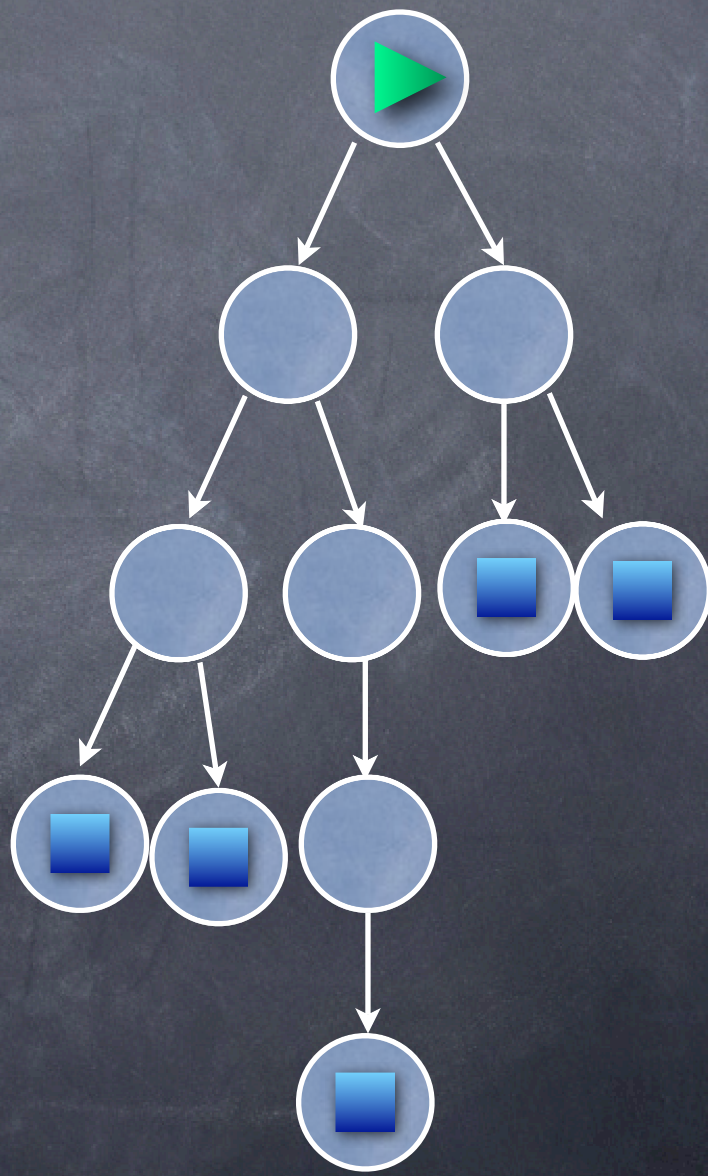

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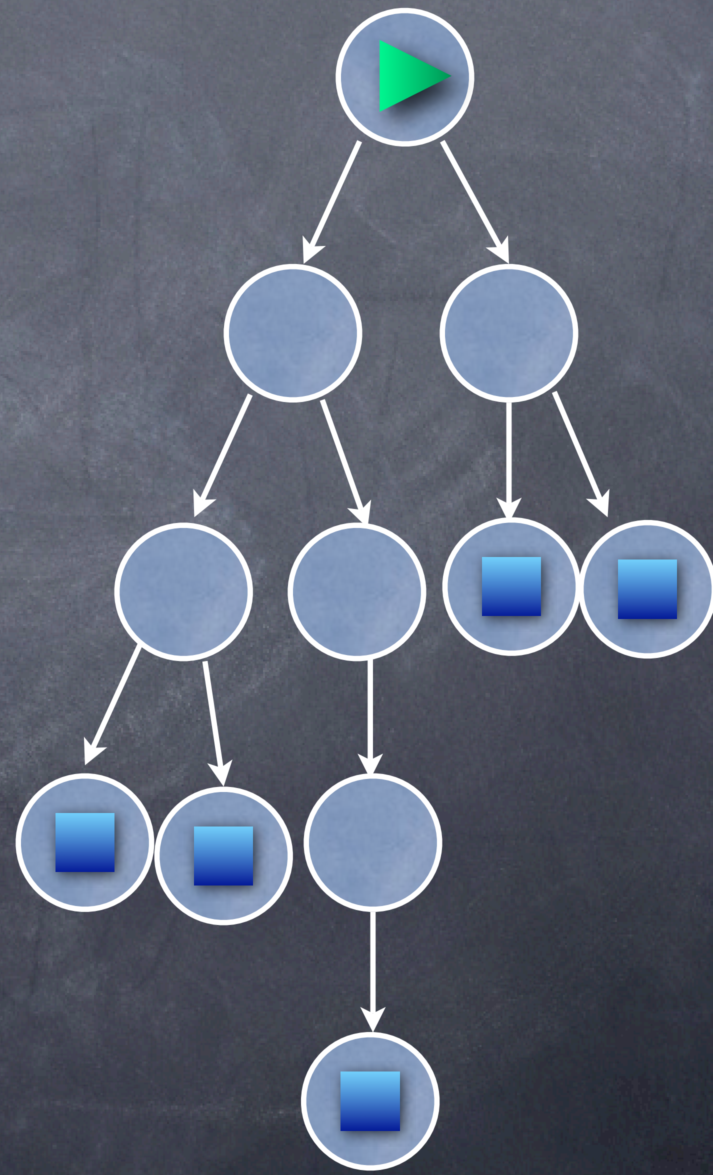

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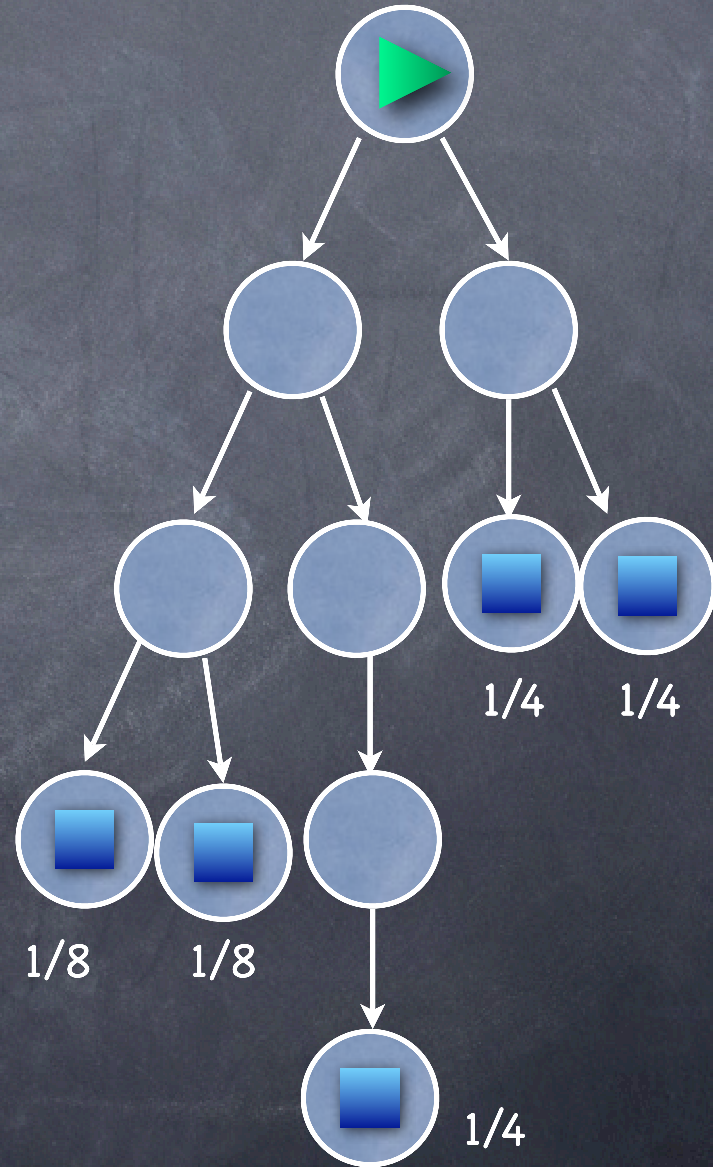
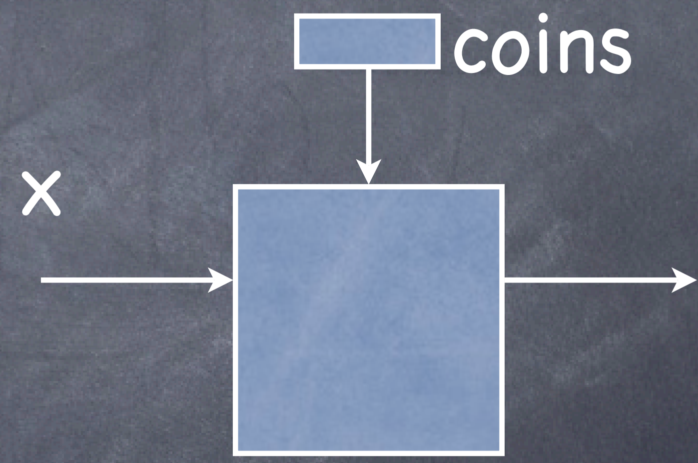
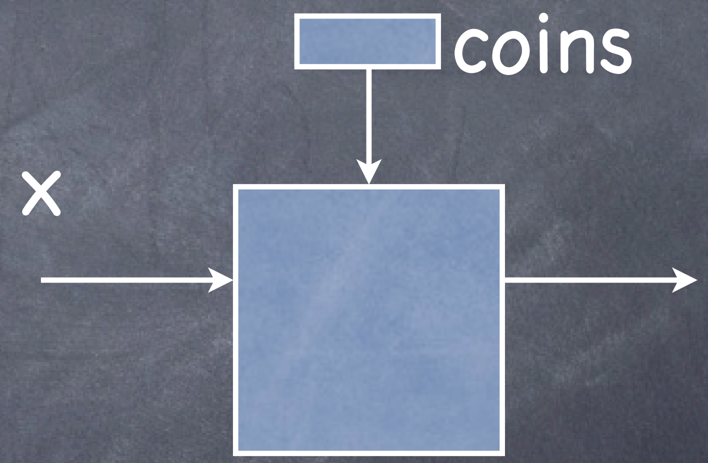


