Non-Uniform Computation

Lecture 10

Non-Uniform Computational Models: Circuits

Non-Uniform Computation

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  - Sometimes will focus on the latter alone
  - Not entirely realistic if the program family is uncomputable or very complex to compute

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    - e.g. advice to decide undecidable unary languages

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- P/log = $\cup_{c,k>0}$ DTIME(kn^c)/k log n

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- P/log contains P
  - Does P/log or P/poly contain NP?

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Use L_2 so that (x,z,pad) in L_2 iff (x,z) in L_1. Can query L_2 with same size instances

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- If no advice worked (one of them was correct), then input not in language

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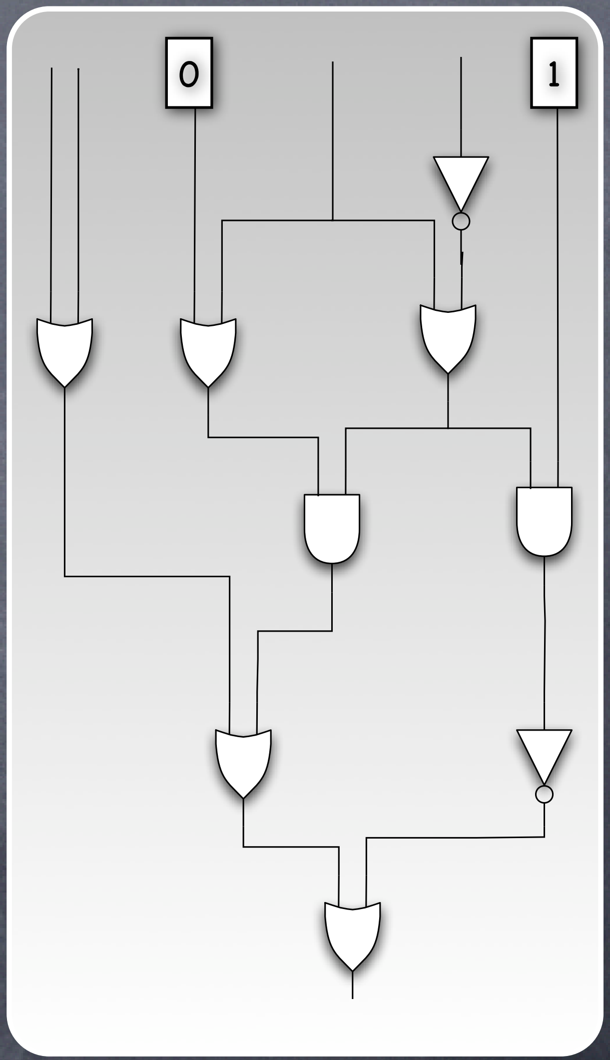
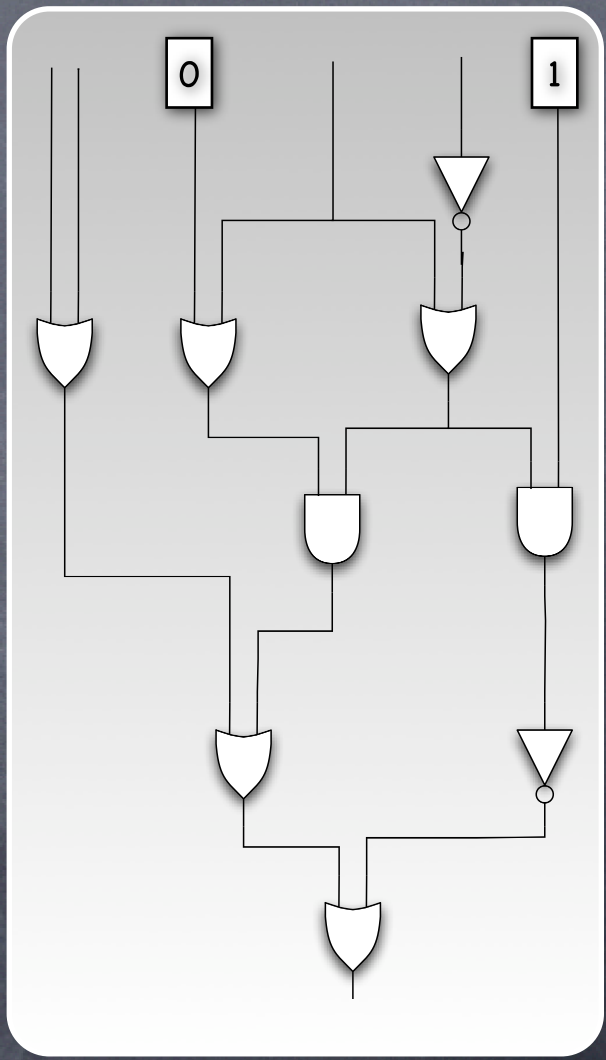
Boolean Circuits



Boolean Circuits

- Directed acyclic graph



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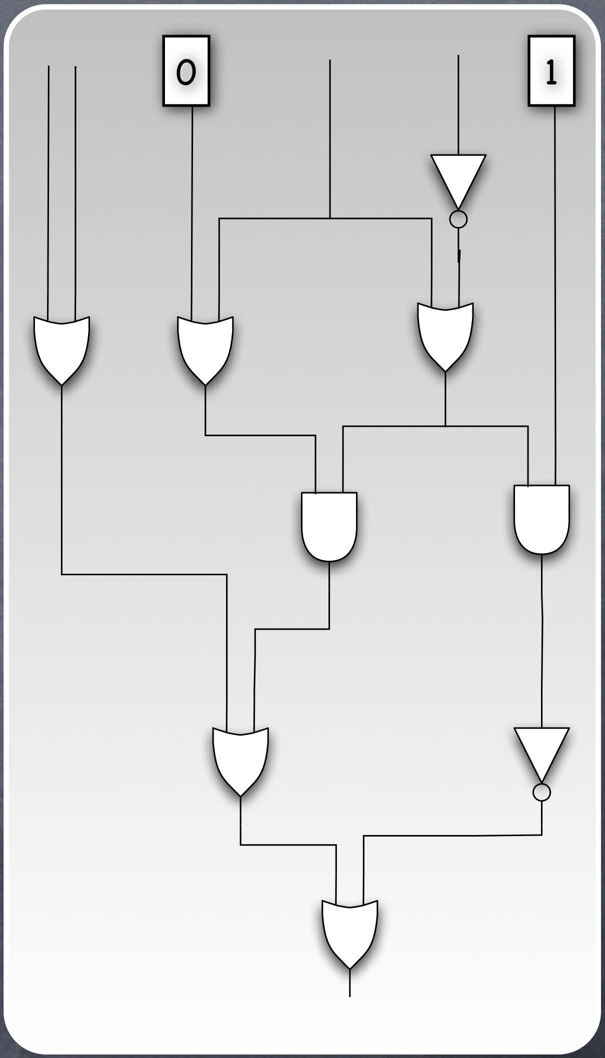

