Computational Complexity

Lecture 9

More of the Polynomial Hierarchy

Alternation

PH is in terms of verification

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- Recall Σ_k^P

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  - Languages $L = \{x|\ \exists w_1 \forall w_2 \ldots Q w_k\ F(x;w_1,w_2,..,w_k)\}$, where F in P

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  - x in L if $\exists w_1\ \forall w_2 \ldots Q w_k$ such that $M(x;w_1,w_2,..,w_k)$ accepts
- Plan: Formulate in terms of a non-deterministic TM (with no certificates)

Verification → Non-determinism

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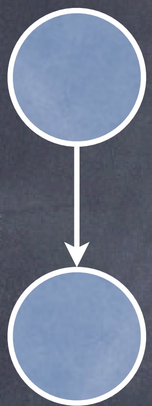

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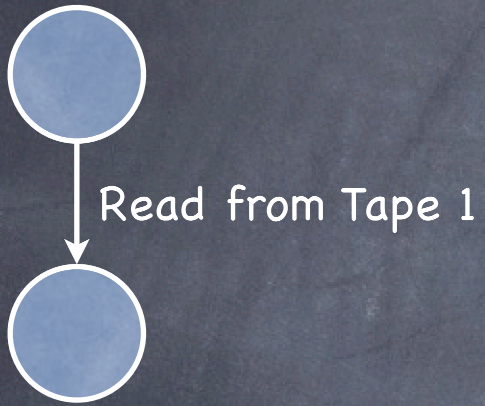

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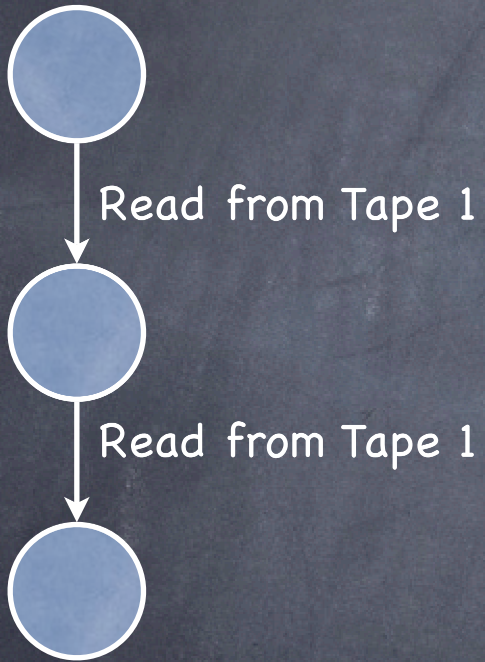

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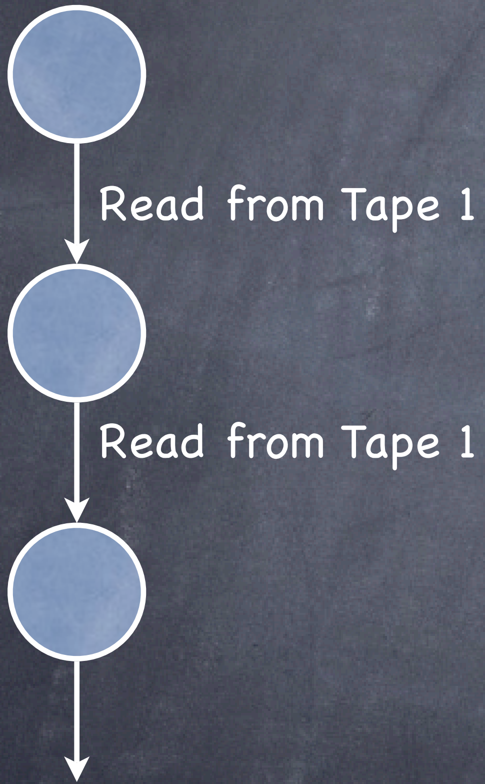

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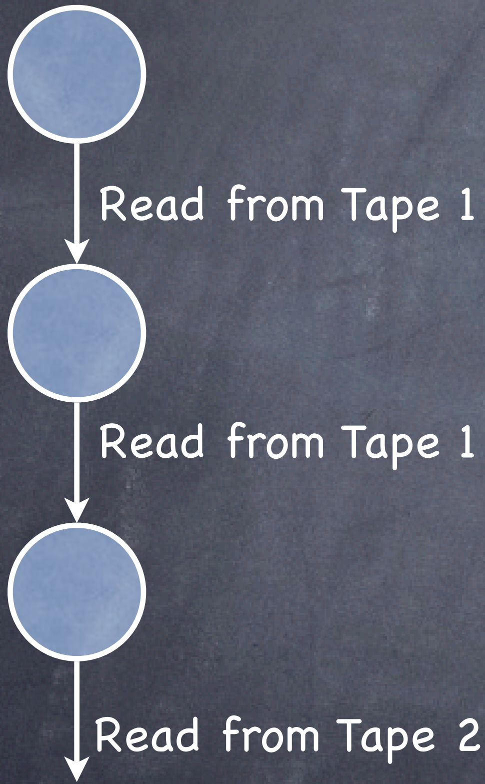

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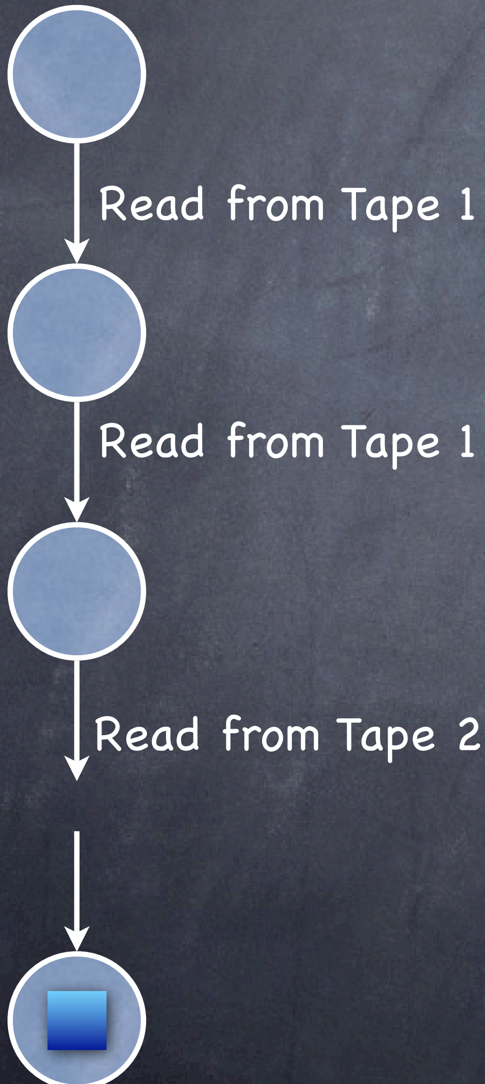