Random Tape



Random Tape

- Random choice: flipping a fair coin



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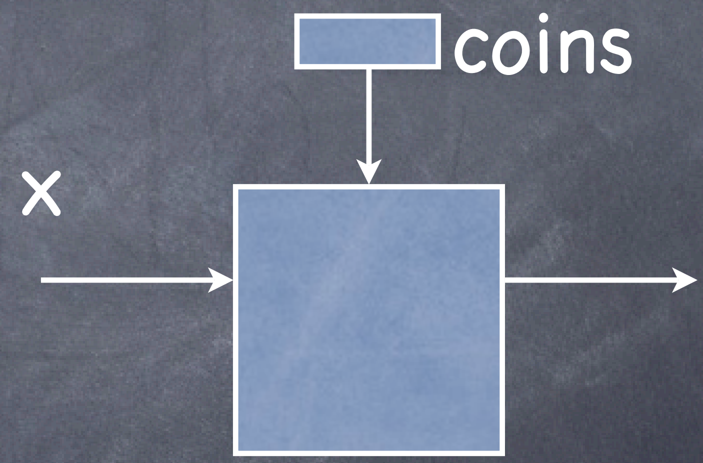

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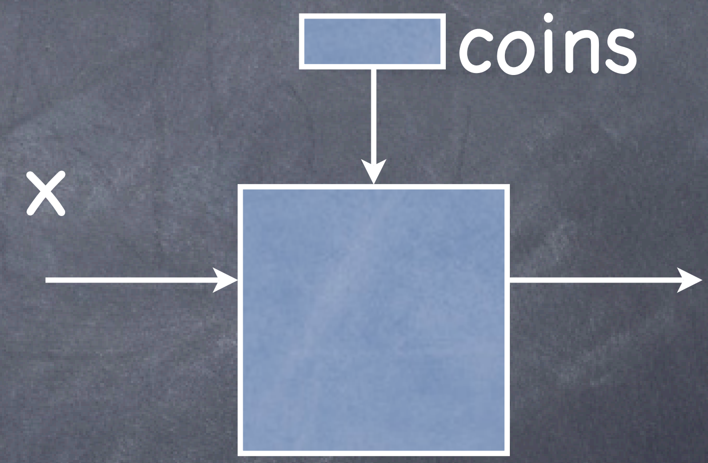
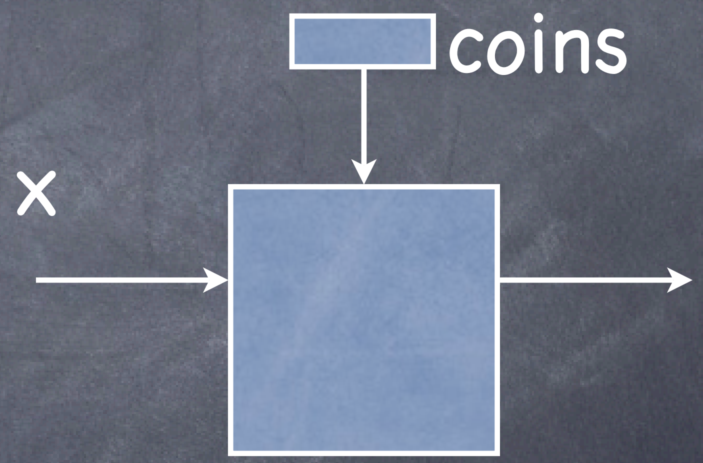


Random Tape

- Random choice: flipping a fair coin
  - Coin flip is written on a read-once "random tape"
  - Enough coin flips made and written on the tape first, then start execution
    - When considering bounded time TMs length of random tape (max coins used) also bounded



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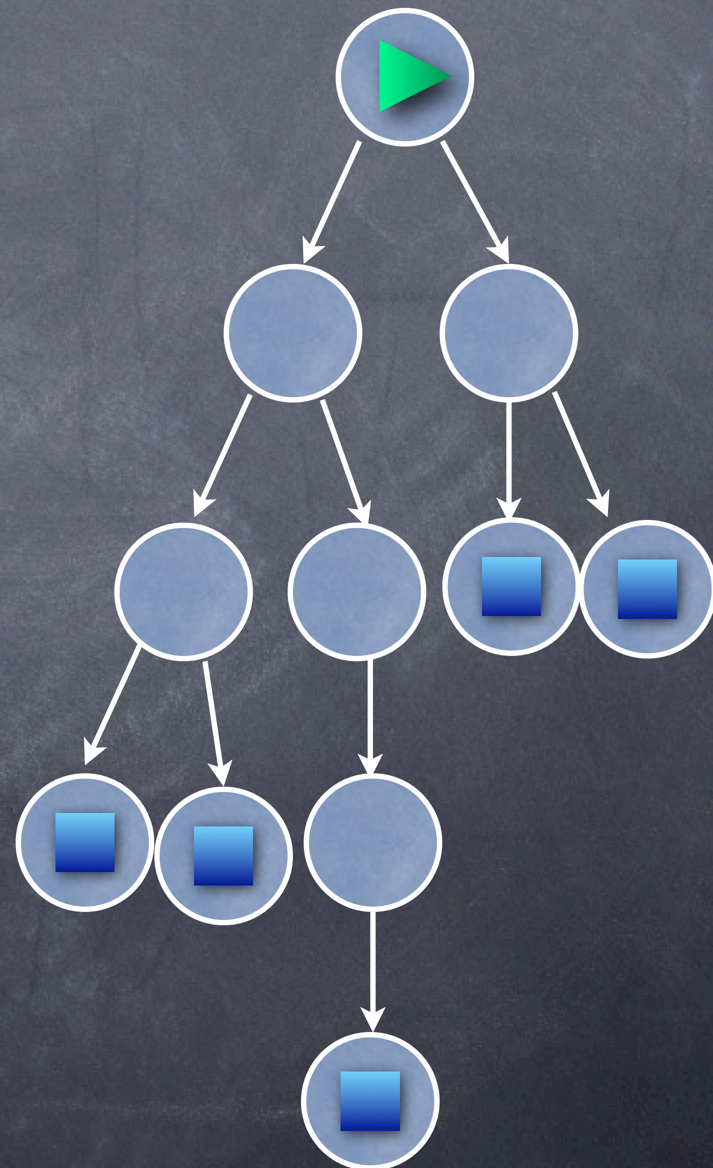

