Circuits

Lecture 11
Uniform Circuit Complexity

Recall

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    - $SIZE(T') \subsetneq SIZE(T)$ if $T=\Omega(t2^t)$ and $T'=O(2^t/t)$
    - Most functions on t bits (that ignore last n-t bits) are in SIZE(T) but not in SIZE(T')

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- Logspace-uniform:
  - An O(log n) space TM can compute the circuit

NC^i and AC^i

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  - $NC^0 \subsetneq AC^0$: Consider L = {1,11,111,...}

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- Open problem: Is NC = P?

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  - Total "work" is size of the circuit

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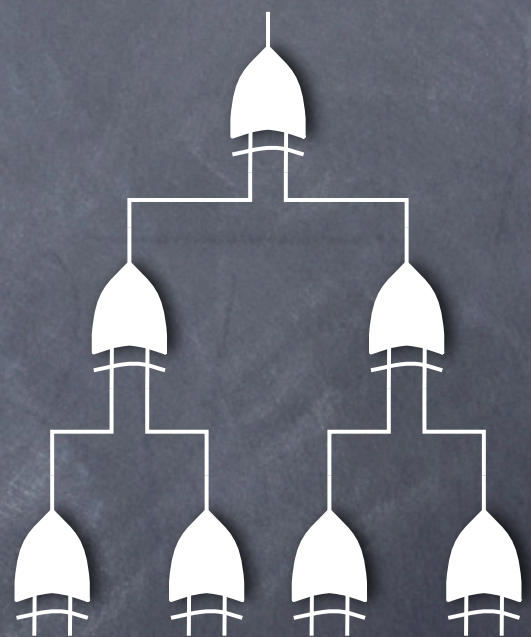

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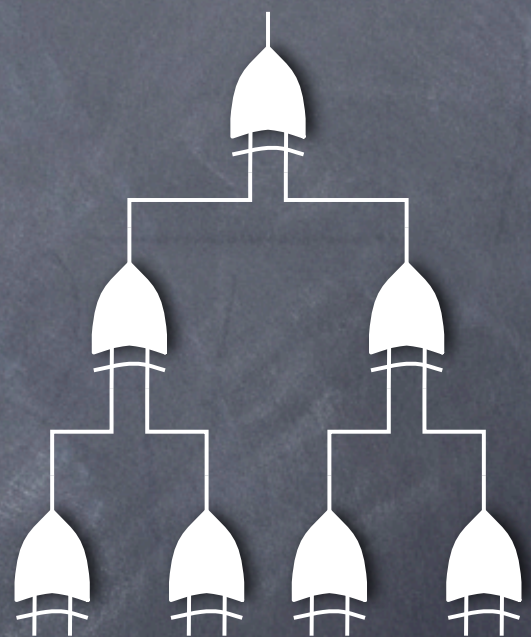

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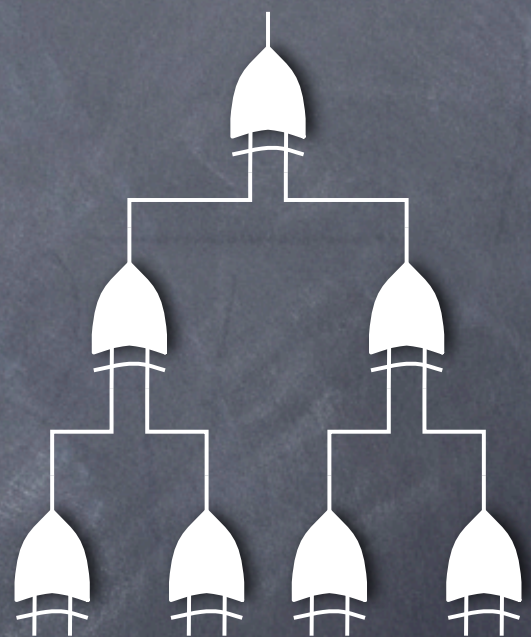

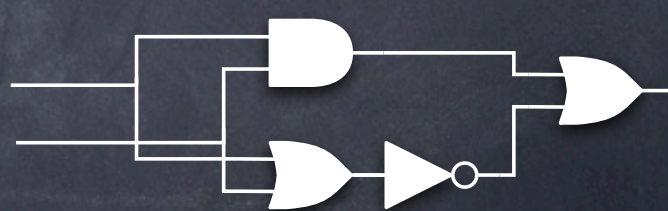

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    - Total depth O(log n)

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    - Length of path = depth of circuit = O(log n)