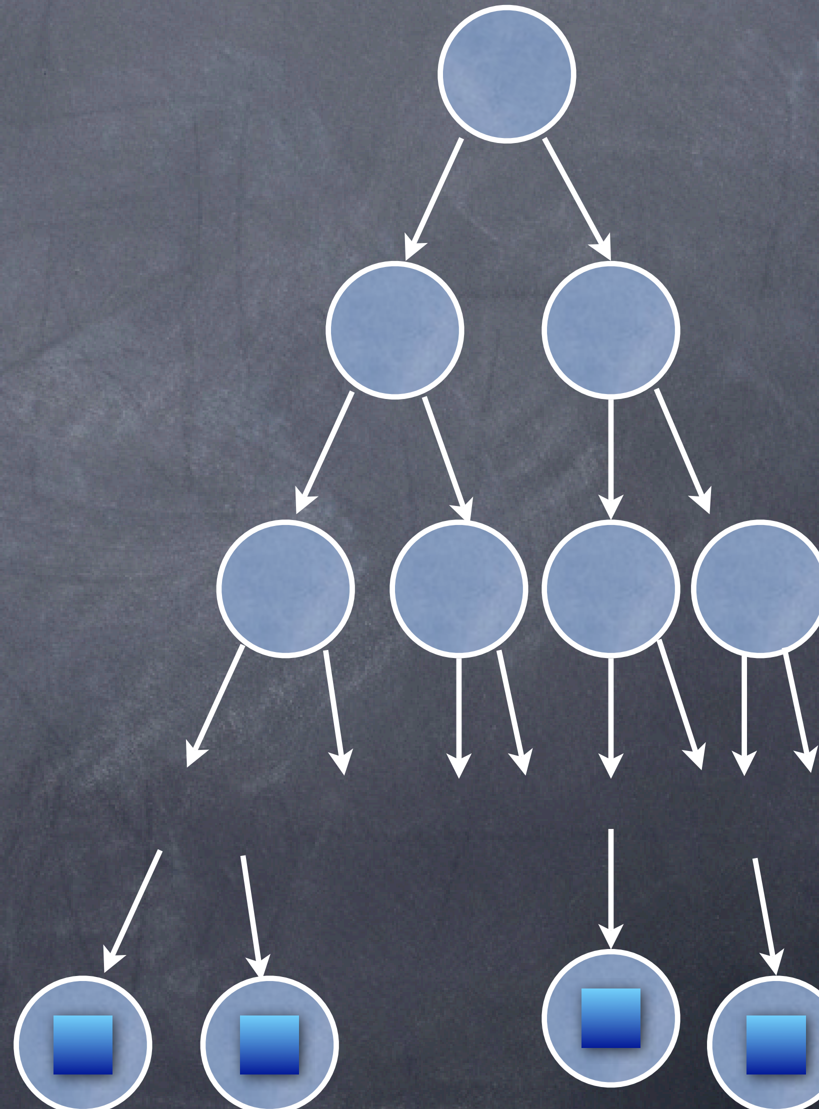

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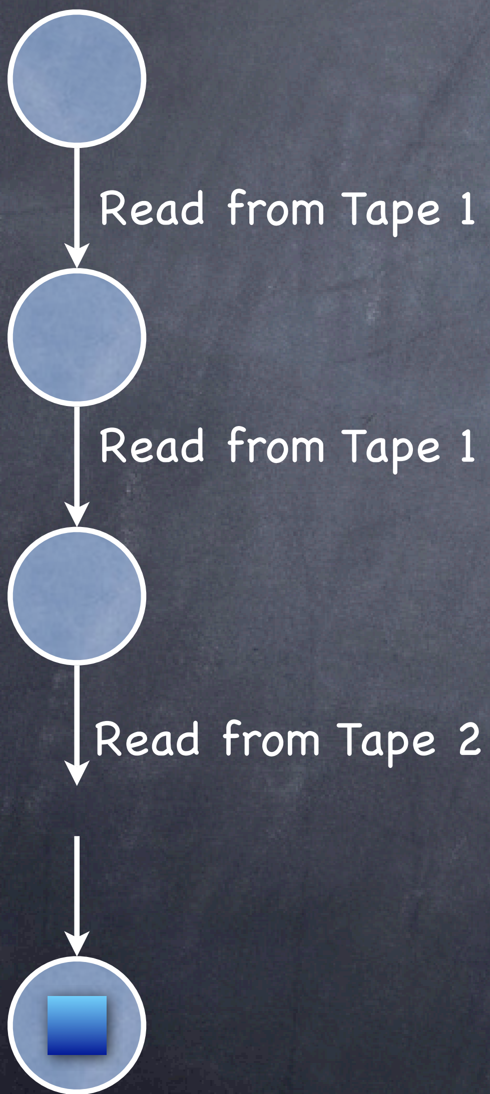