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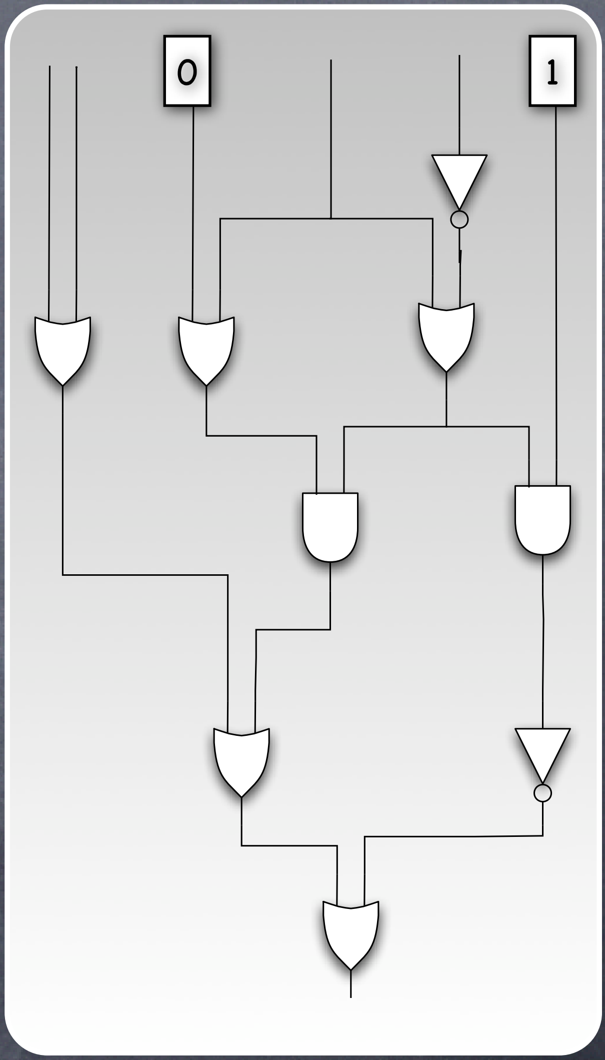

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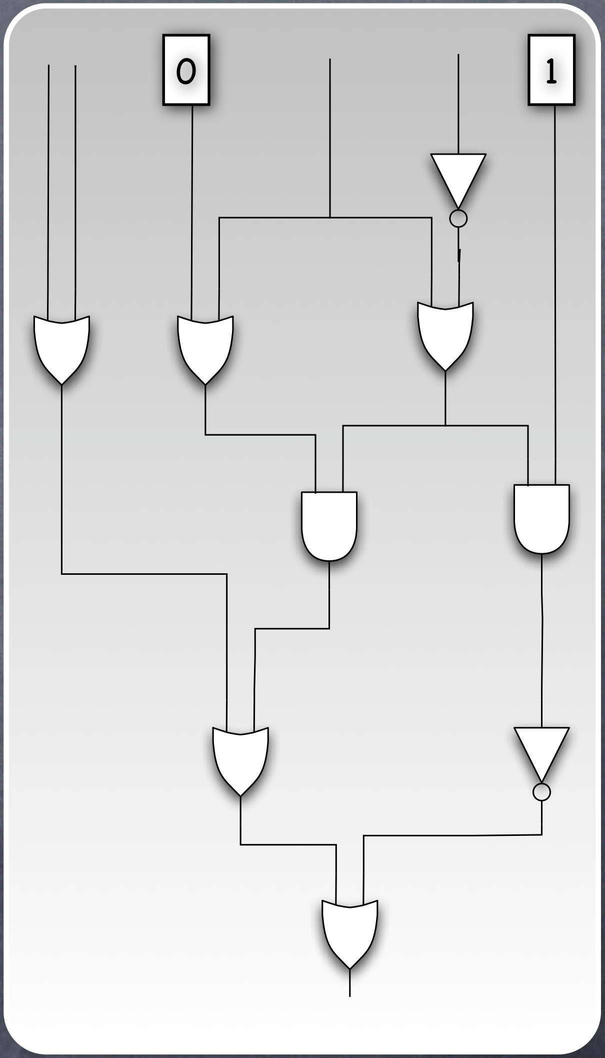

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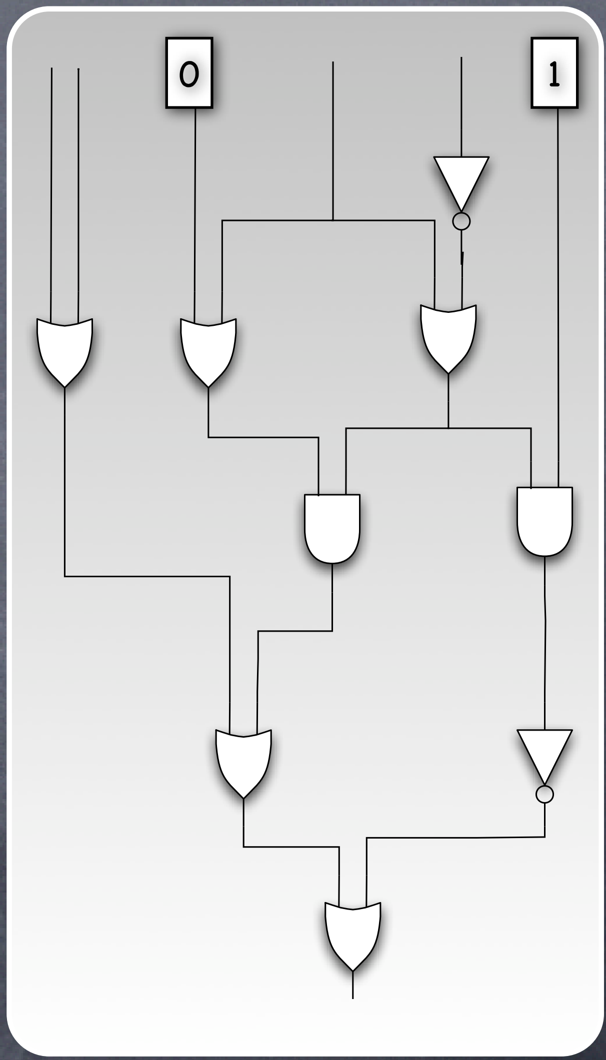

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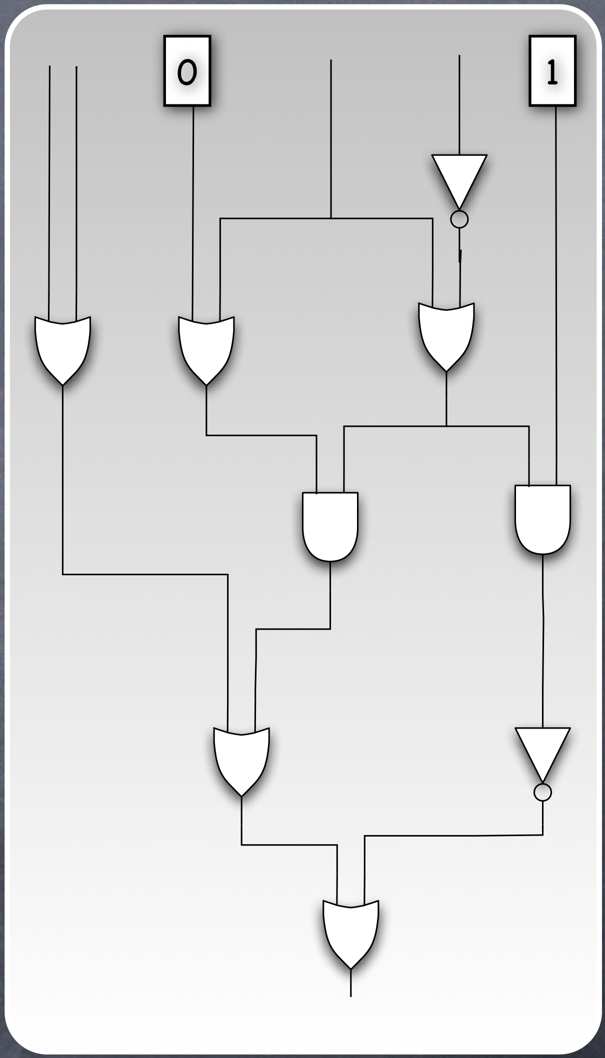