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  - Combining the NC^1 circuits for reduction and the AC^1 circuit for PATH, we get an AC^1 circuit

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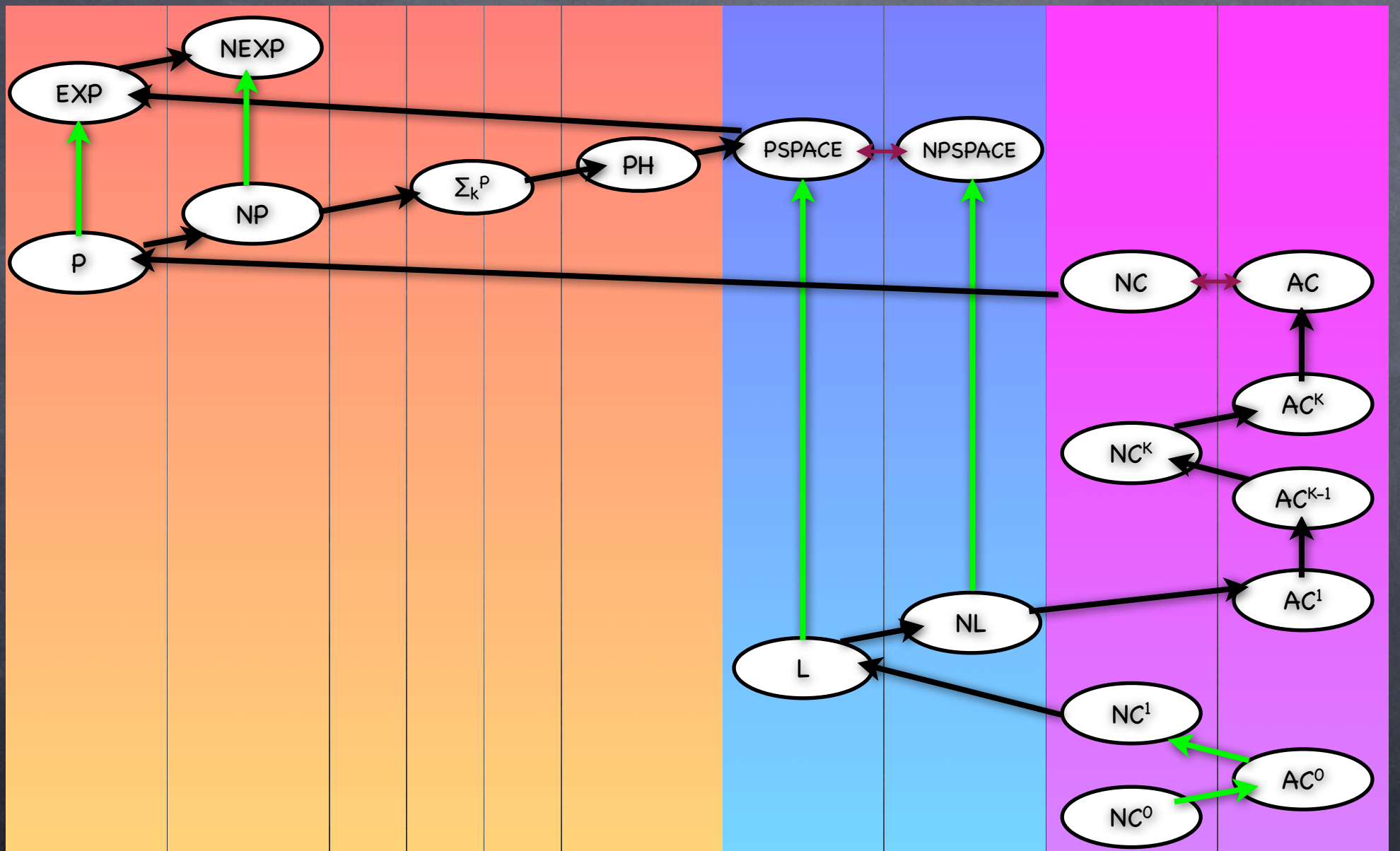
- Open: Is NC = P? (Can all polynomial time decidable languages be sped up to poly-log time using parallelization?)

Zoo


NEXP
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PSPACE
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PH
Σ_k^P
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  - Coincides with EXP (Why?)