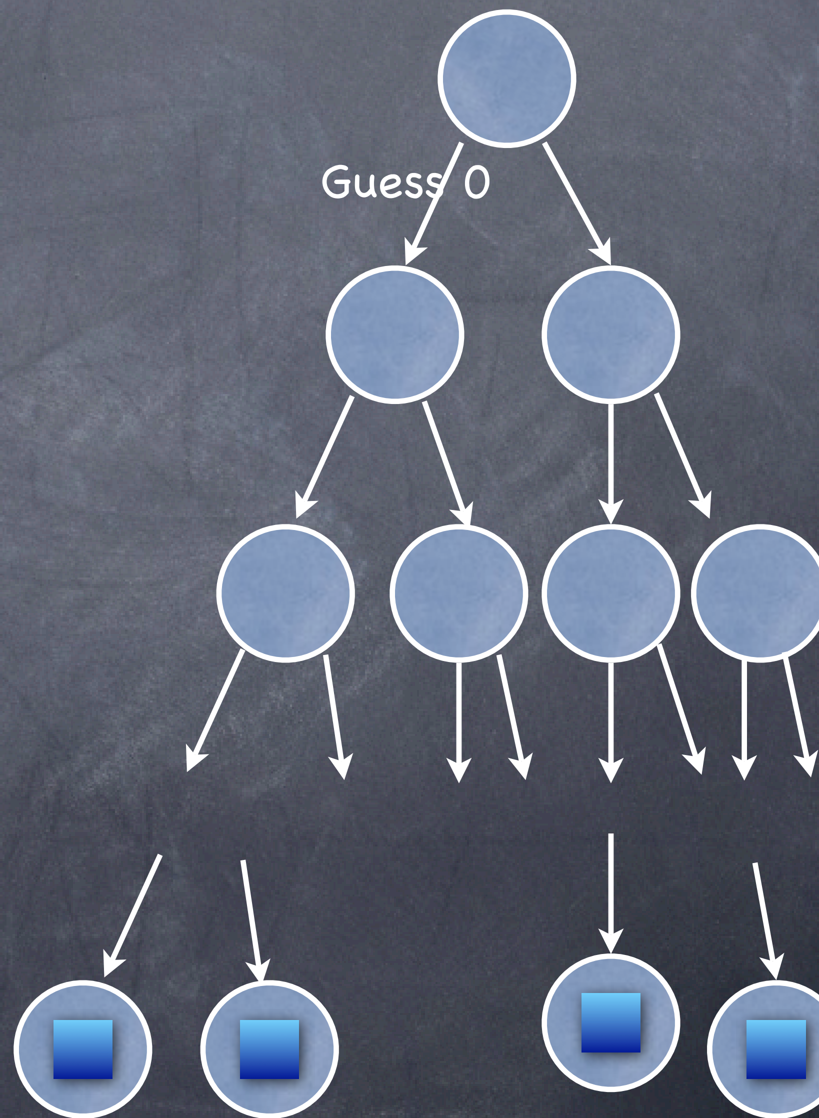

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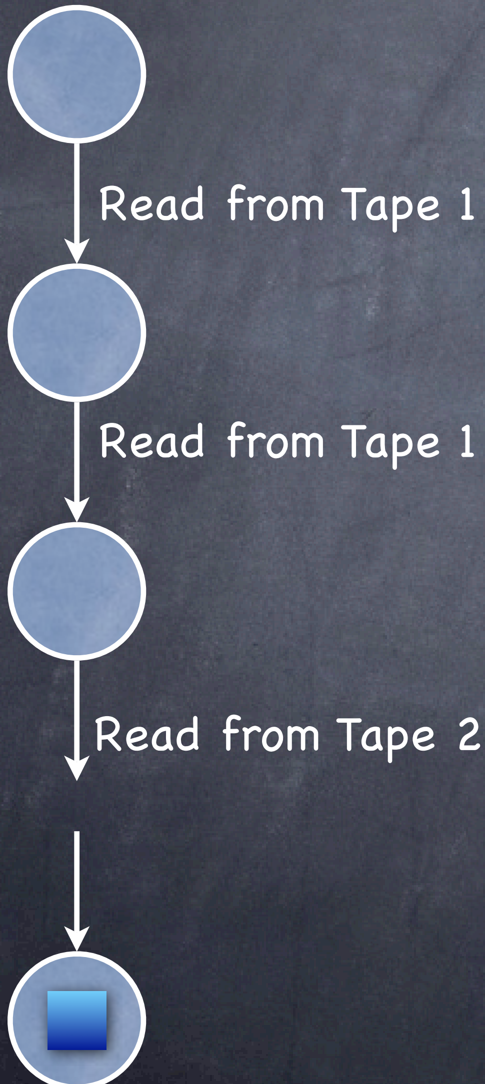