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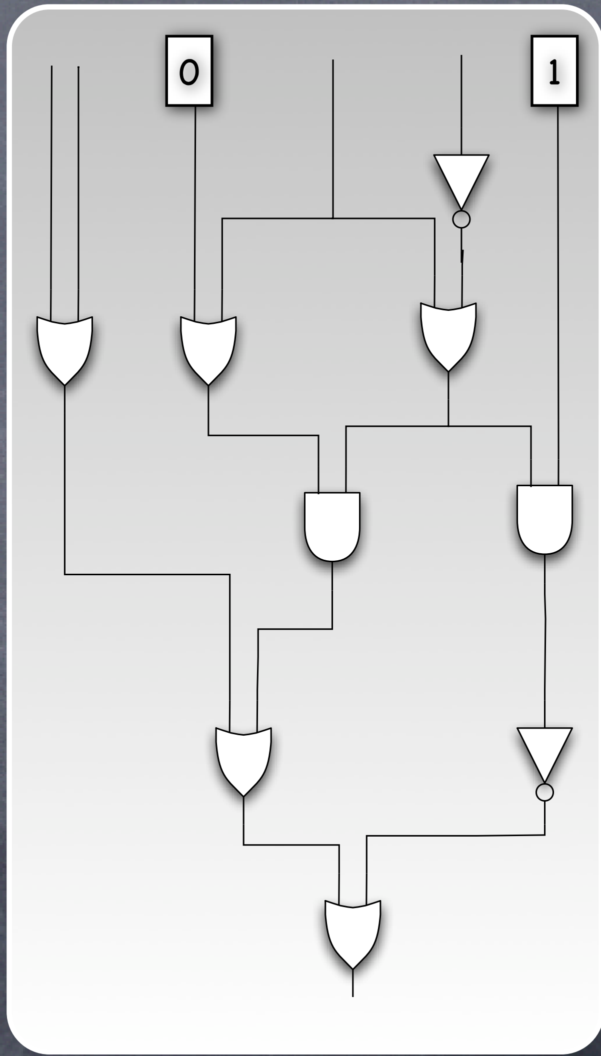

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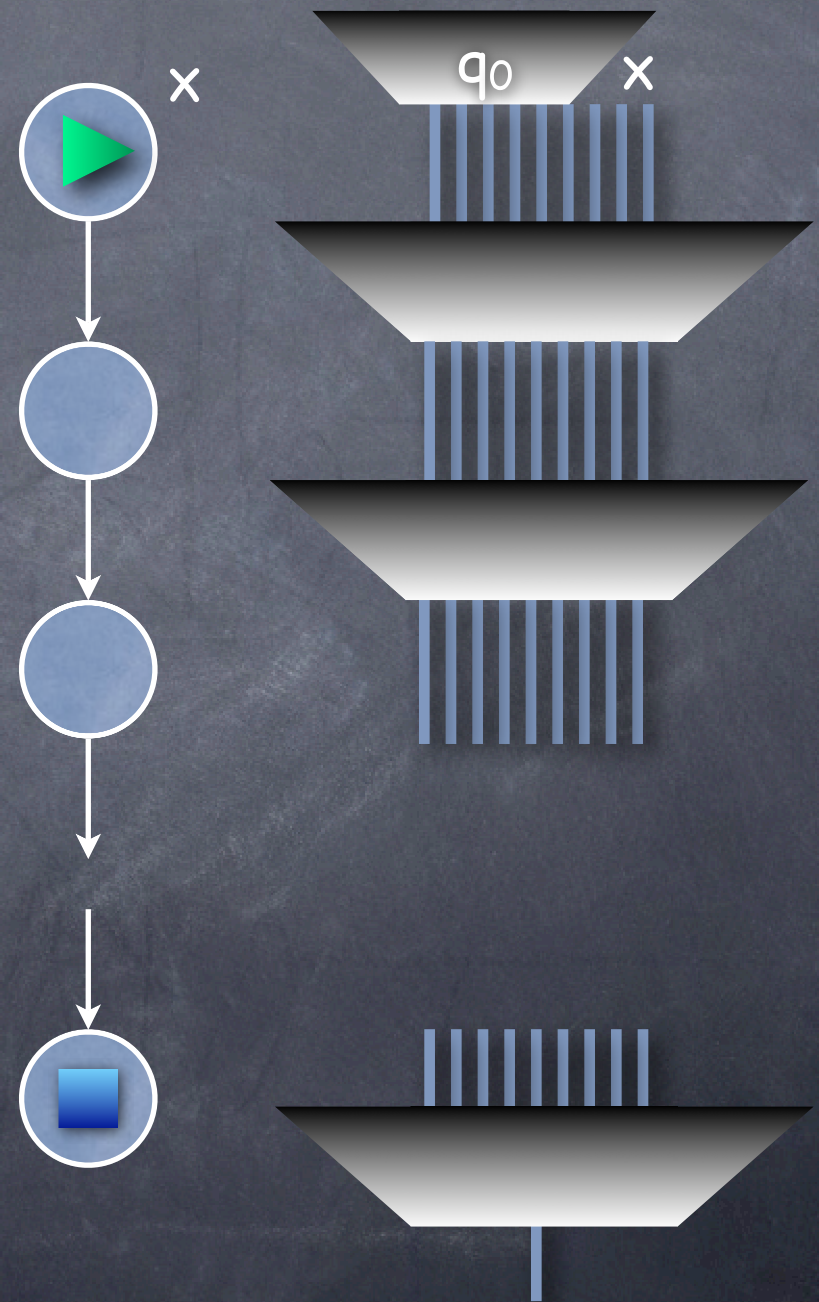