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  - Each edge has a polynomially long label, and quantified variables take values from the same domain. Checking if edge is a correct wire in poly time (uniformity)

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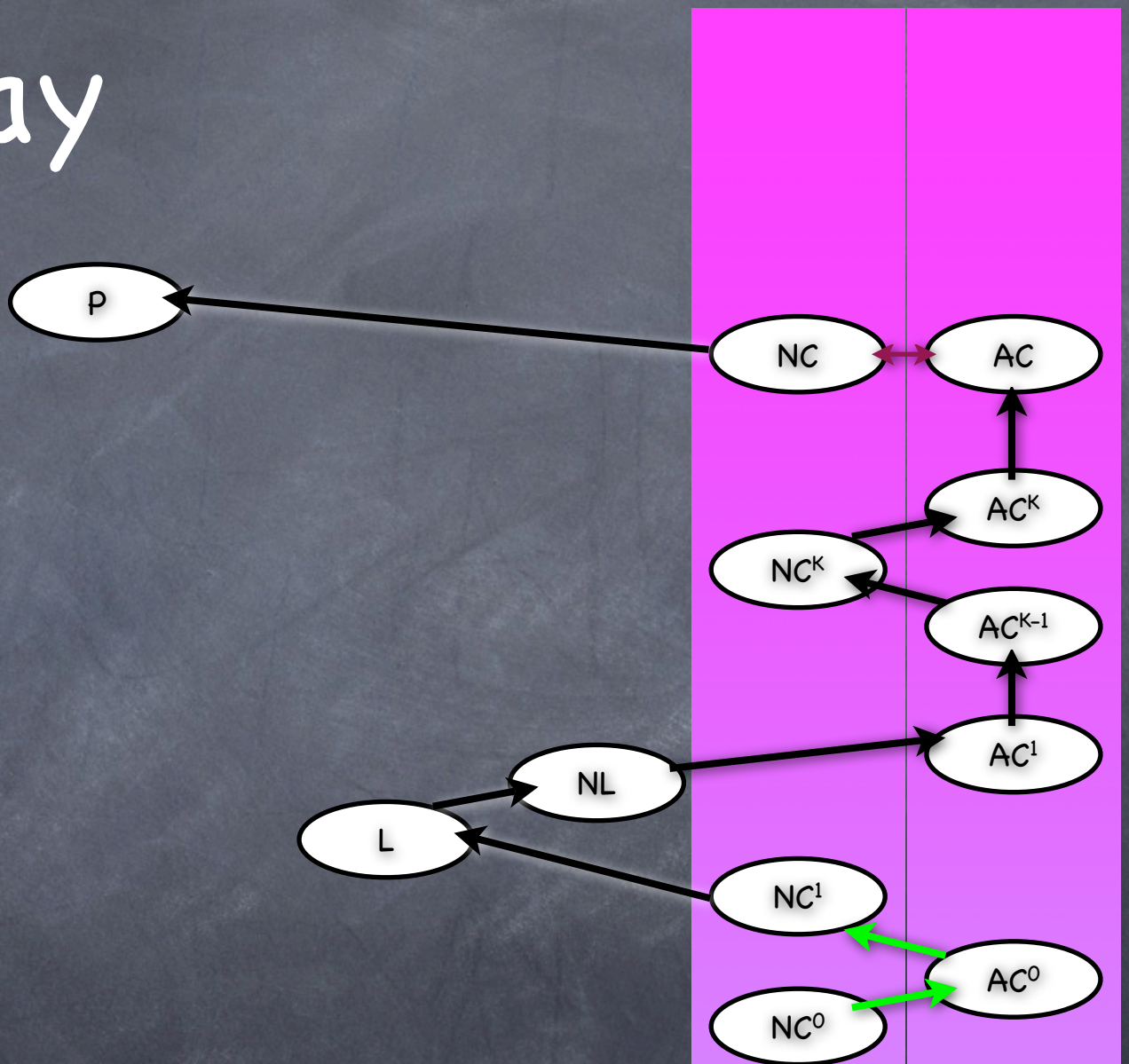
  - Circuit can be implicitly computed in polynomial time. Size $2^{O(\text{total length of variables})}$