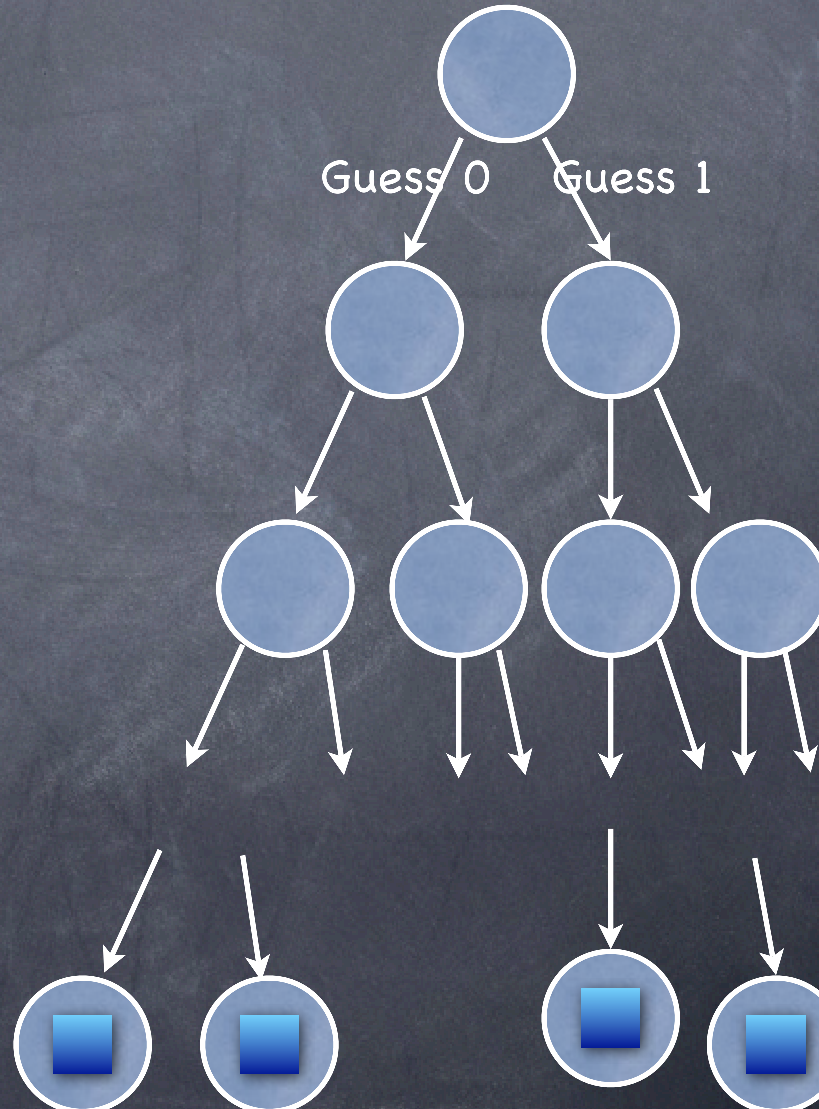

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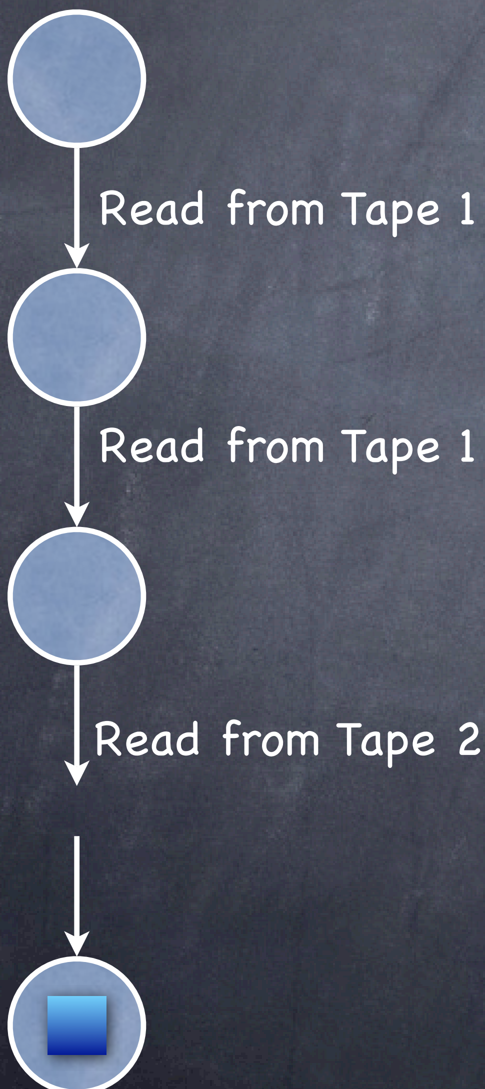

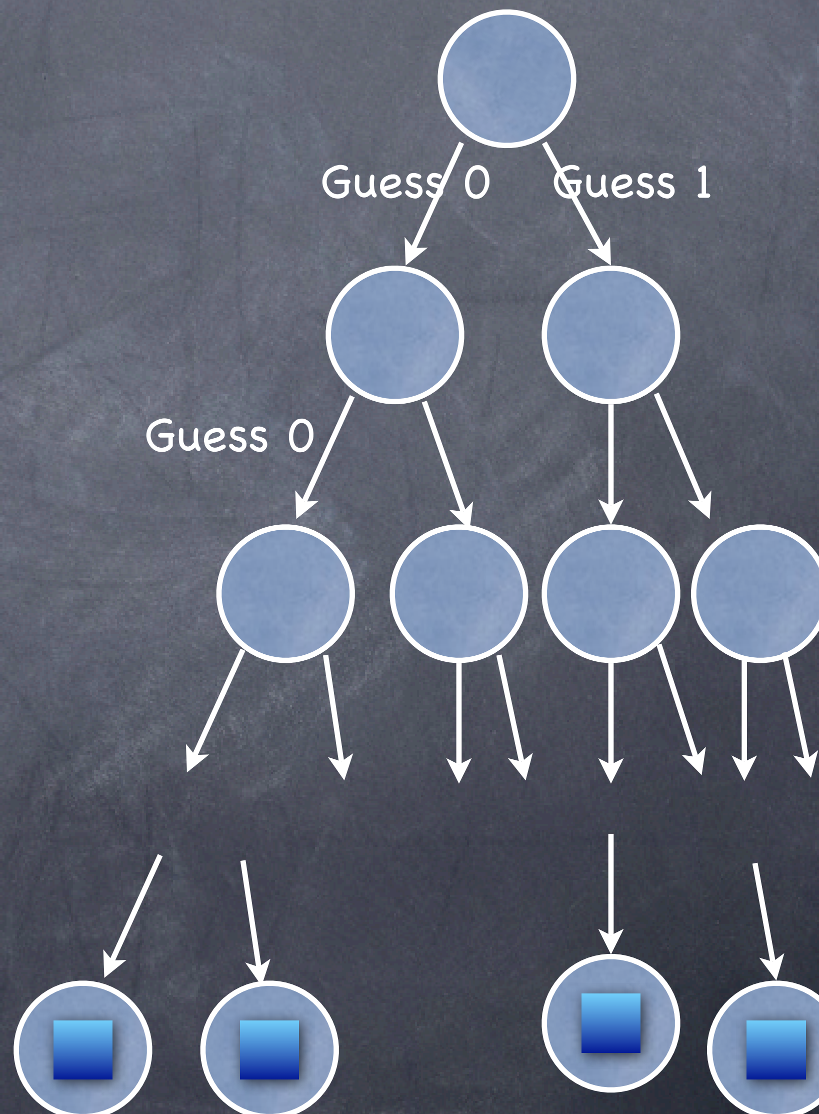

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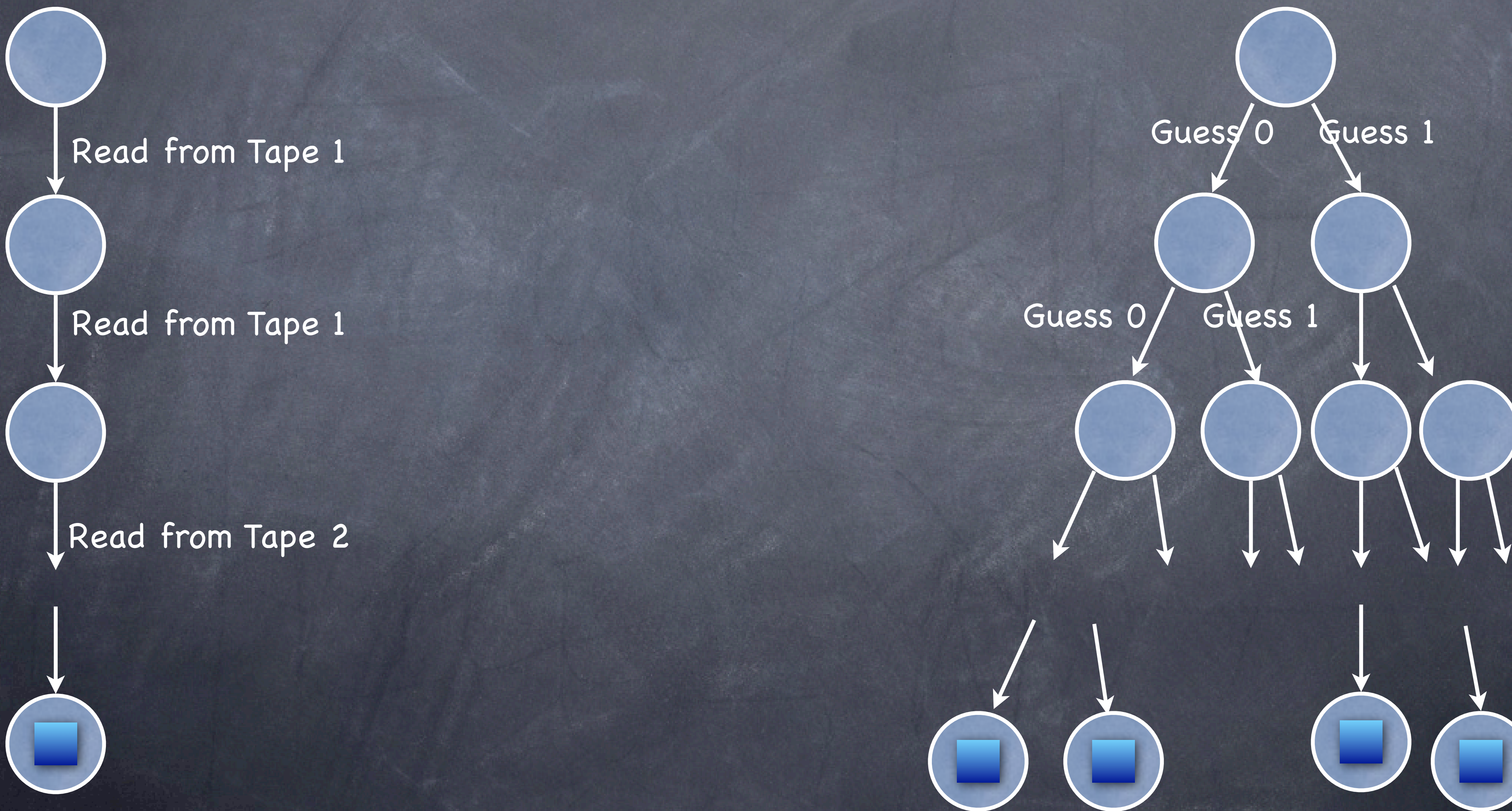