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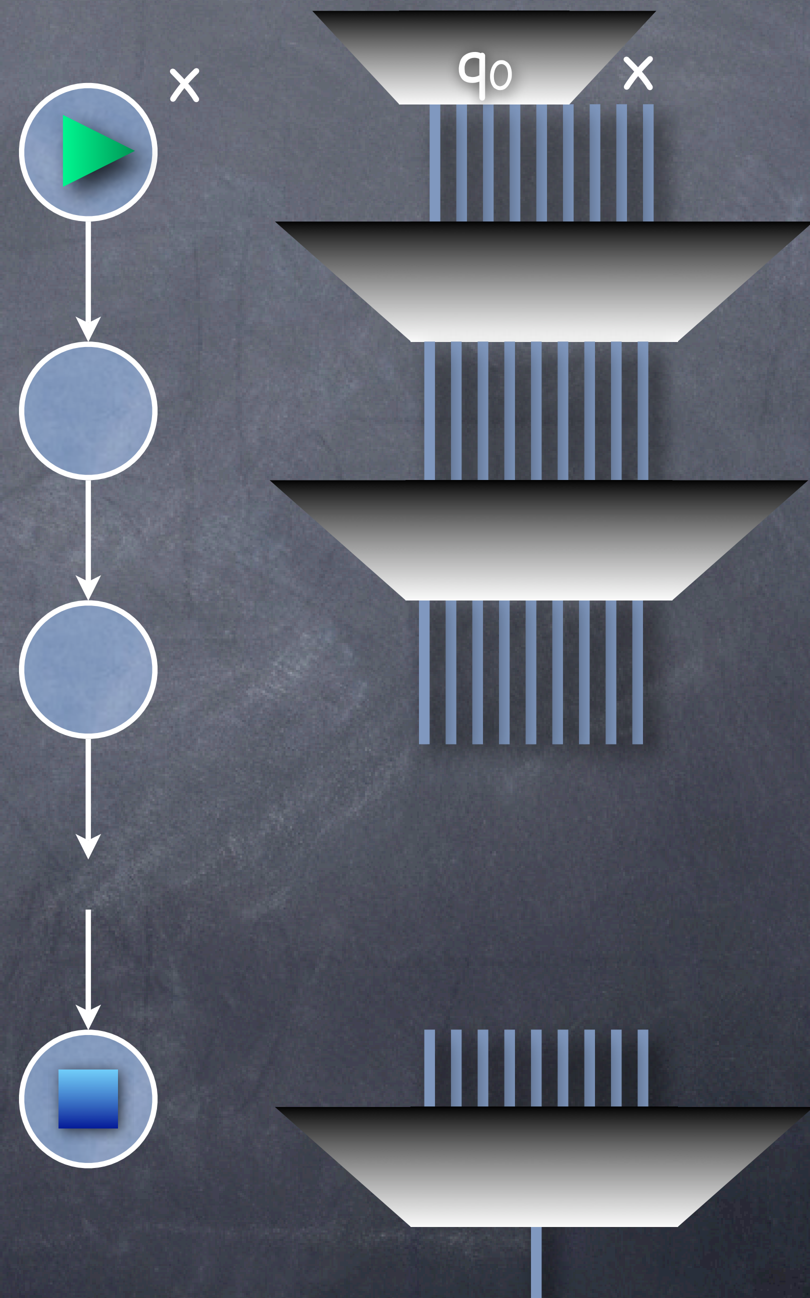

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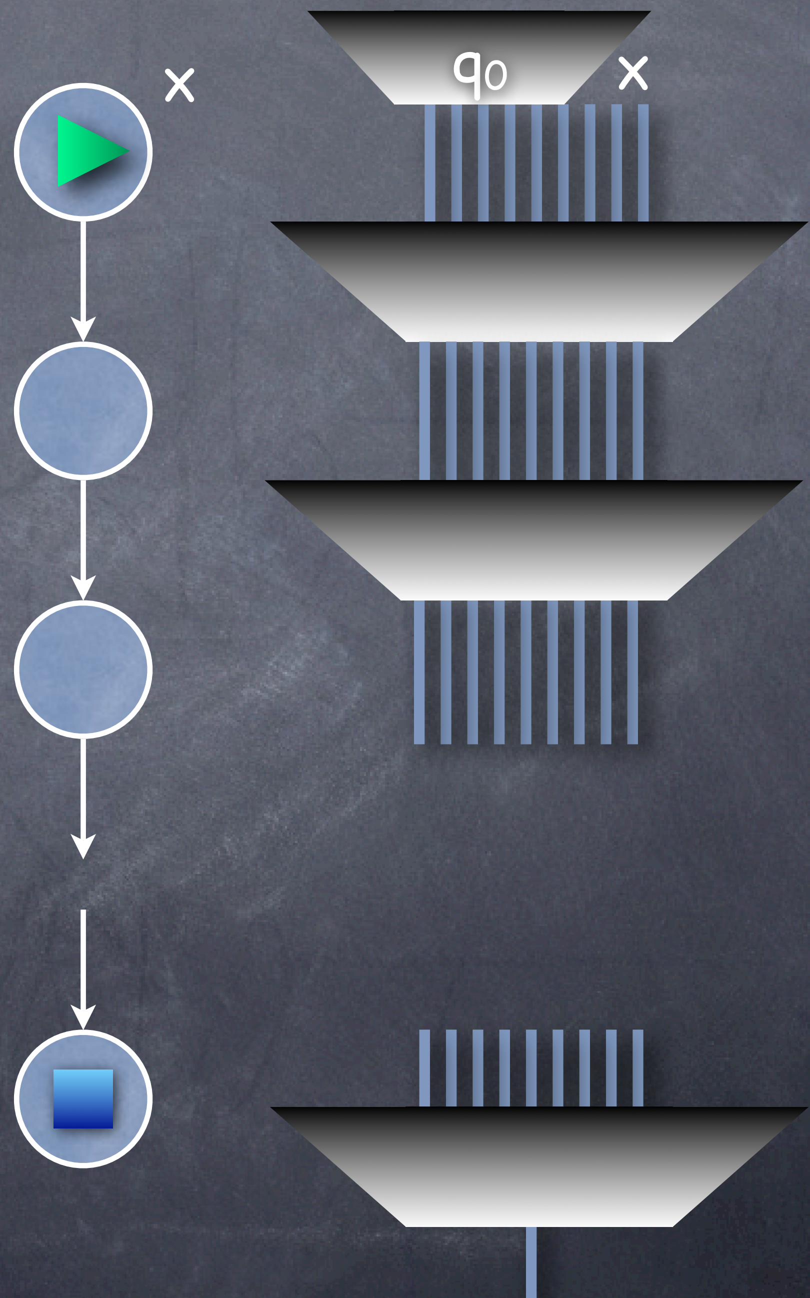

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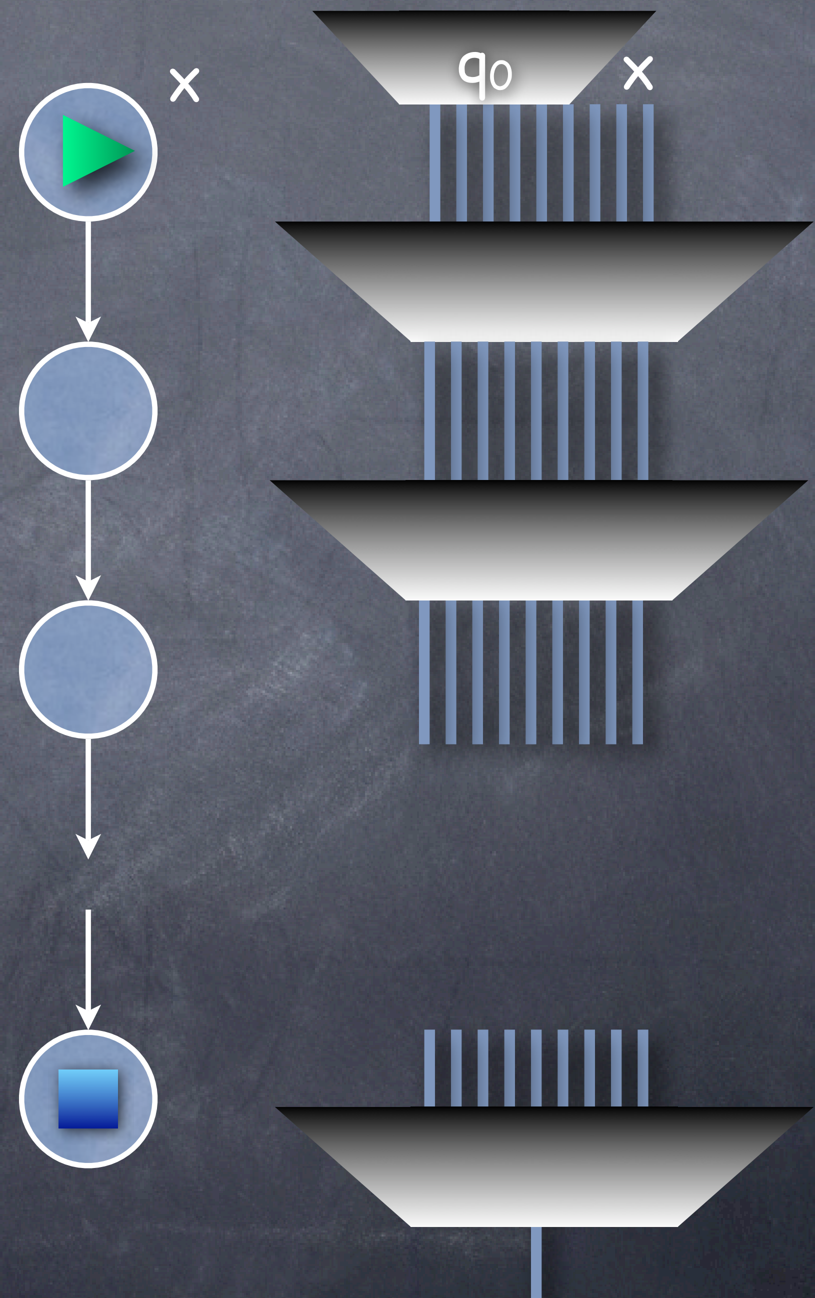