Today



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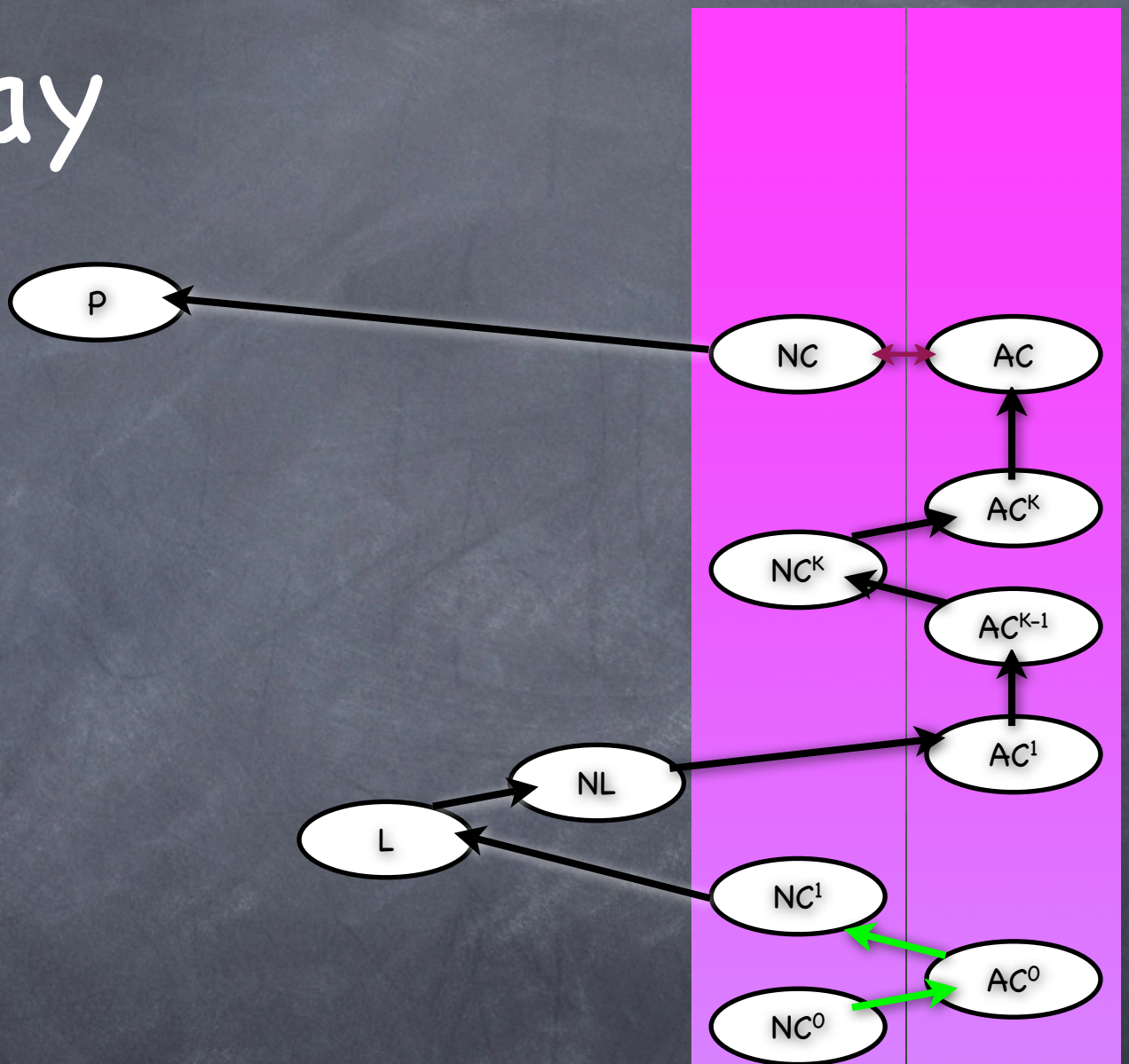