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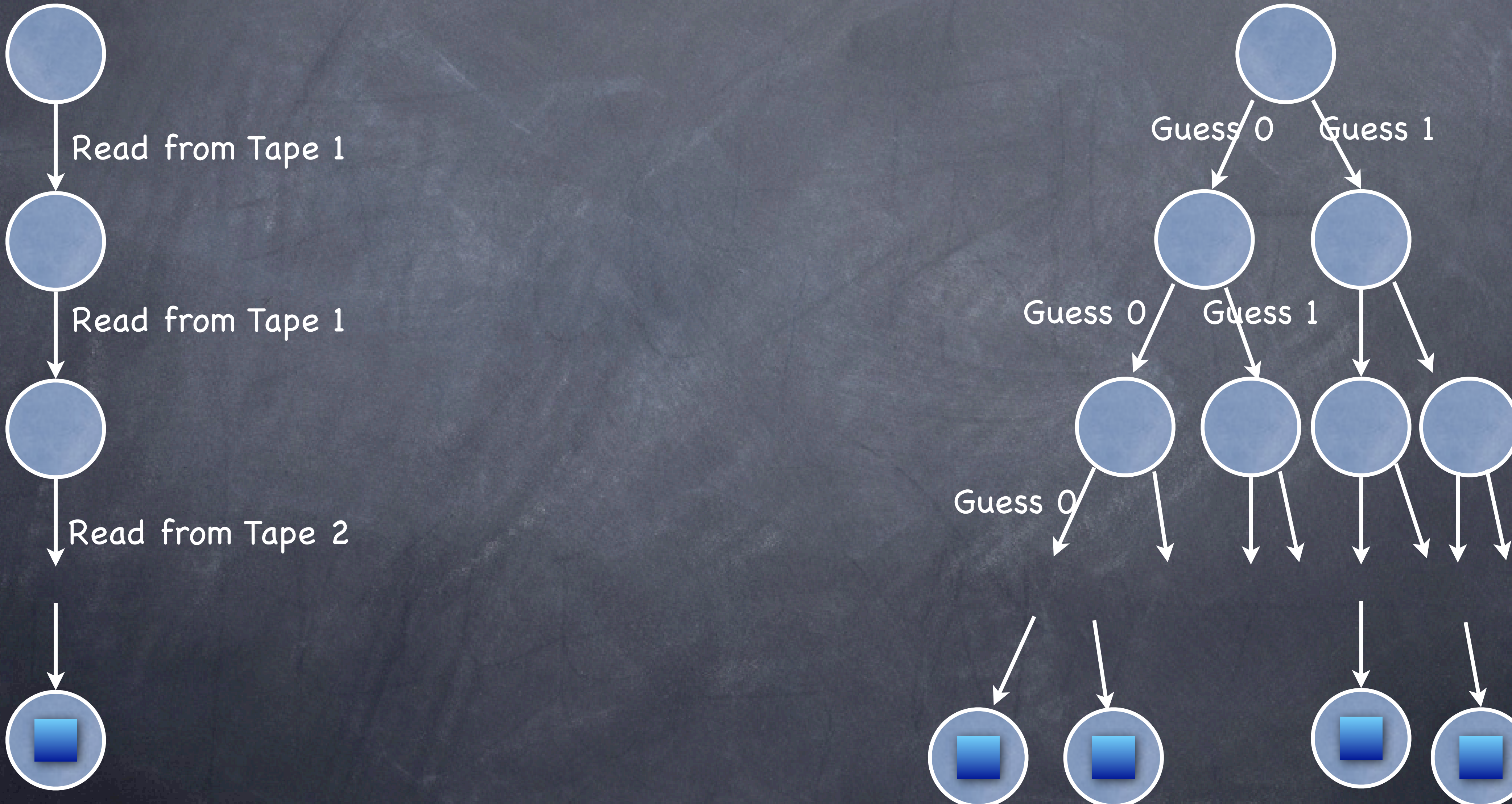

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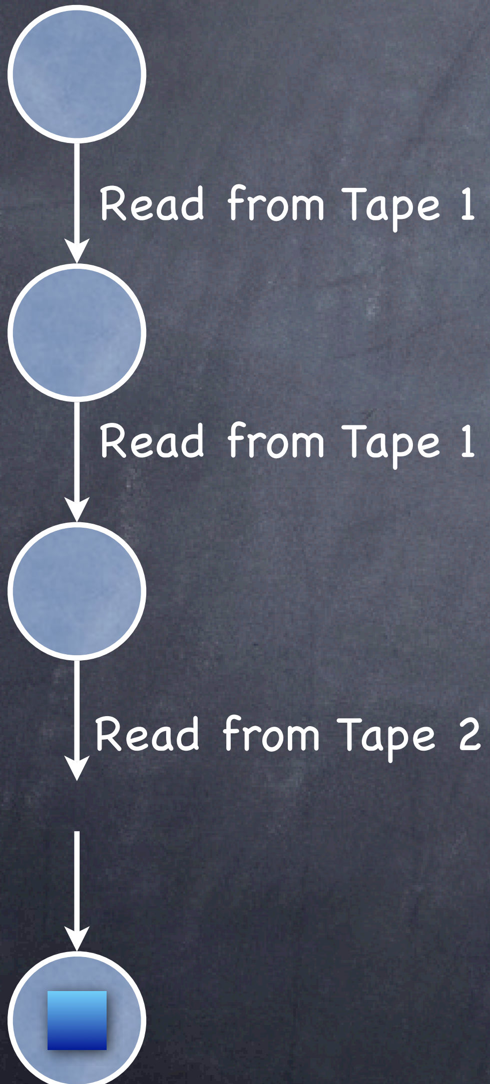