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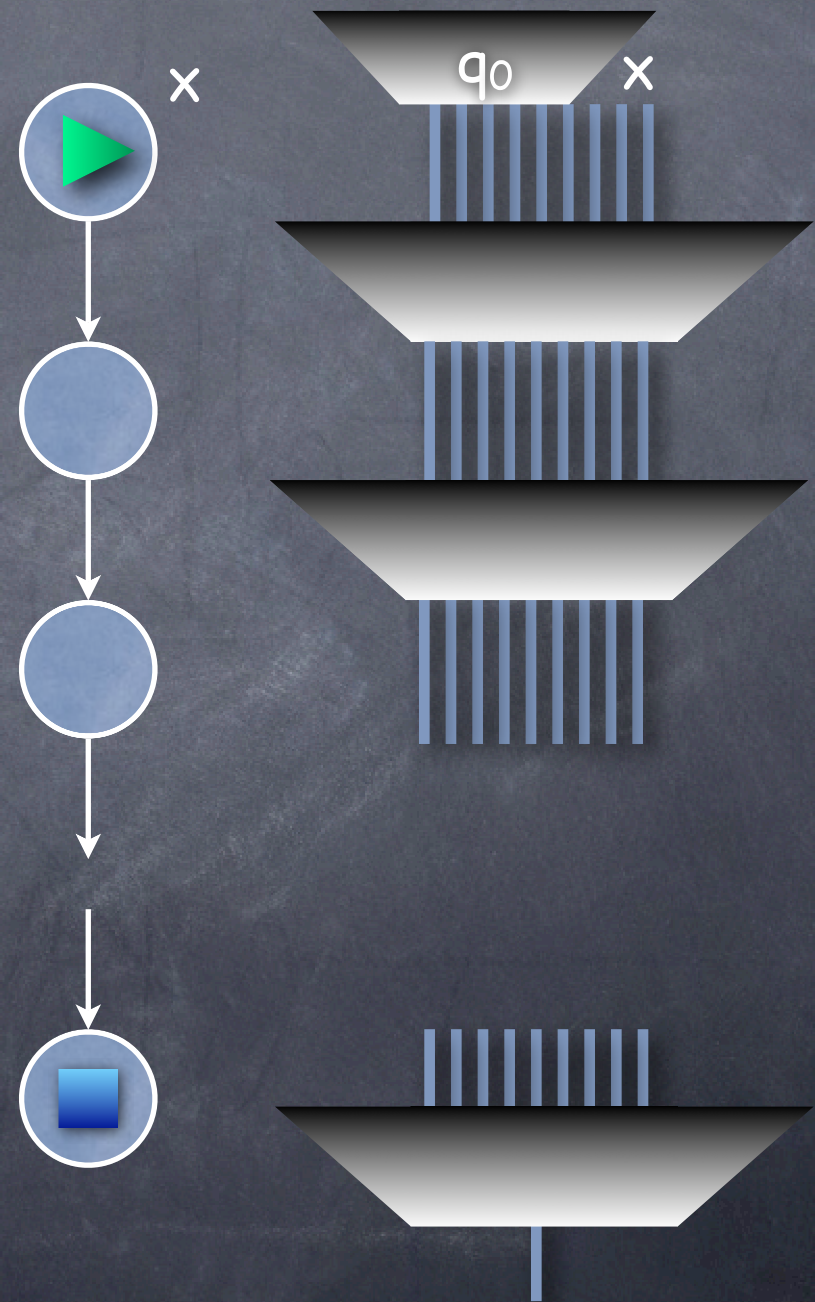

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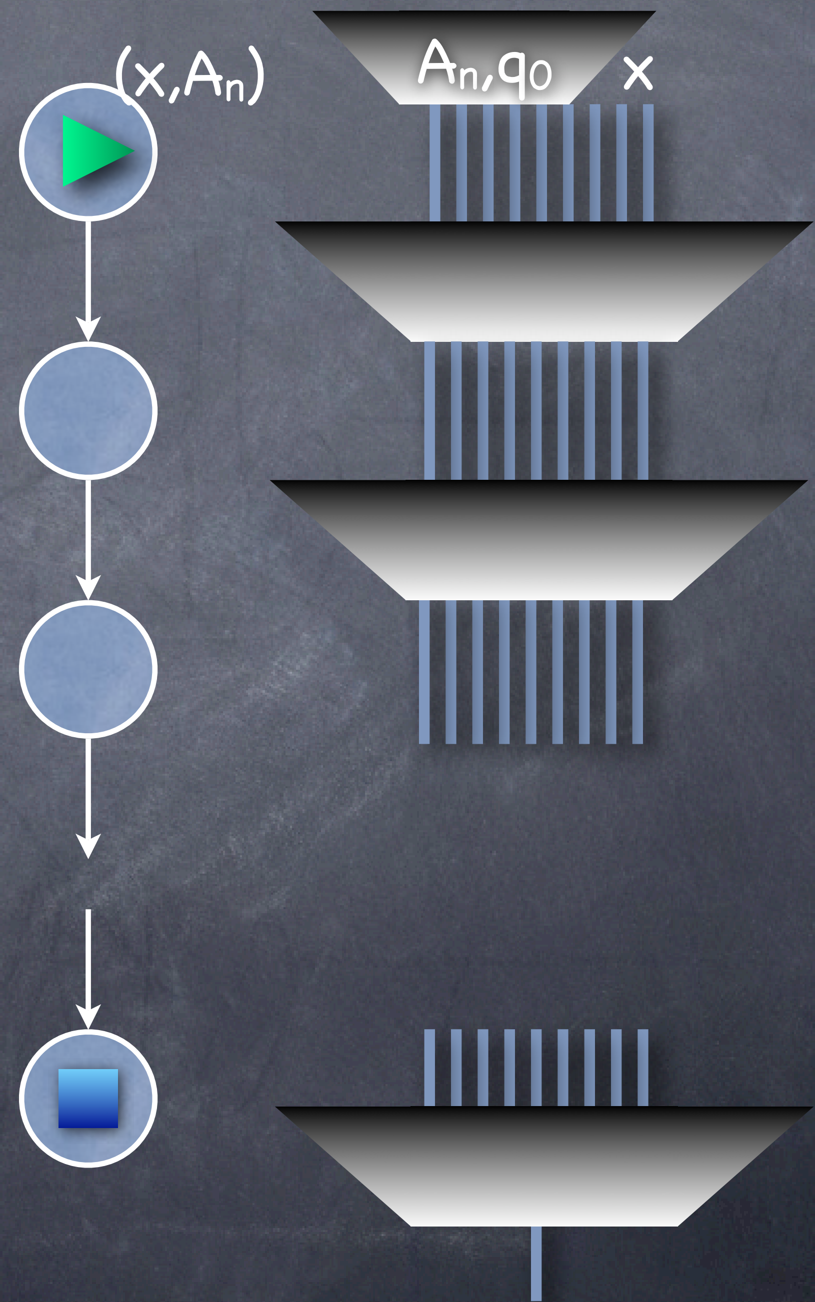