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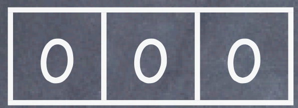
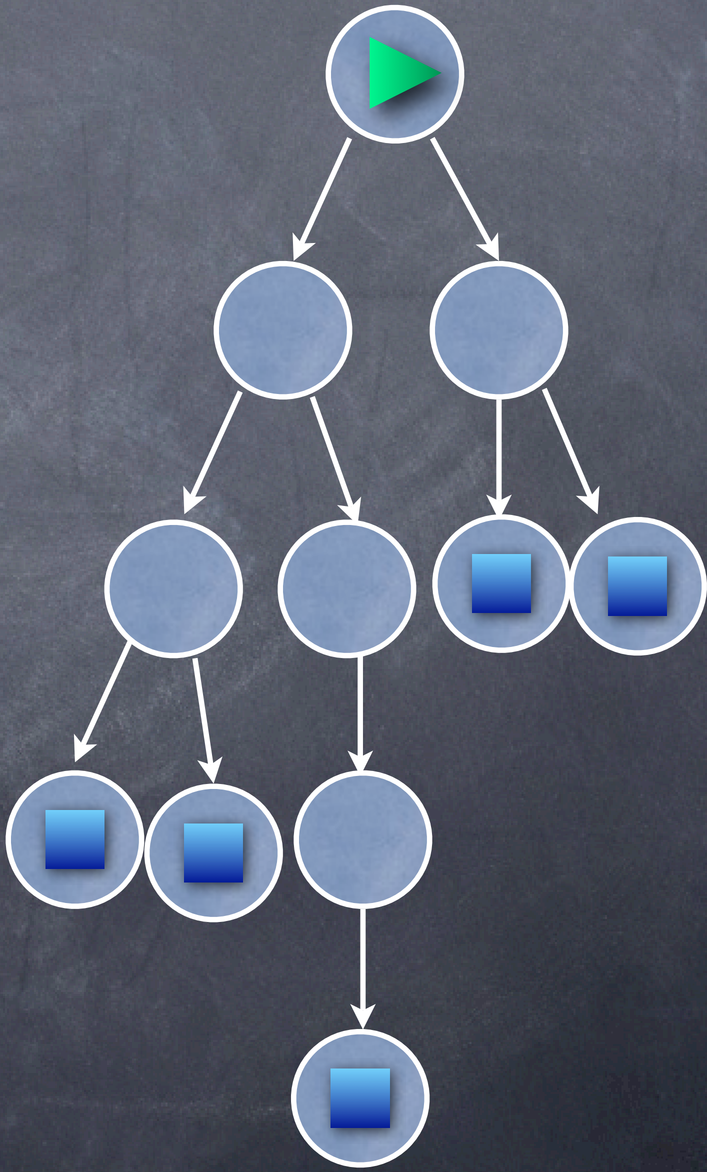
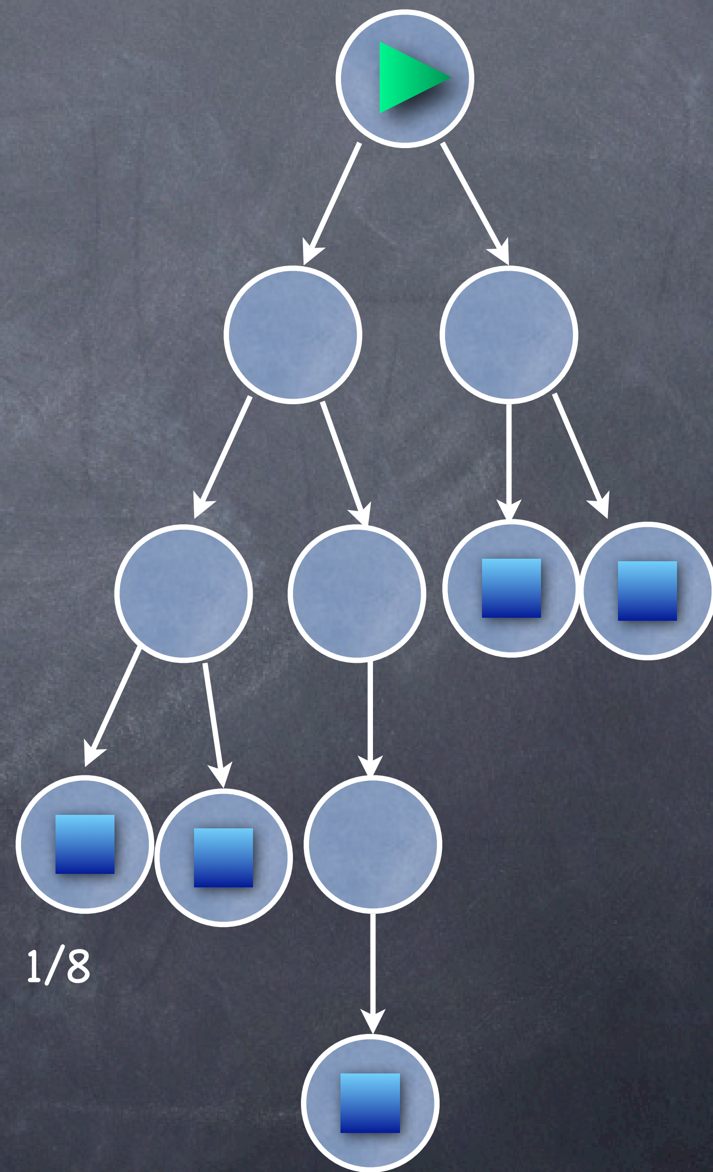