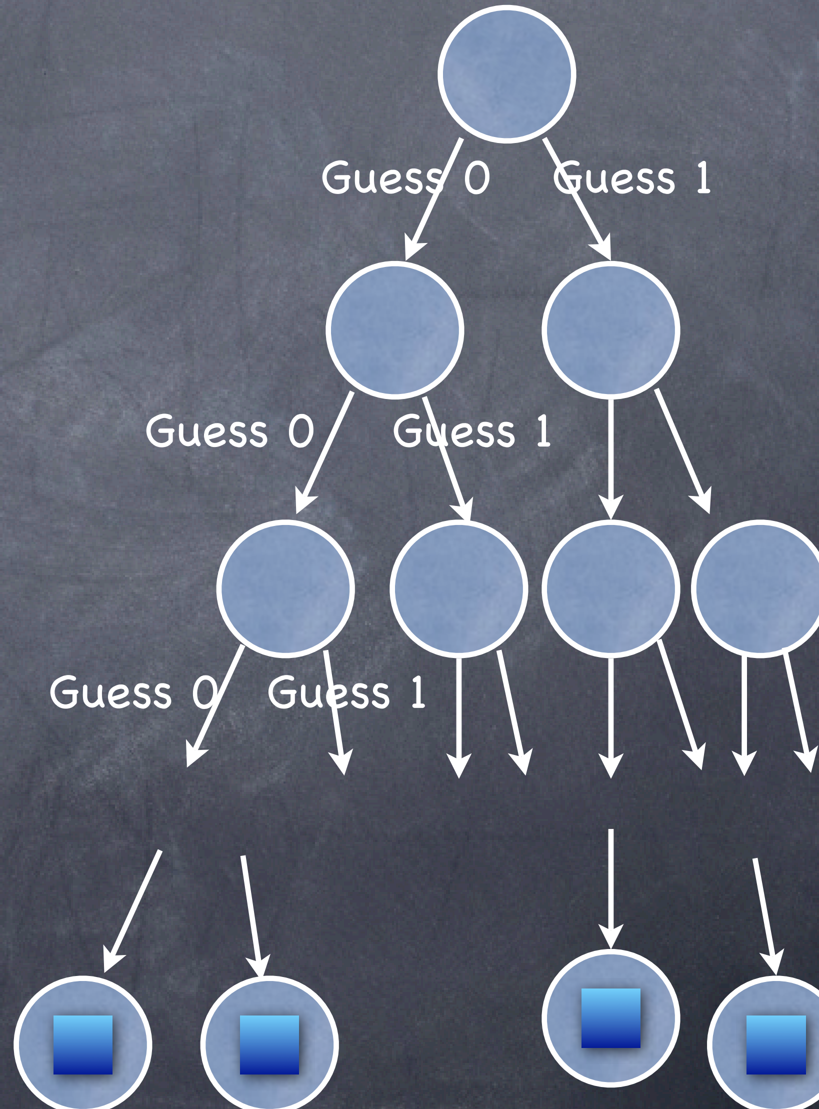

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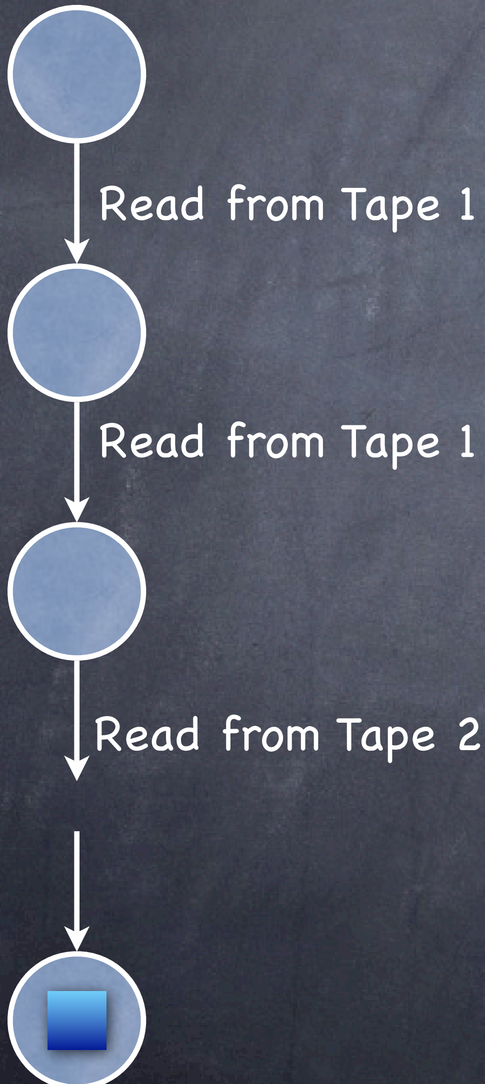