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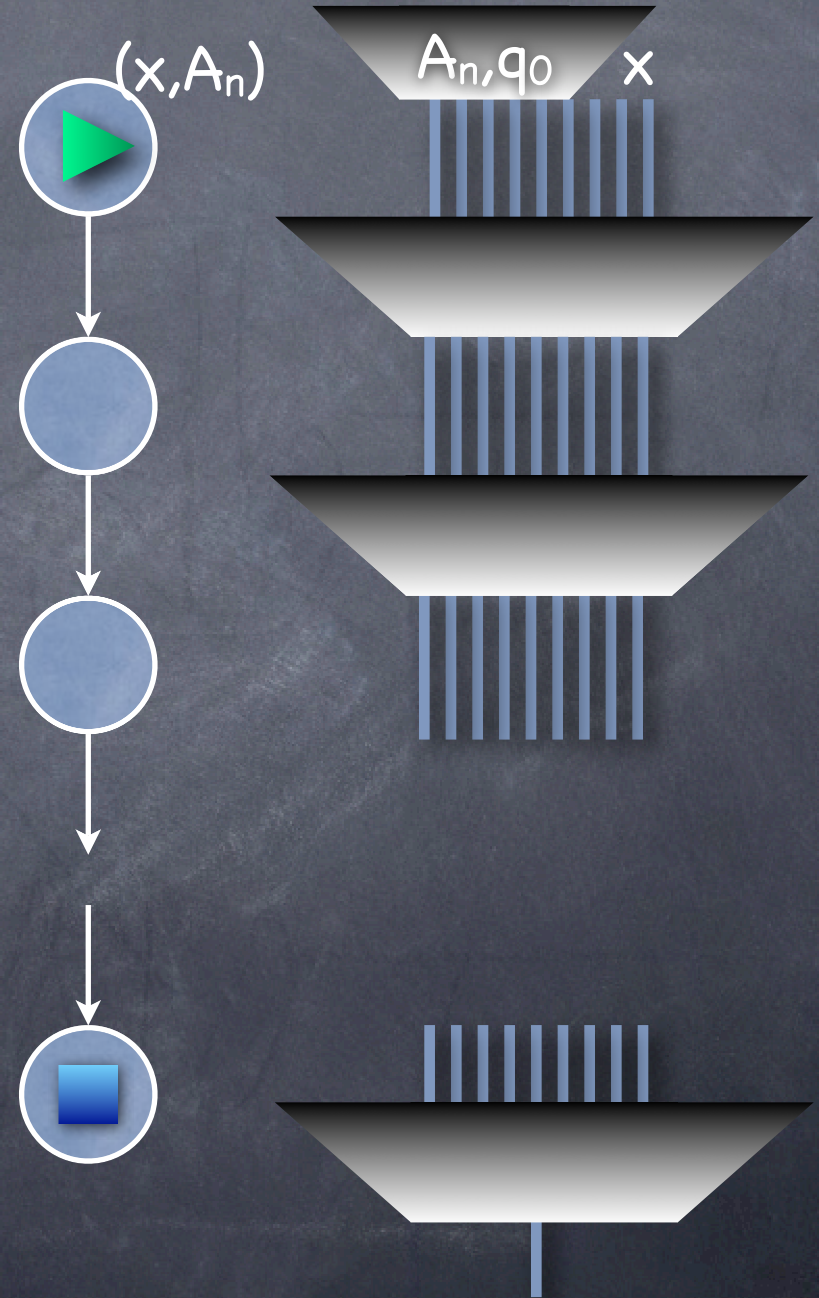

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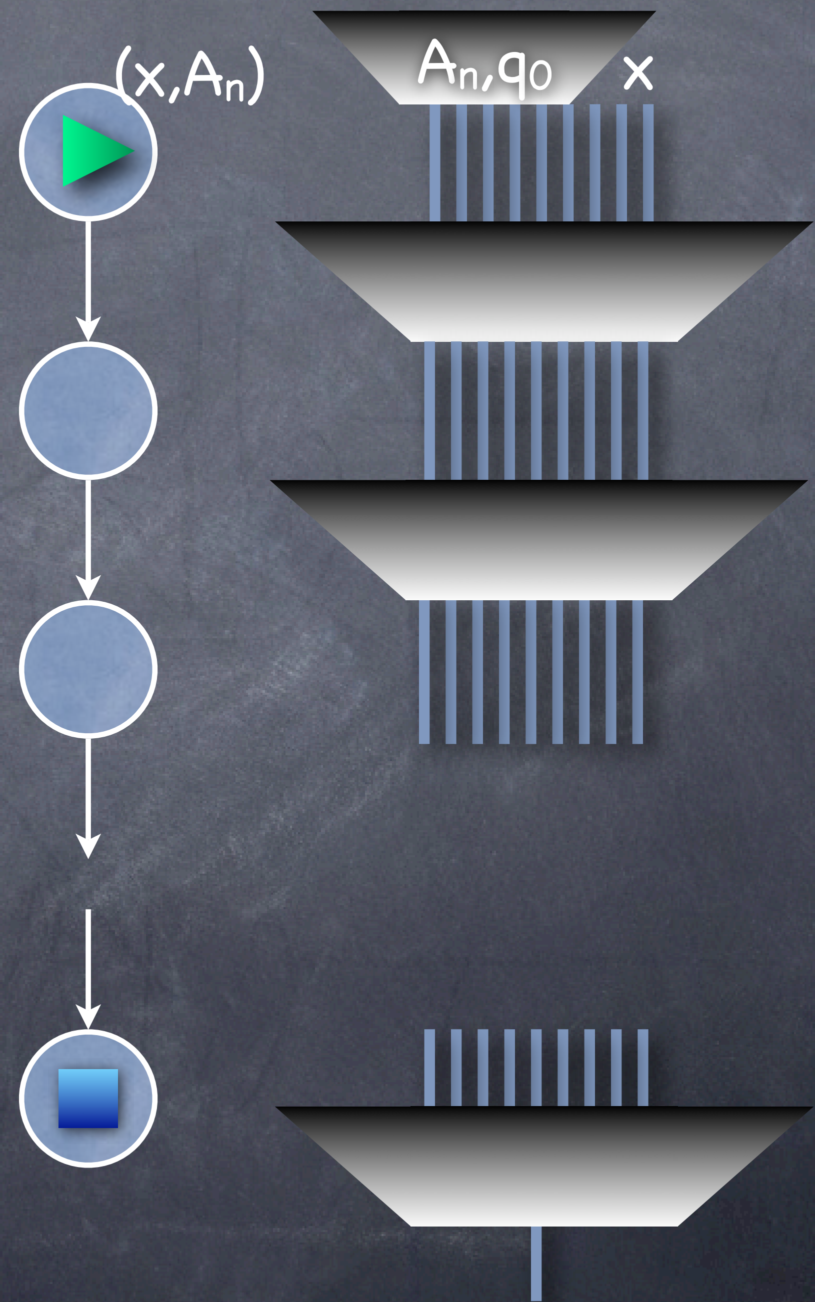