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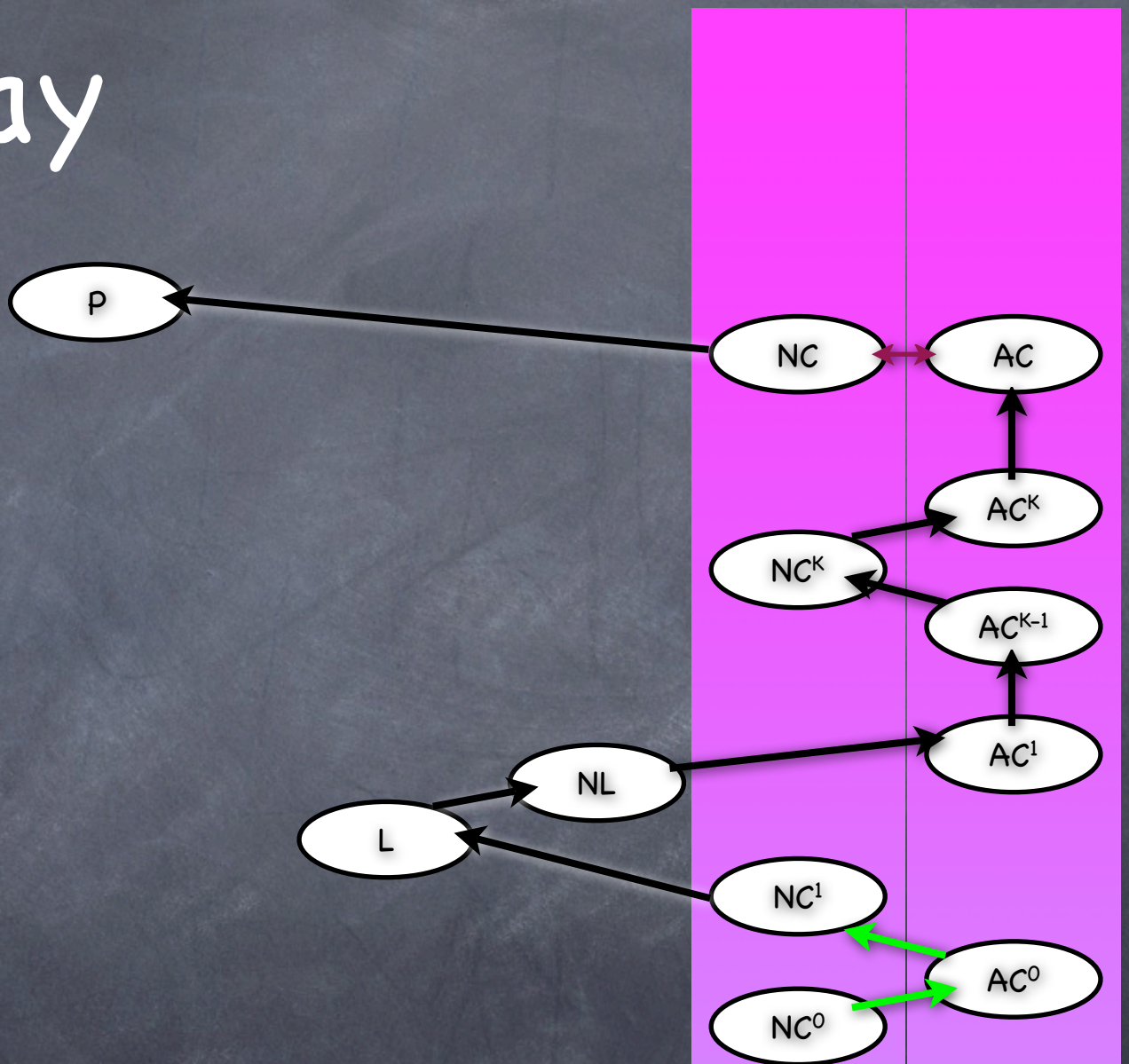