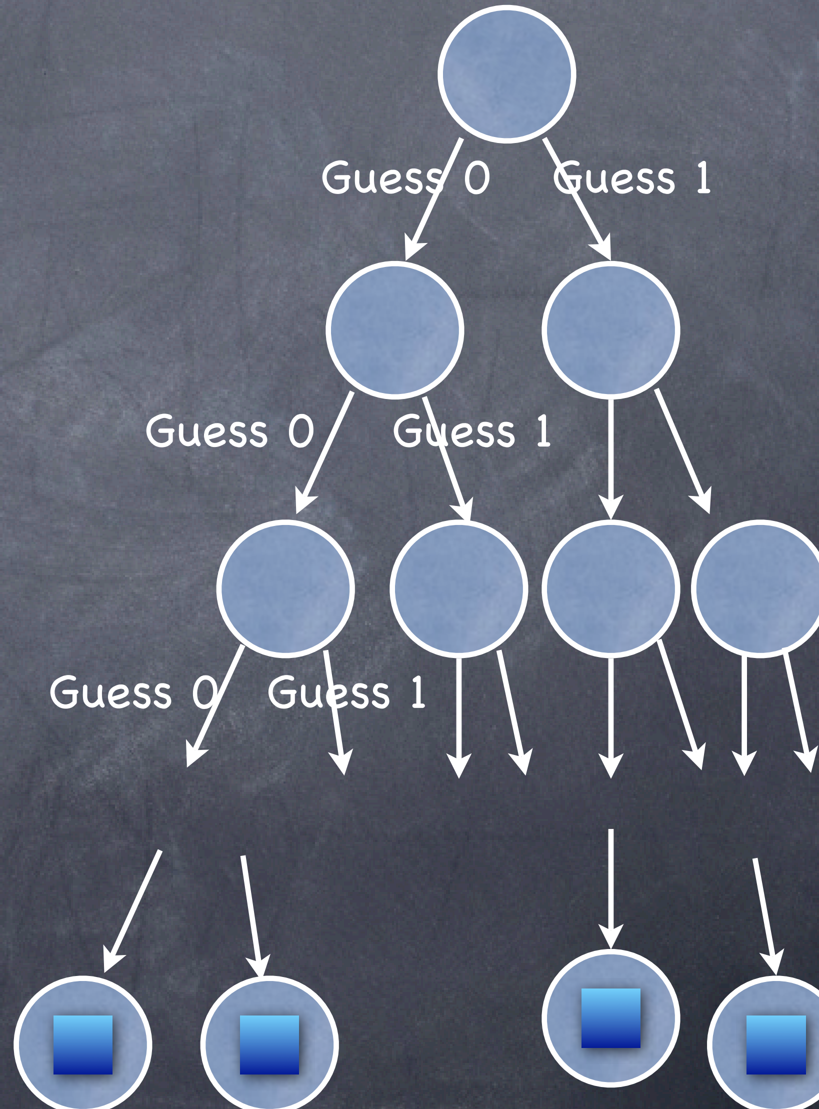

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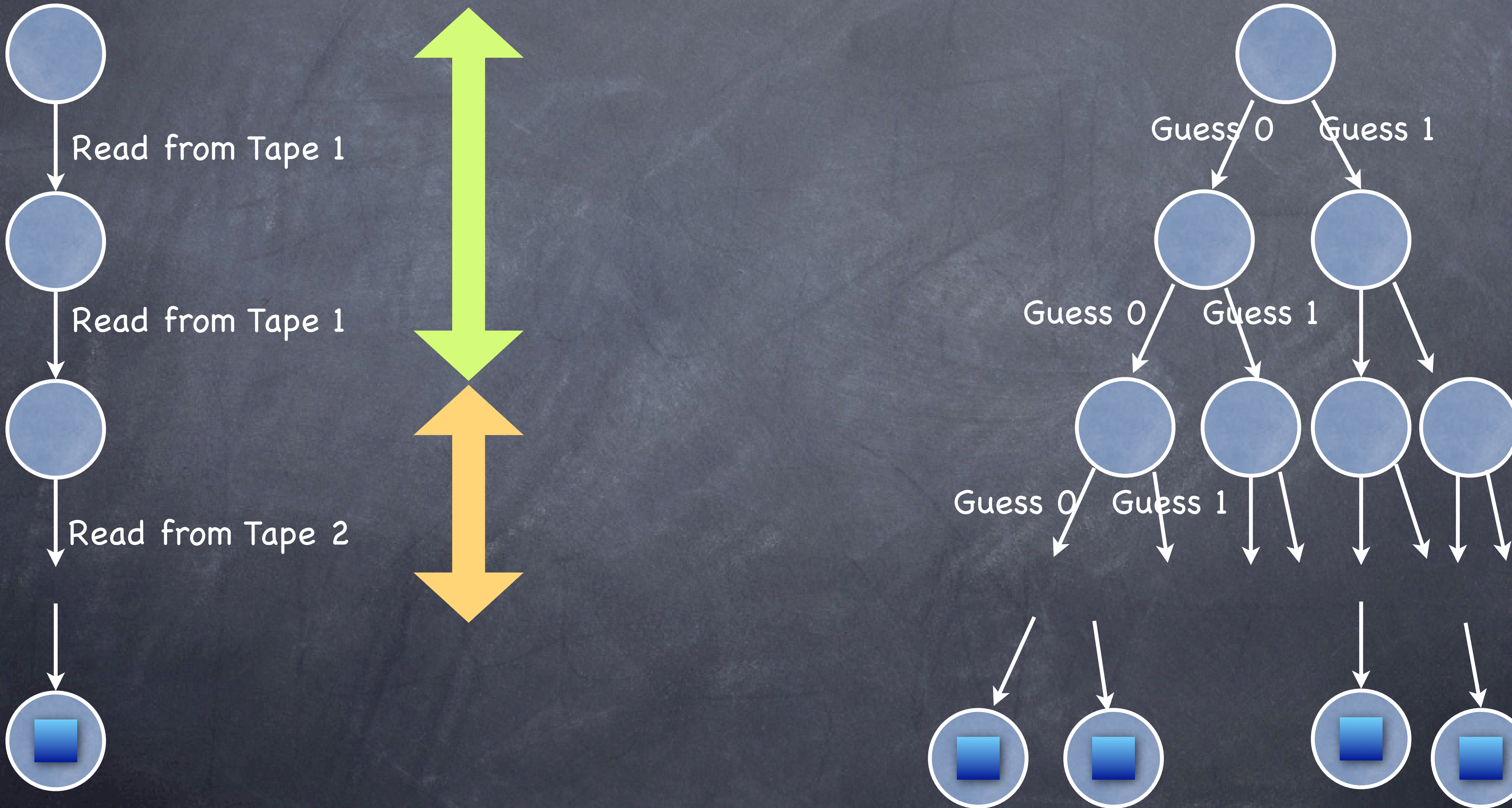