Random Tape

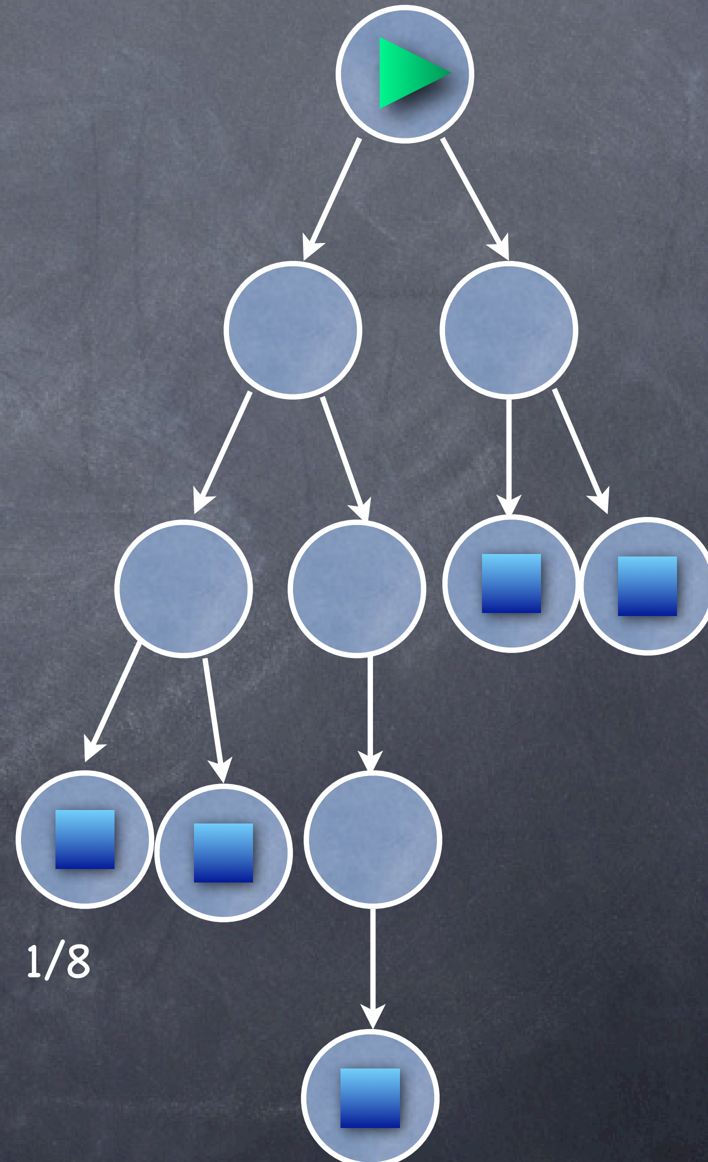
0	0	0



Random Tape

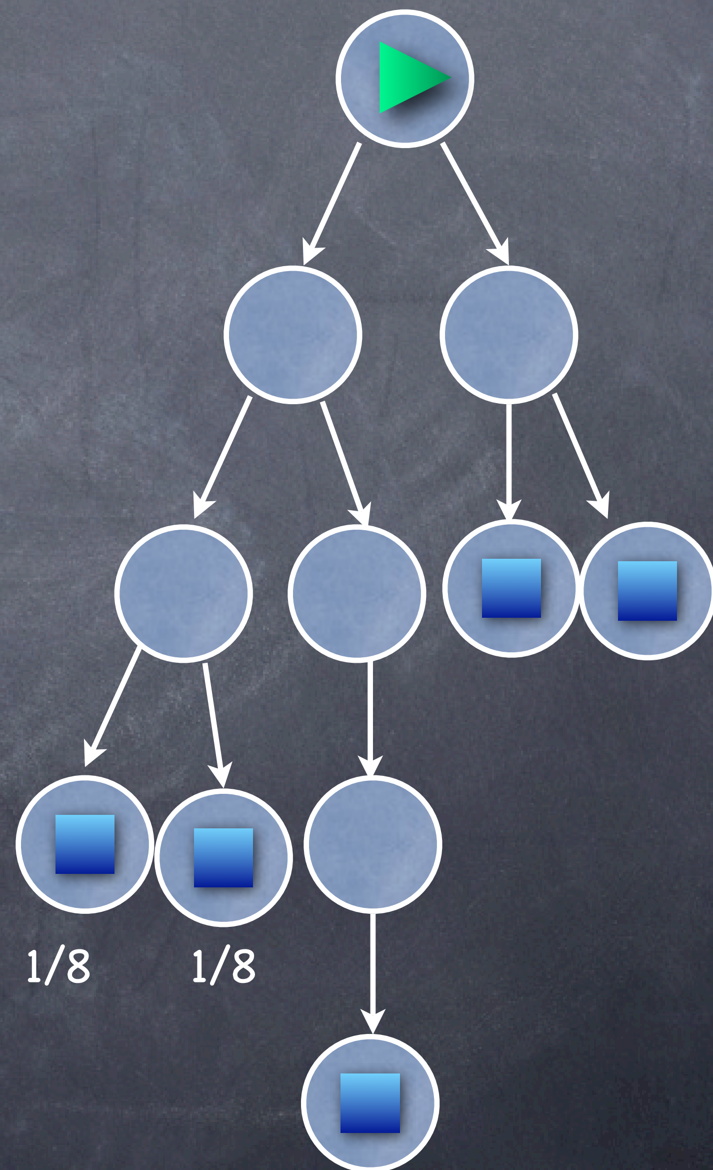
0	0	0

0	0	1



Random Tape

0	0	0
0	0	1

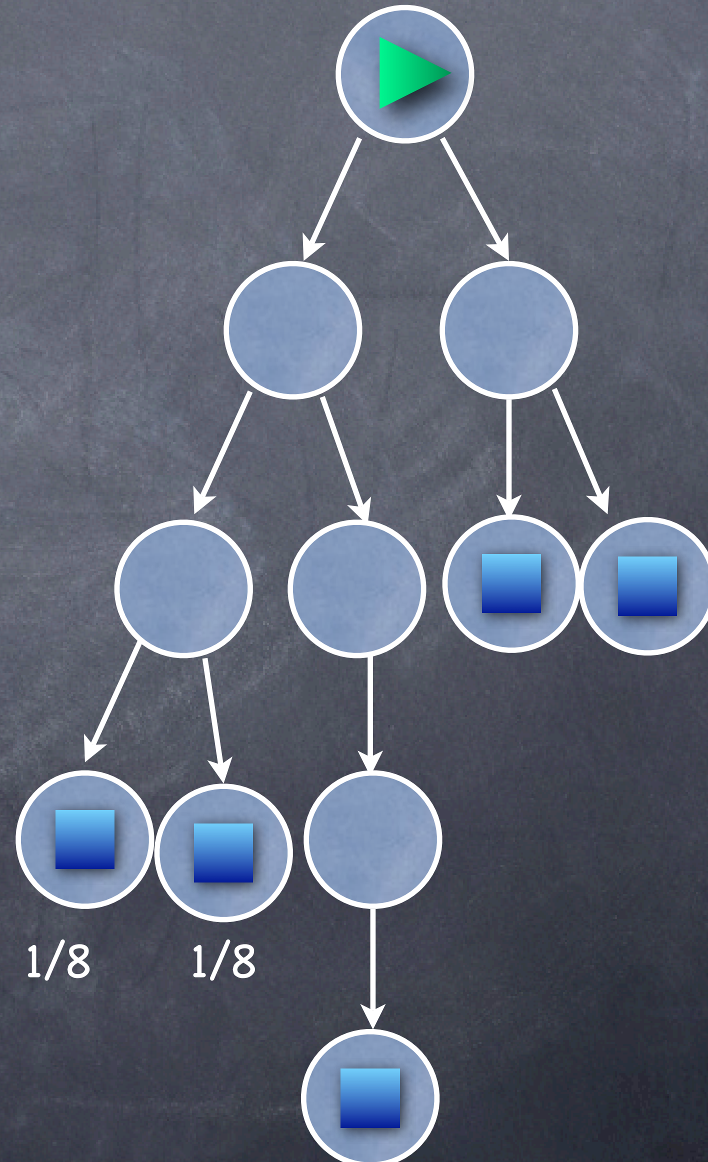


Random Tape

0	0	0

0	0	1

0	1

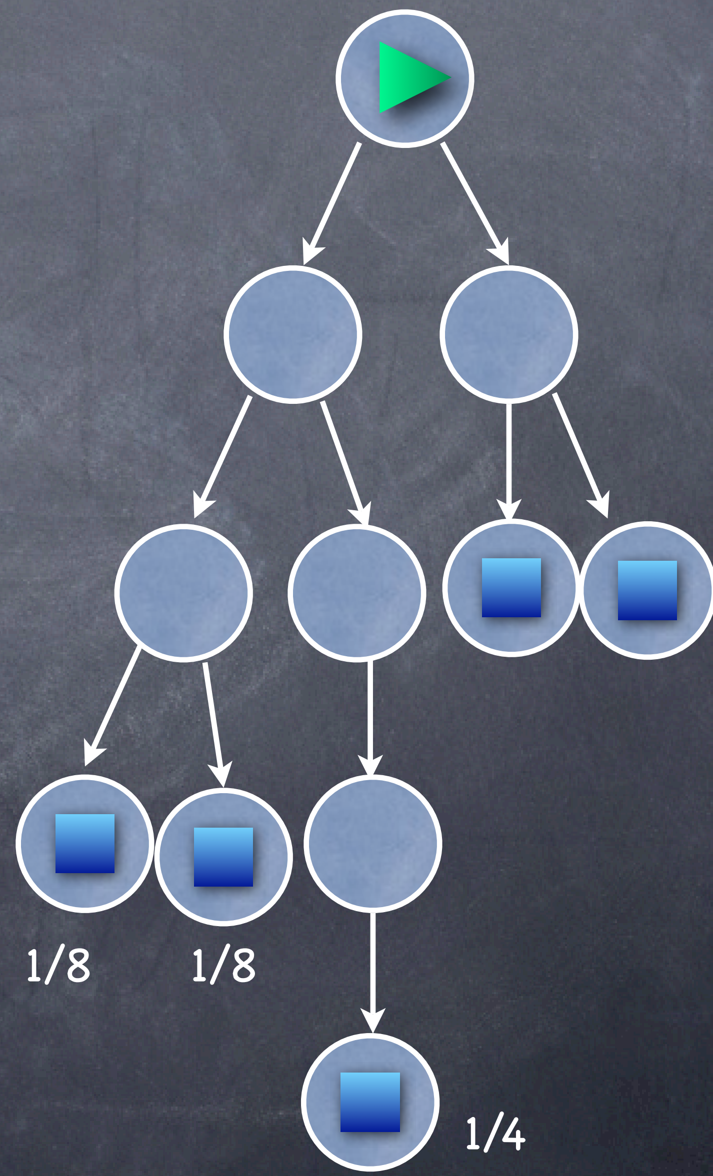


Random Tape

0	0	0

0	0	1

0	1



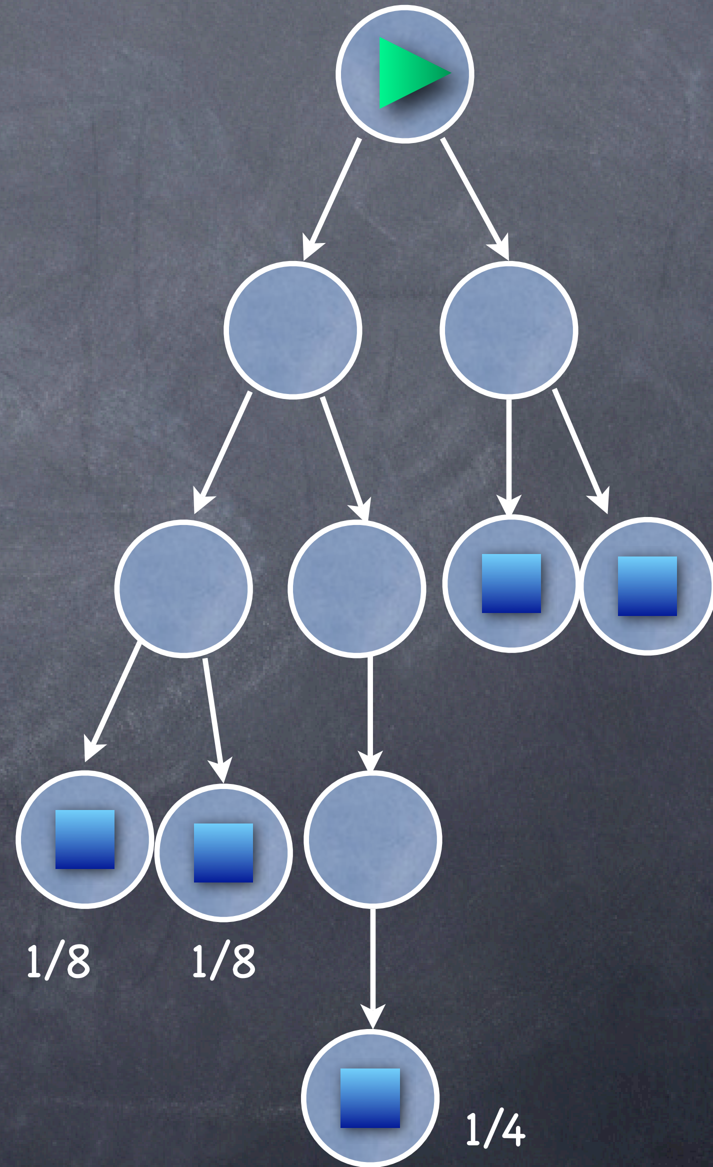
Random Tape

0	0	0

0	0	1

0	1

1	0



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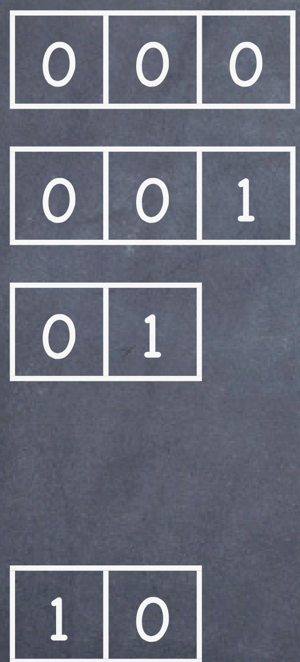