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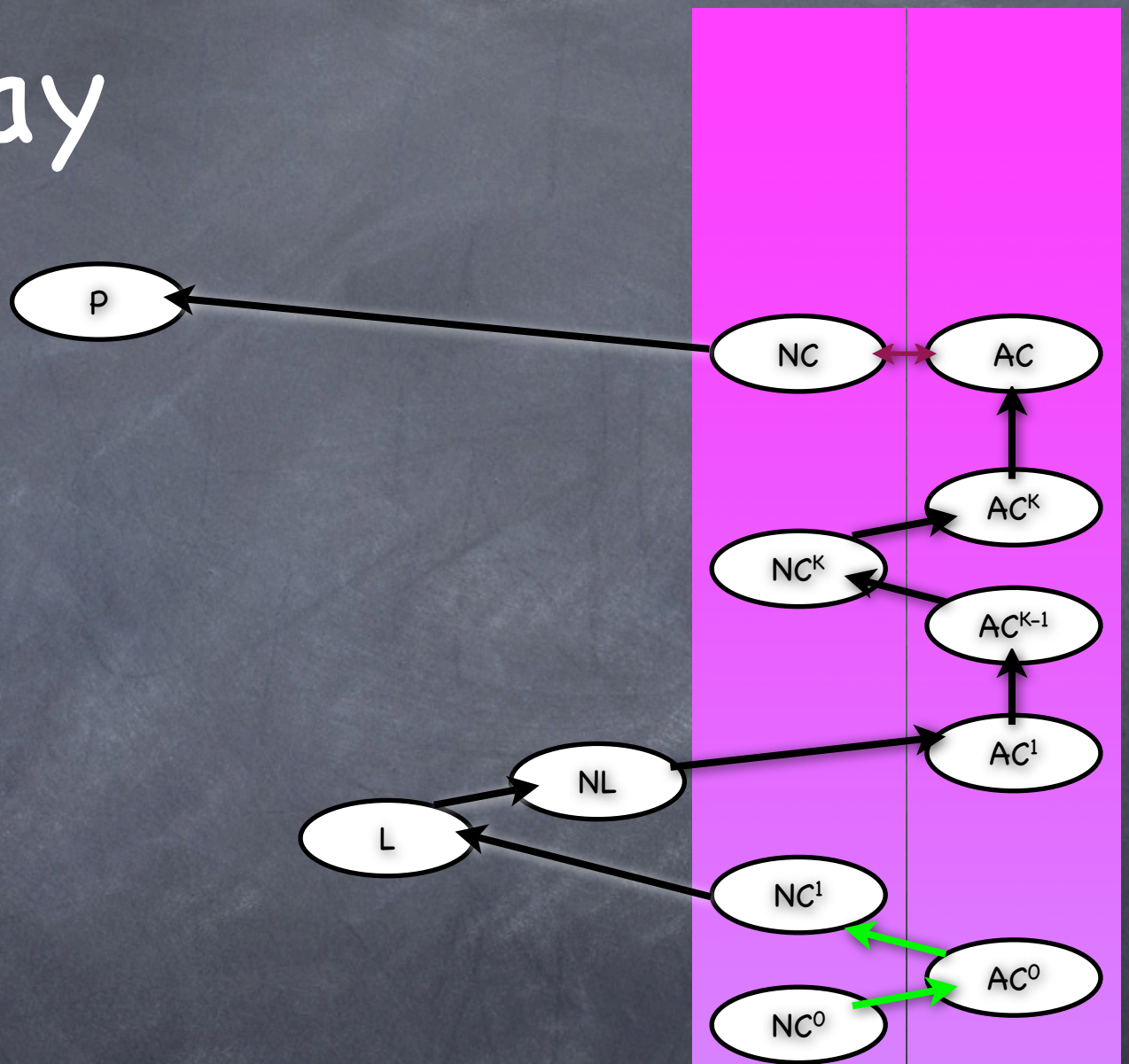

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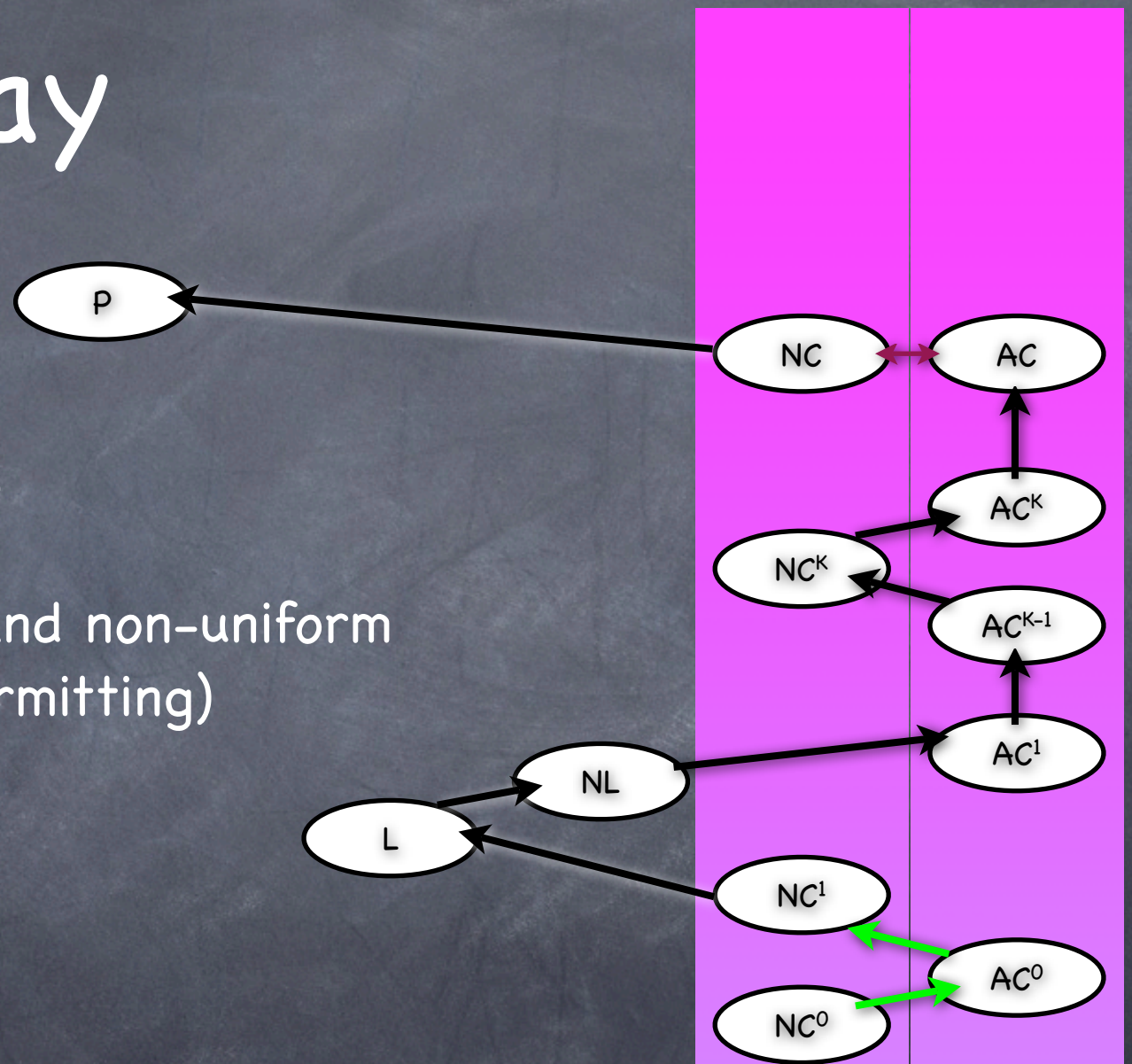