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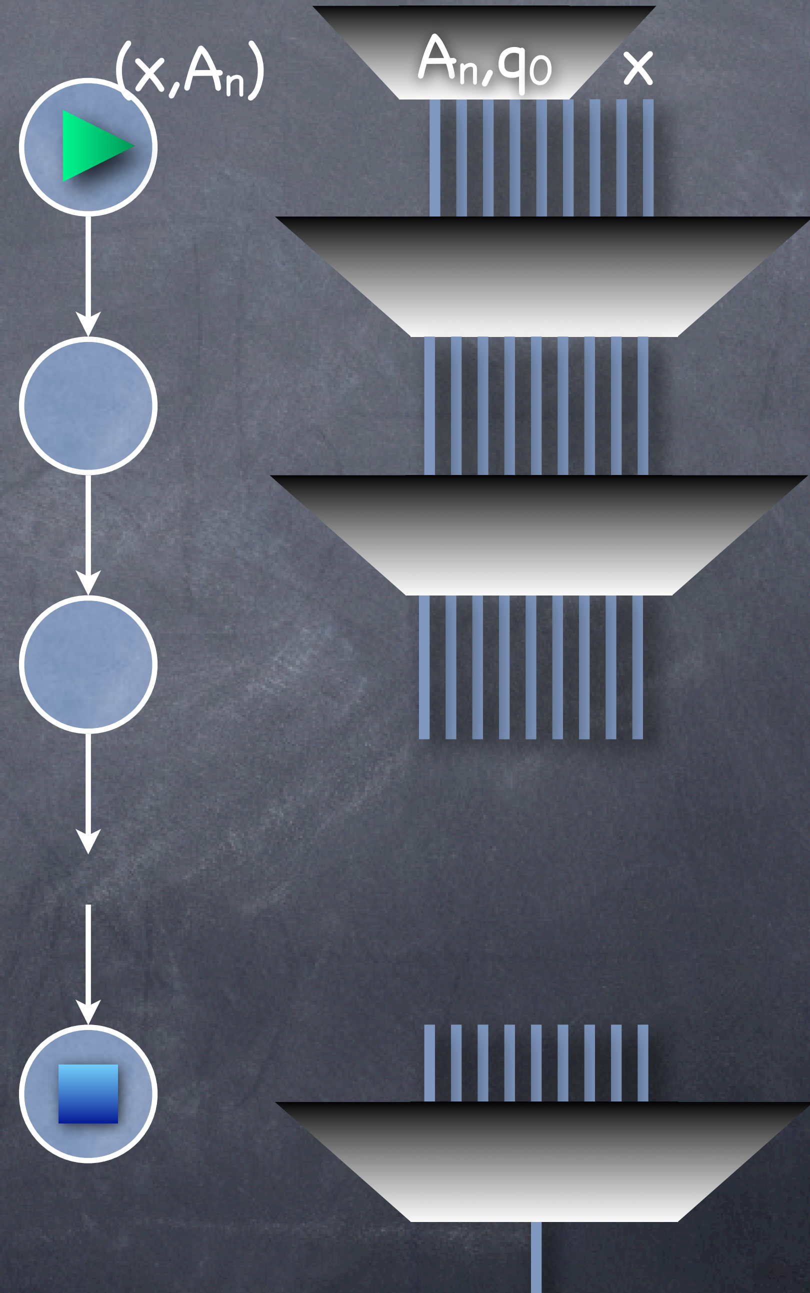

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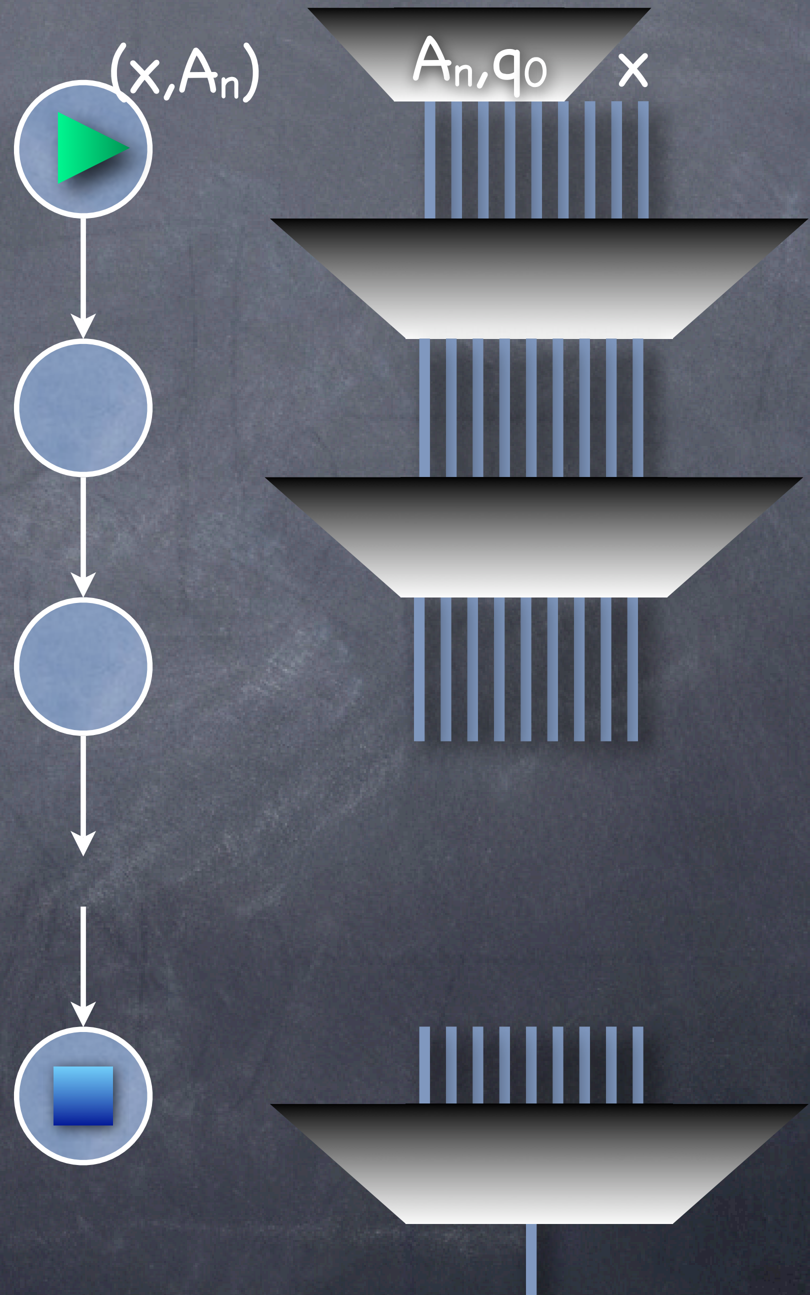