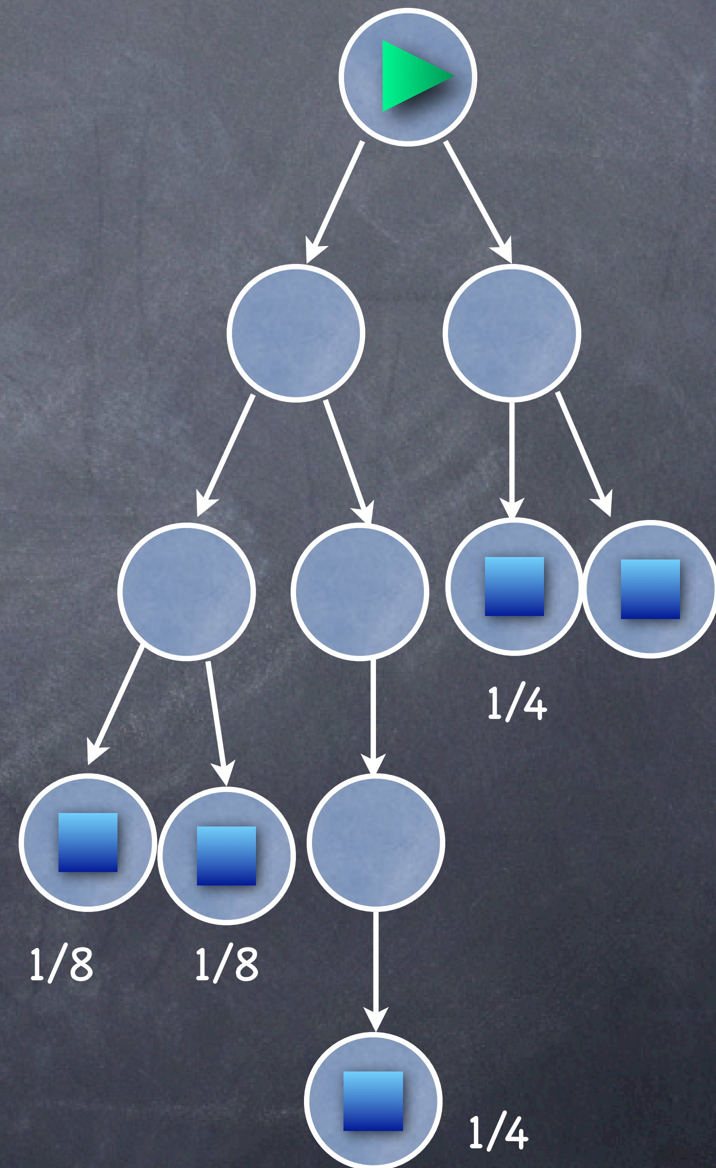

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0	0	0

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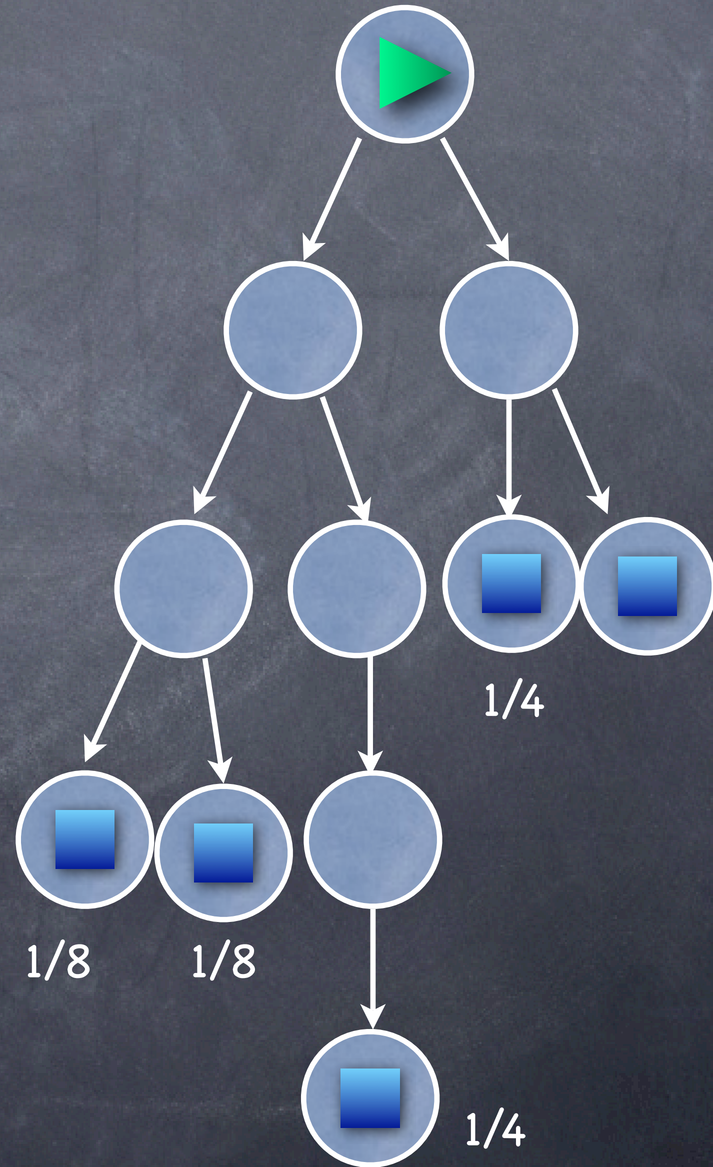

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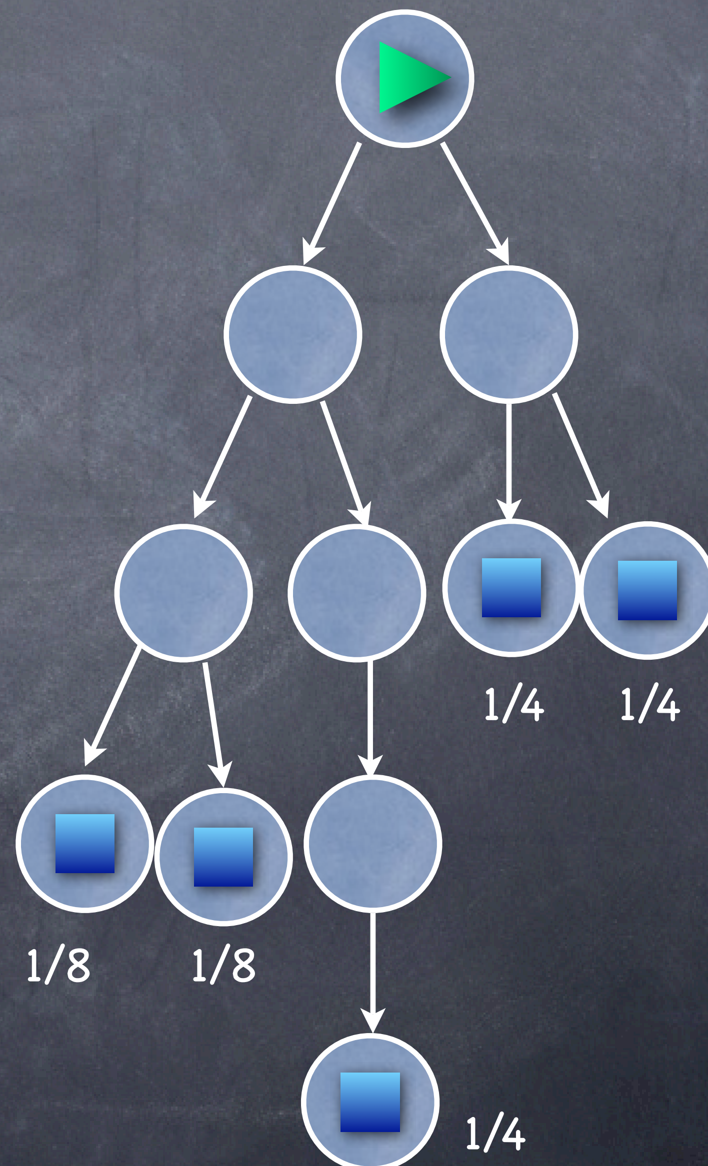
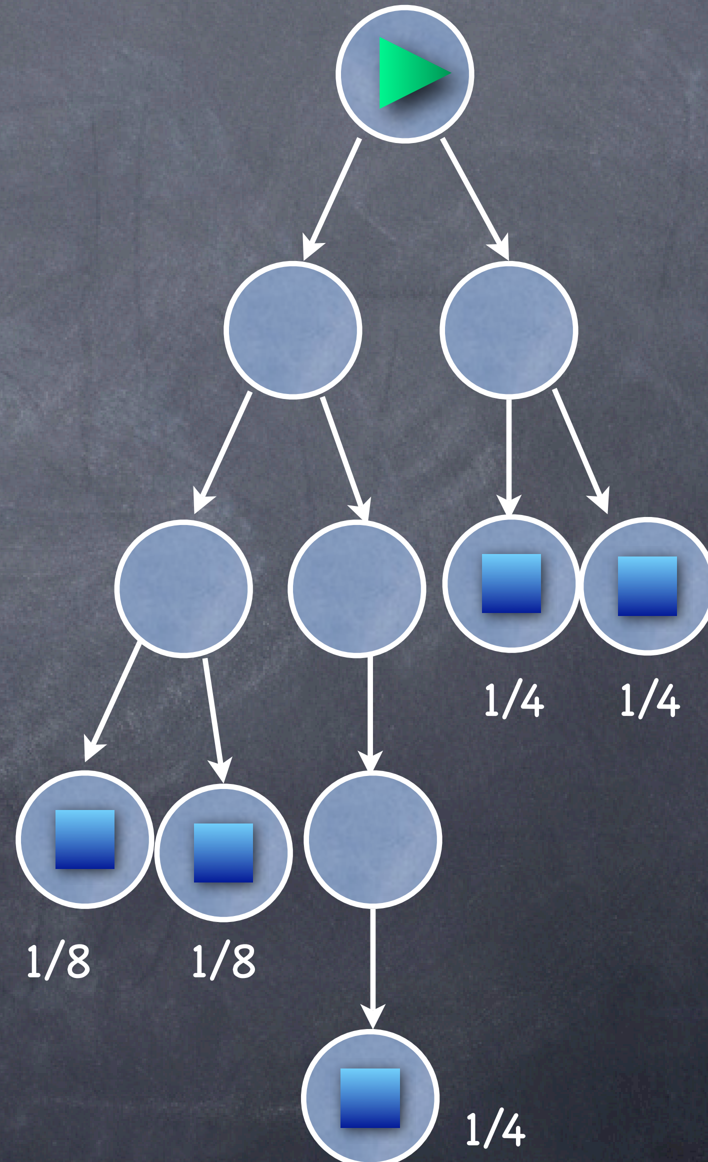