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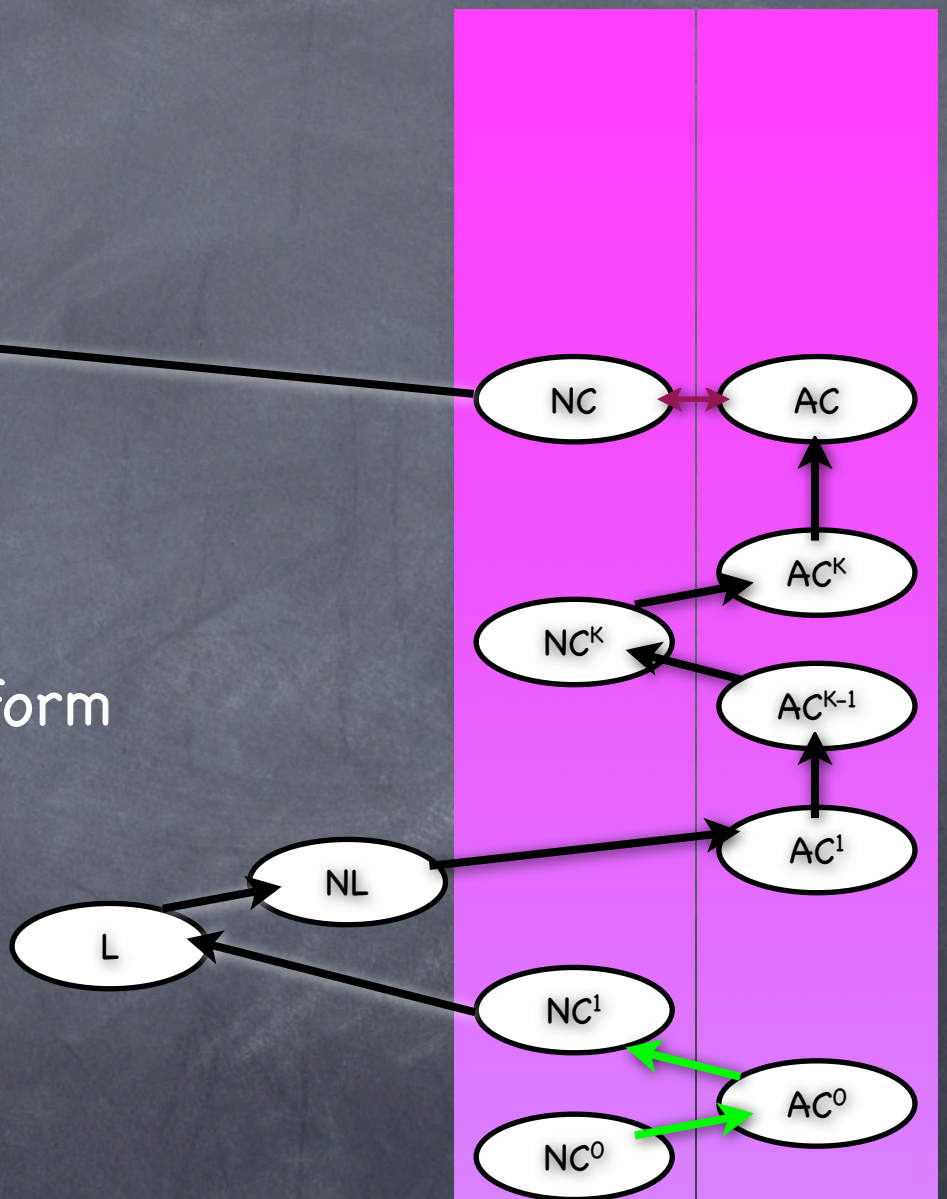