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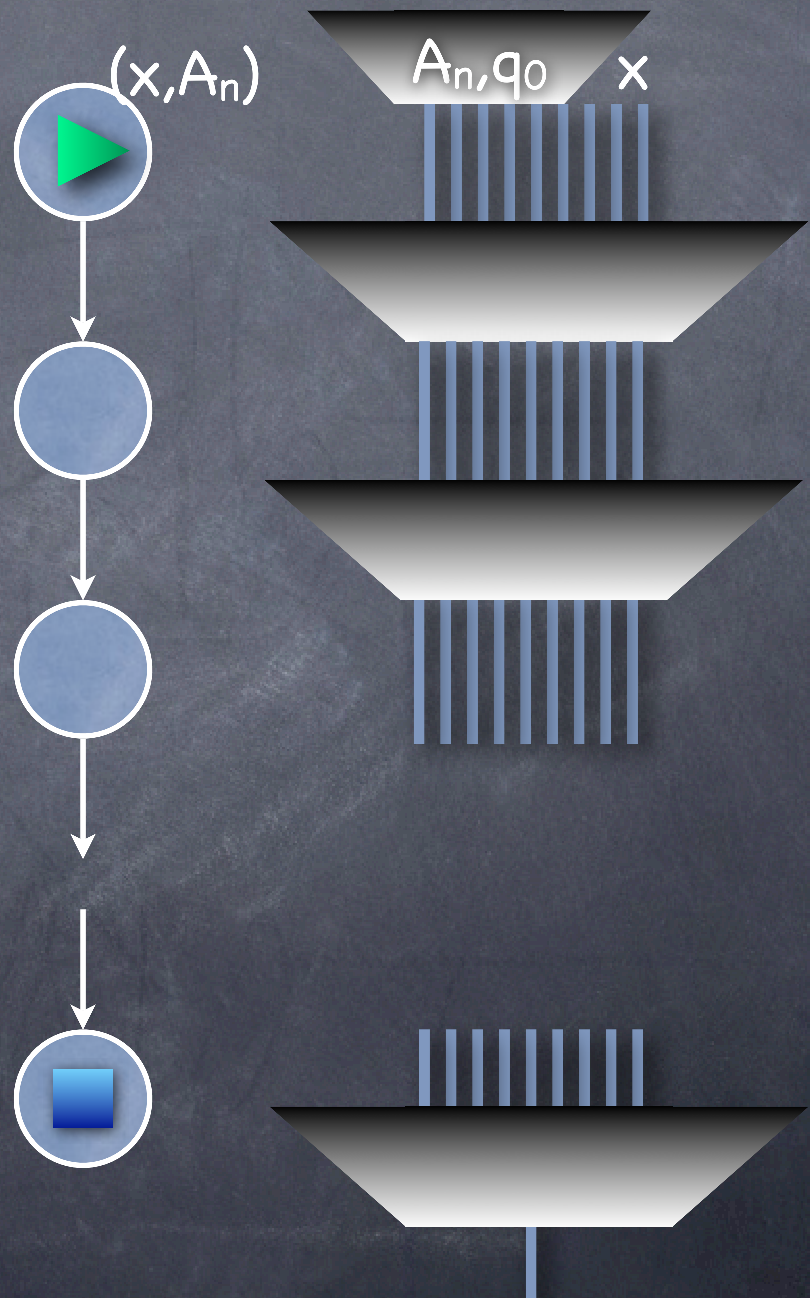

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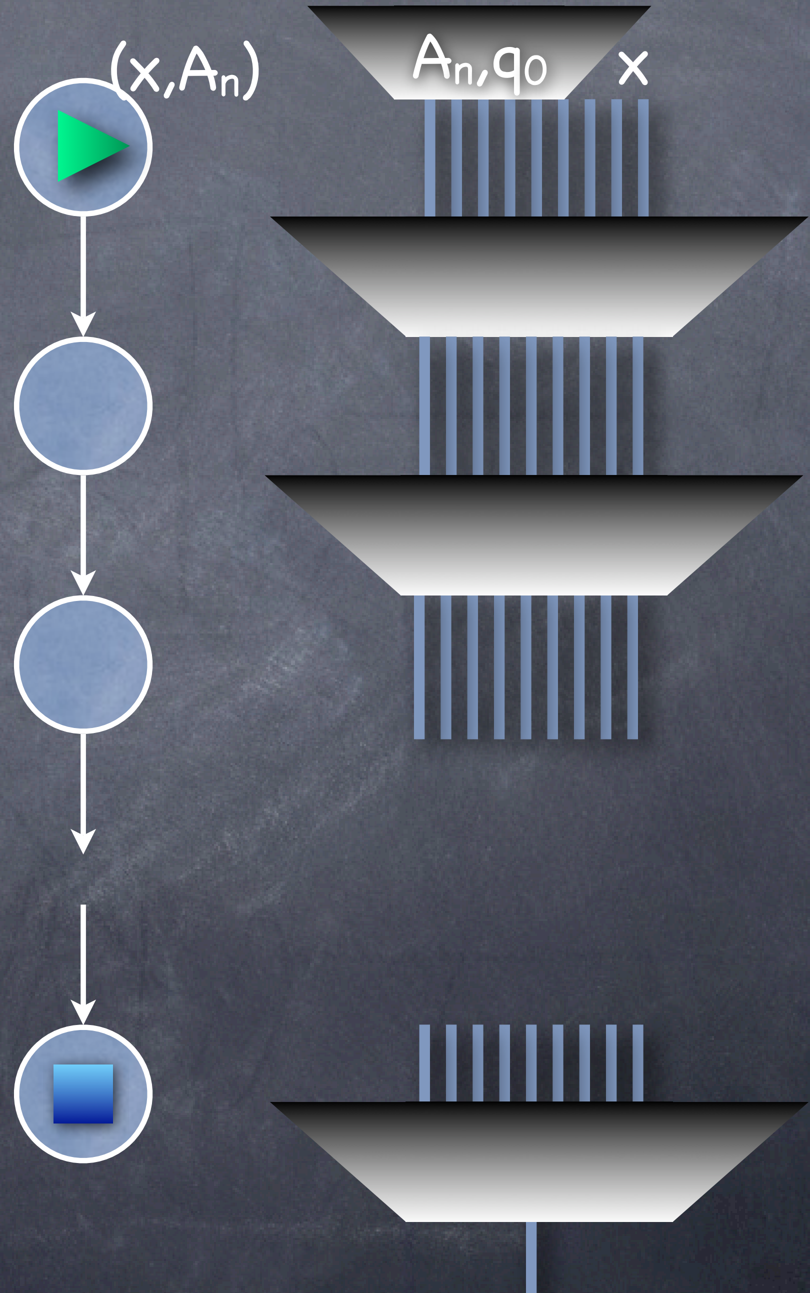
    - $|A_n|$ comparable to size of circuit C_n



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  - P/poly ⊆ SIZE(poly): Transformation from Cook's theorem, with advice string hardwired into circuit

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- SIZE(T′) $\subsetneq$ SIZE(T) if T=Ω(t2^t) and T′=O(2^t/t)
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    - All of them in SIZE(T), most not in SIZE(T′)

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- Logspace-uniform:
  - An O(log n) space TM can compute the circuit

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  - Similarly AC^i and $AC = \cup_{i>0} AC^i$

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- So NC = AC

NC and P

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  - Build the circuit in logspace (so poly time) and evaluate it in time polynomial in the size of the circuit

NC and P

- $NC \subseteq P$
  - Build the circuit in logspace (so poly time) and evaluate it in time polynomial in the size of the circuit
- Open problem: Is $NC = P$?

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