Random Tape

0 0 0	0 0 0
0 0 1	0 0 1
0 1	0 1 0
	0 1 1
1 0	1 0 0
	1 0 1
1 1	1 1 0
	1 1 1



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  - Accept if pr[yes] > 2/3; reject if pr[yes] = 0 — RTM

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- (Not standard nomenclature!)

PTIME, BPTIME and RTIME

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  - Just like NTIME(T)
- BPTIME(T) = class of languages decided by BPTMs in time T
  - Similarly PTIME(T) and RTIME(T)

PP, BPP and RP

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- RP = $\bigcup_{c>0}$ RTIME($O(n^c)$)
  - co-RP

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    - i.e., if co-RTM says yes, can be wrong

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  - PTMs and co-PTMs differ on accepting inputs with Pr[yes]=1/2

co-BPP, co-PP

- BPP = co-BPP
  - co-BPTMs are same as BPTMs
- In fact PP = co-PP
  - PTMs and co-PTMs differ on accepting inputs with Pr[yes]=1/2
  - But can modify a PTM so that Pr[M(x)=yes] ≠ 1/2 for all x, without changing language accepted

PP = co-PP

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- Modifying a PTM M to an equivalent PTM M′, so that for all x $\Pr[M'(x)=\text{yes}] \neq 1/2$

PP = co-PP

- Modifying a PTM M to an equivalent PTM M′, so that for all x Pr[M′(x)=yes] ≠ 1/2
  - Consider M′(x): w.p. 1/2 run M(x); w.p. 1/2, ignore input and say yes w.p. 1/2 - ϵ, and say no w.p. 1/2 + ϵ

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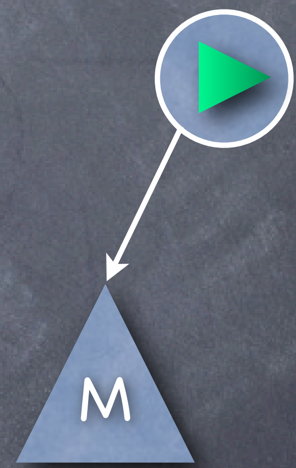

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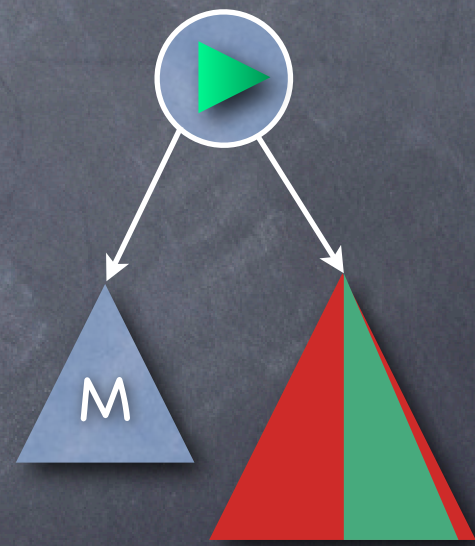

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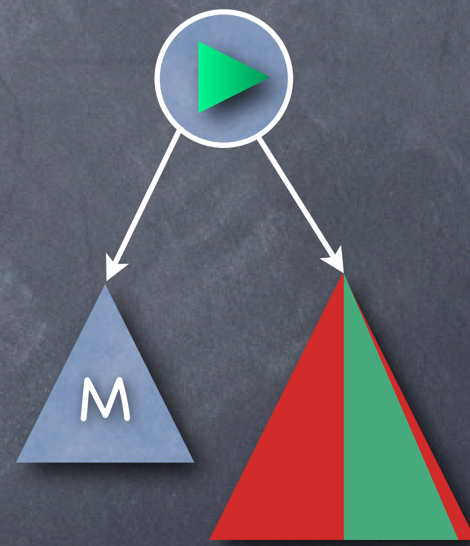
    - pr[M′(x)=yes] = pr[M(x)=yes]/2 + (1/2 - ϵ)/2



PP = co-PP

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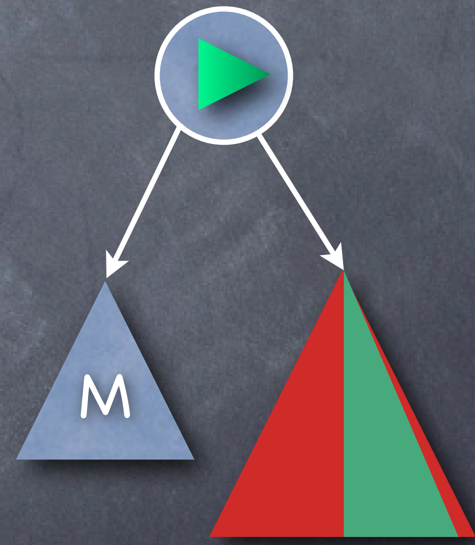

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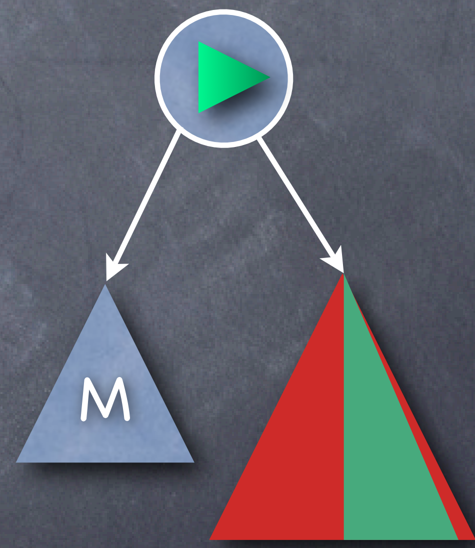

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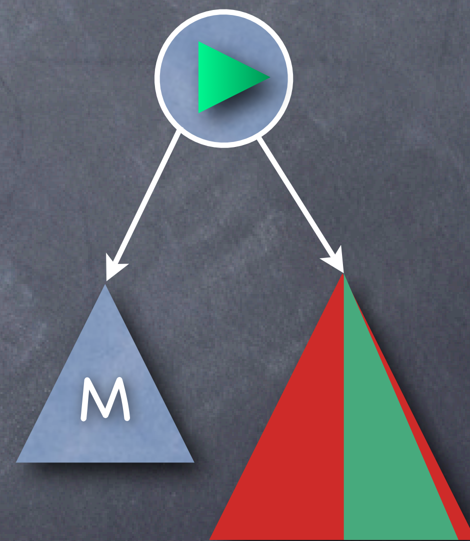