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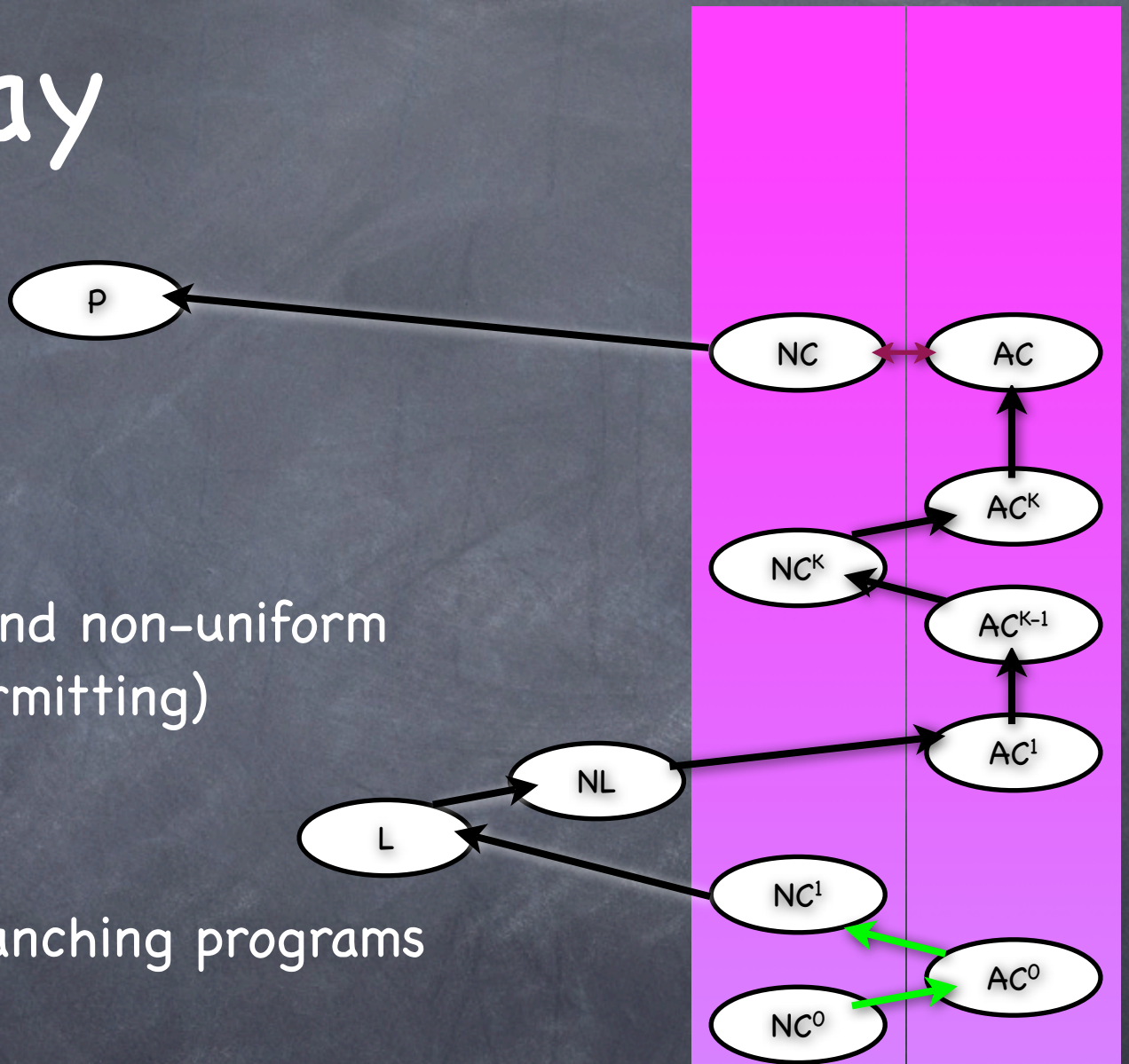

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