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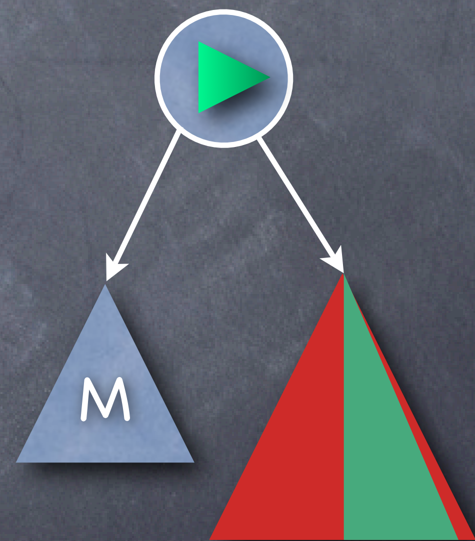
      - M′ tosses at most m+2 coins



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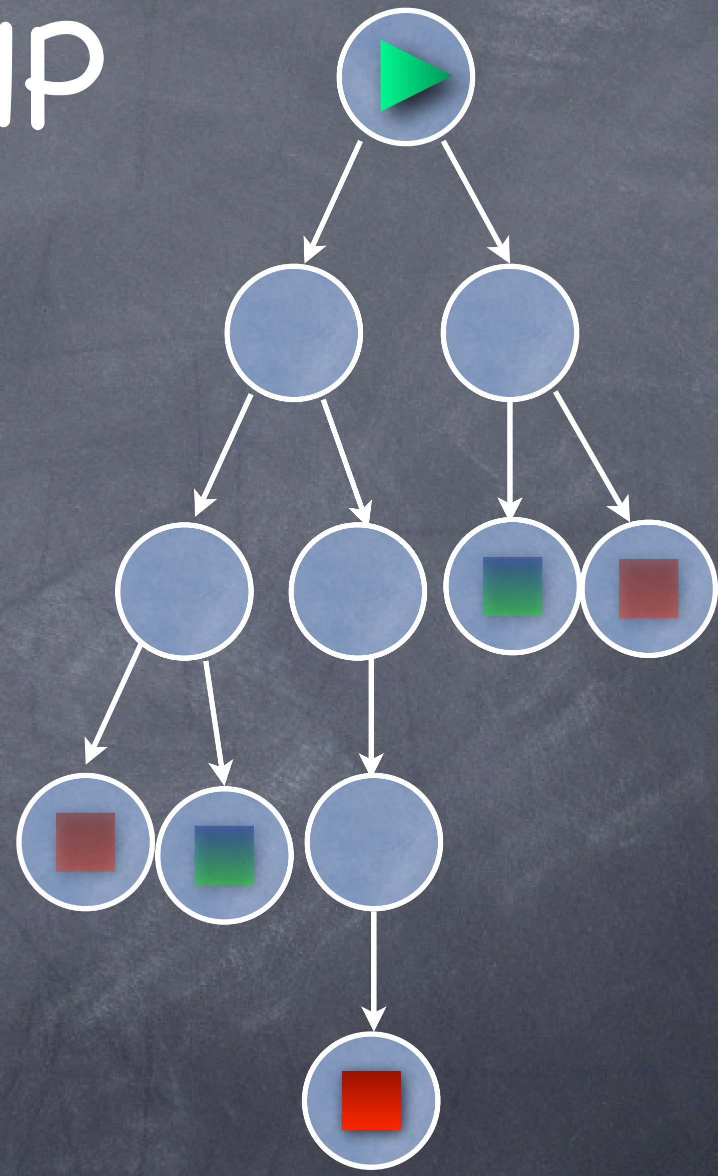

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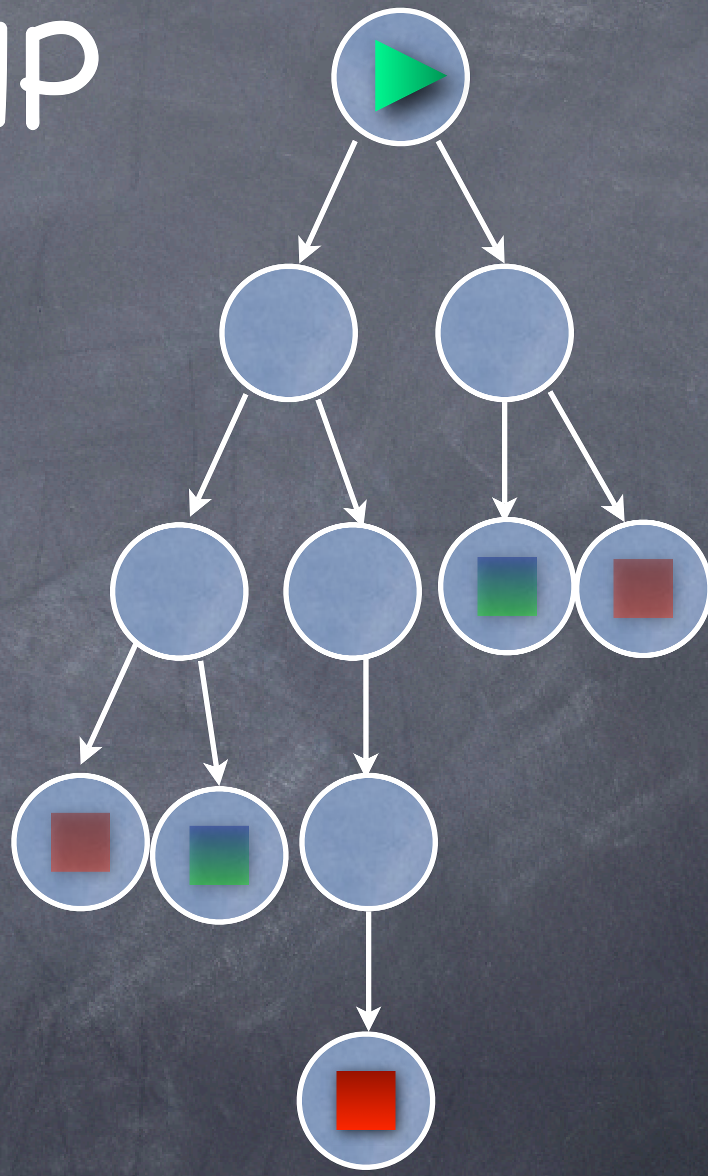

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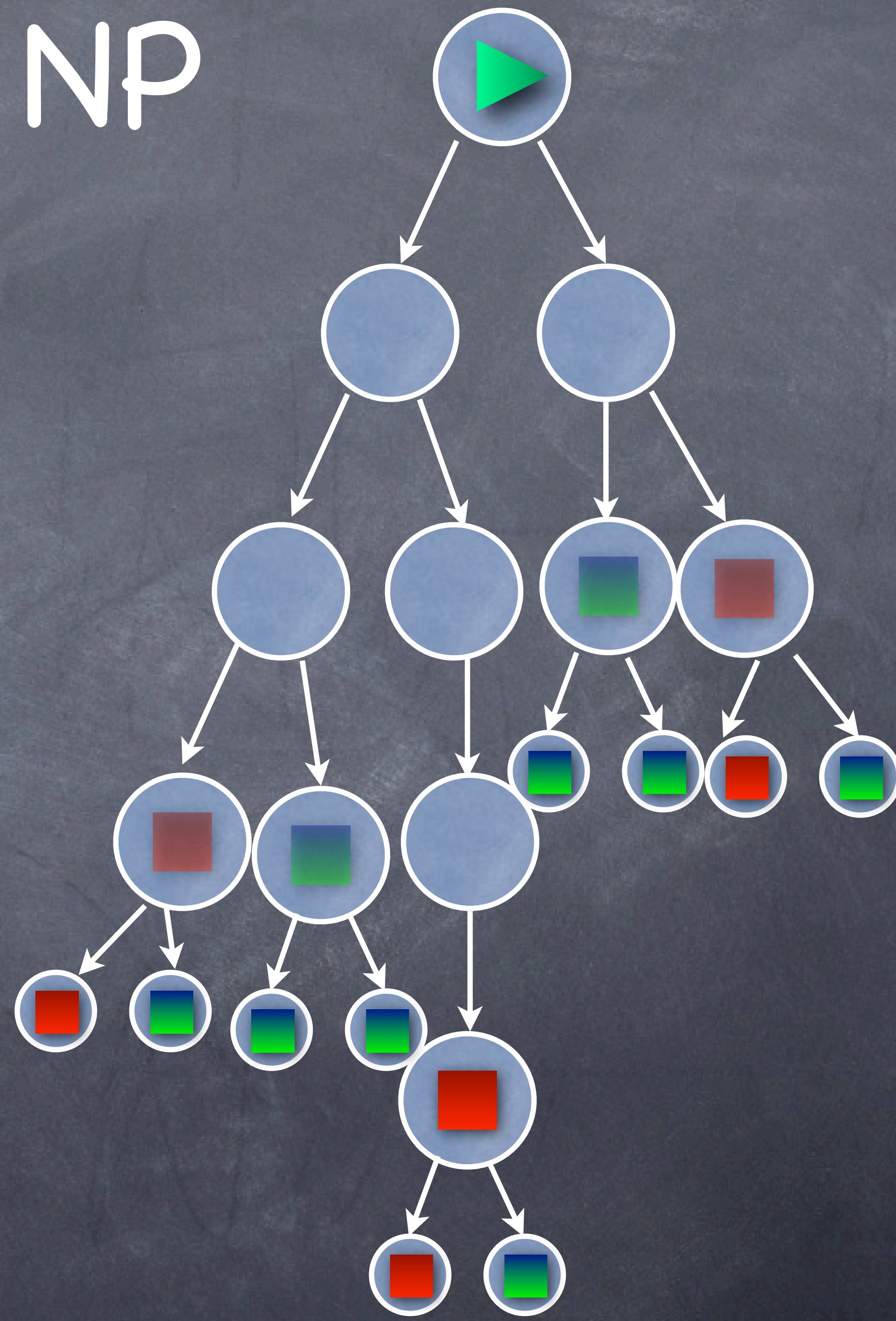

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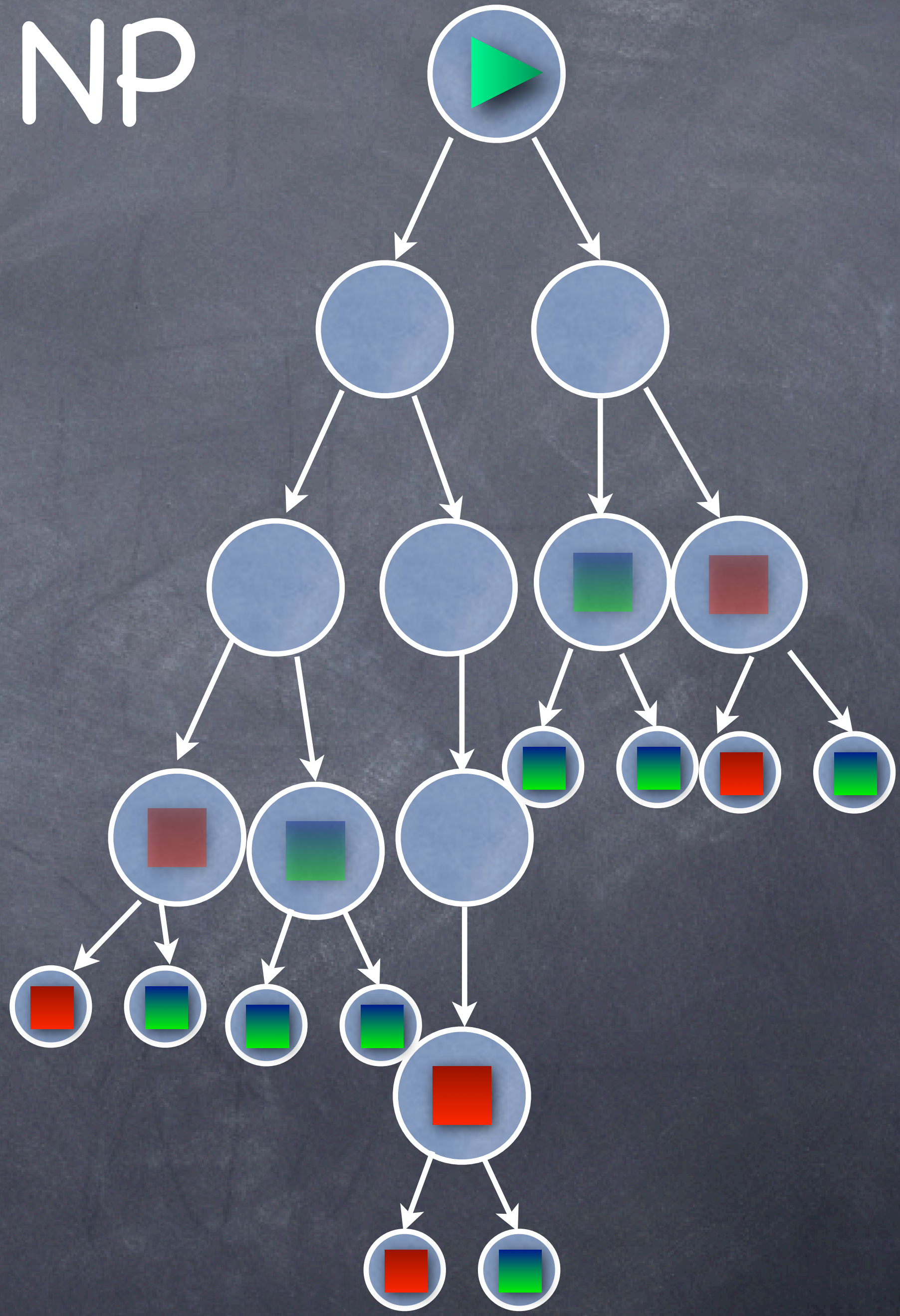

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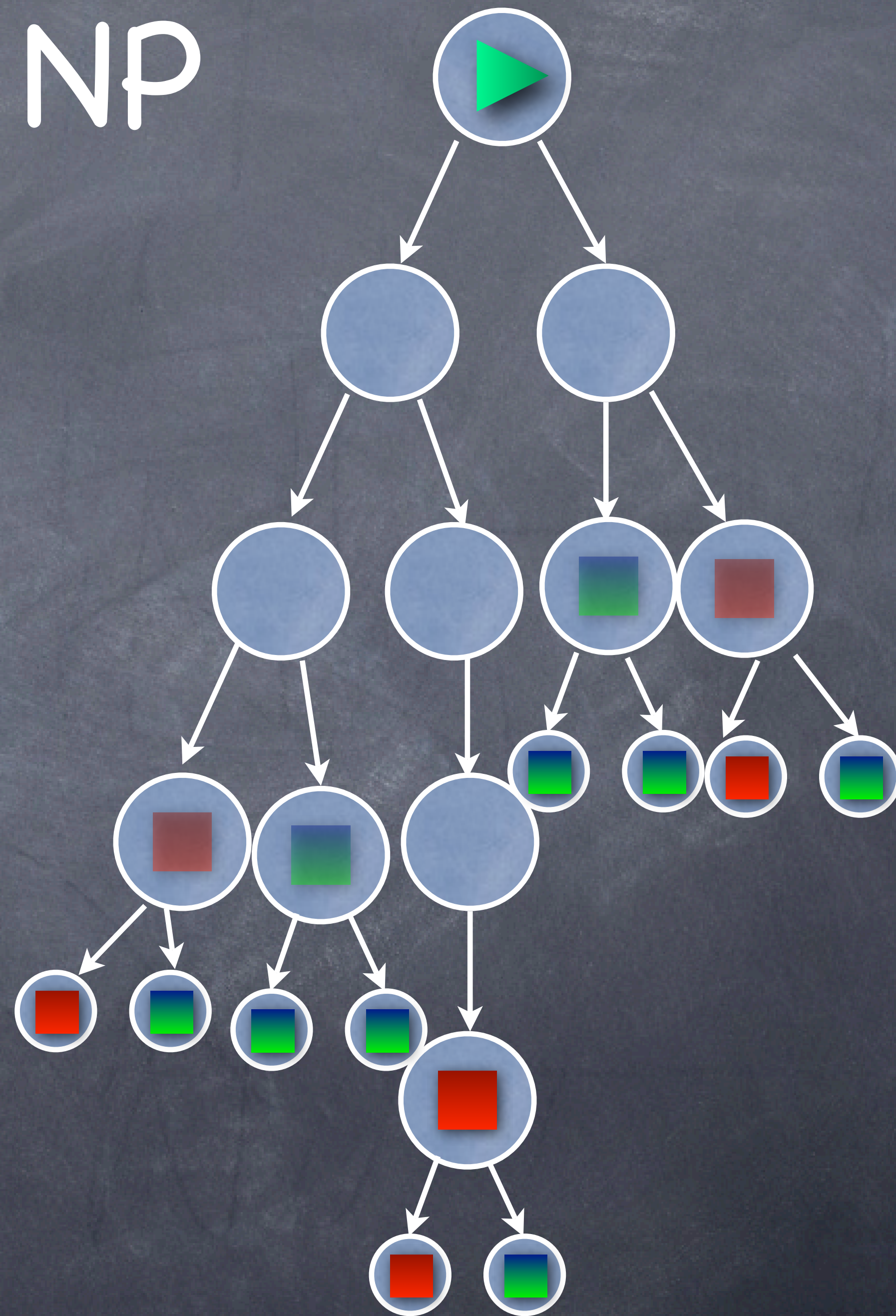
- Accepting gap can be exponentially small



Bounding Probability Gap

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    - For $\delta = n^{-c}$, $t=n^{d+c}$ is polynomial

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    - Pr[|estimate - real| ≥ δ/2] ≤ $2^{-\Omega(t.\delta^2)}$
      - t = $O(n^d/\delta^2)$ enough for Pr[error] ≤ 2^{-n^d}

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- Next: more on BPP and relatives