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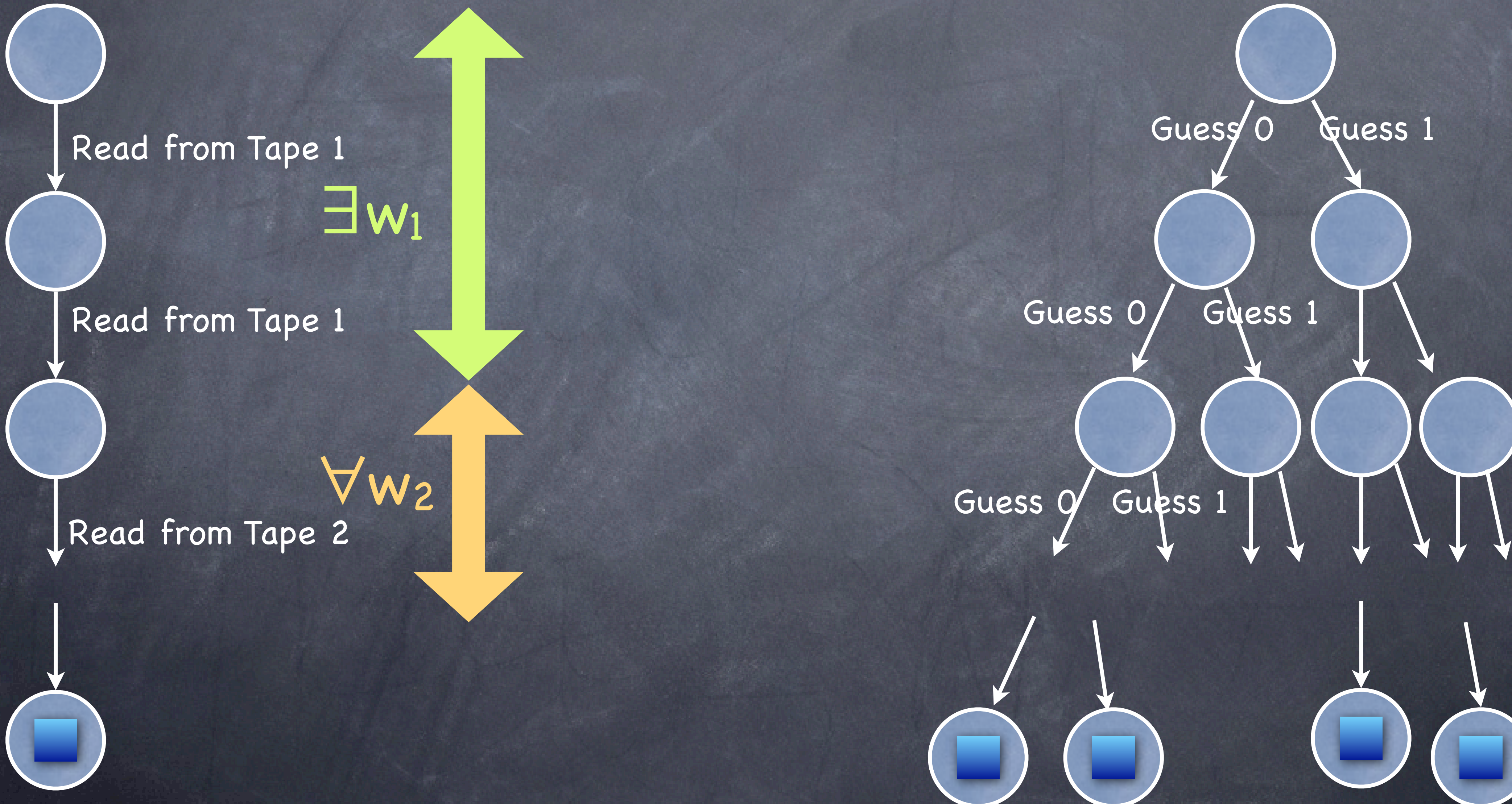

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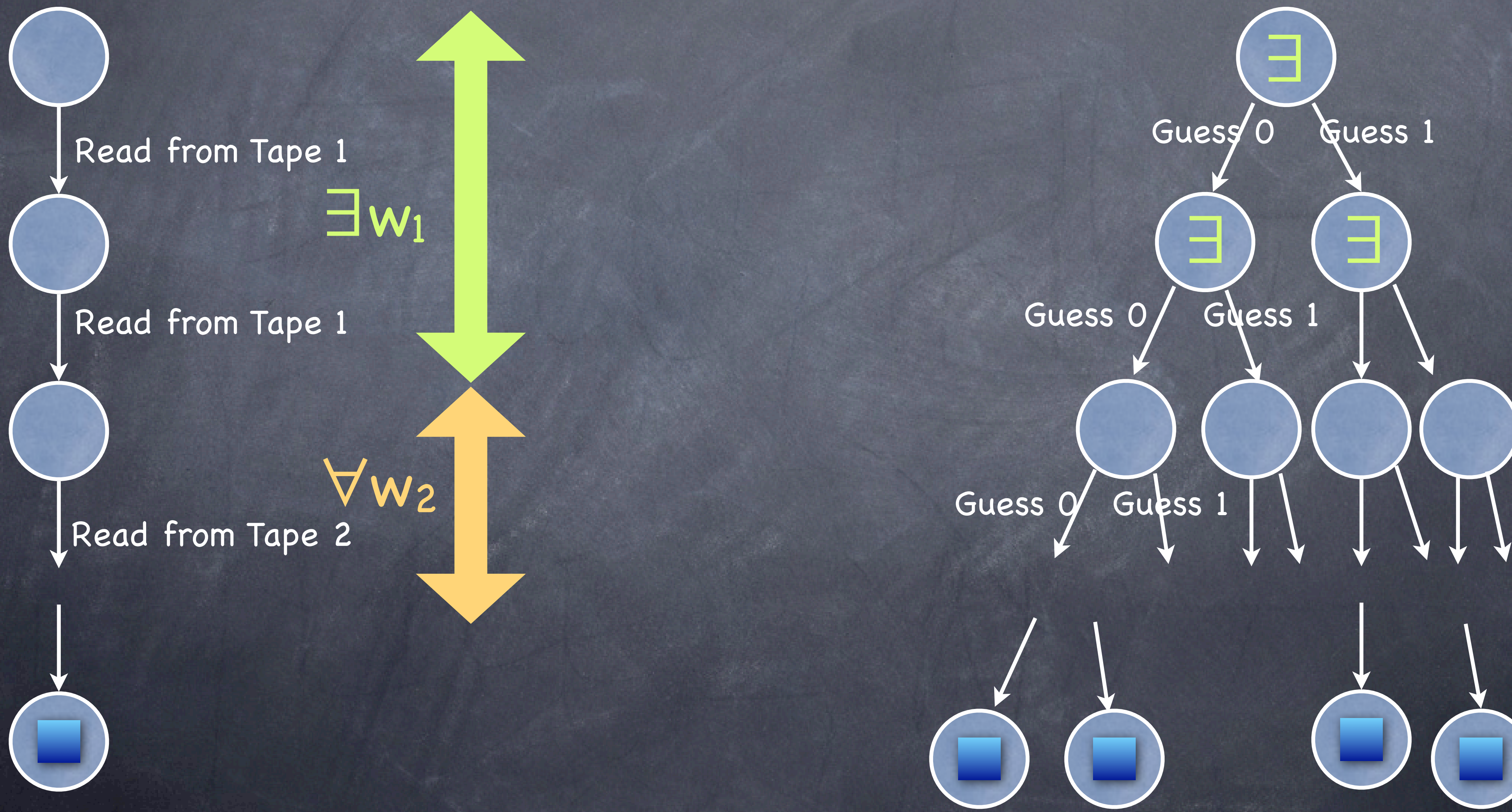

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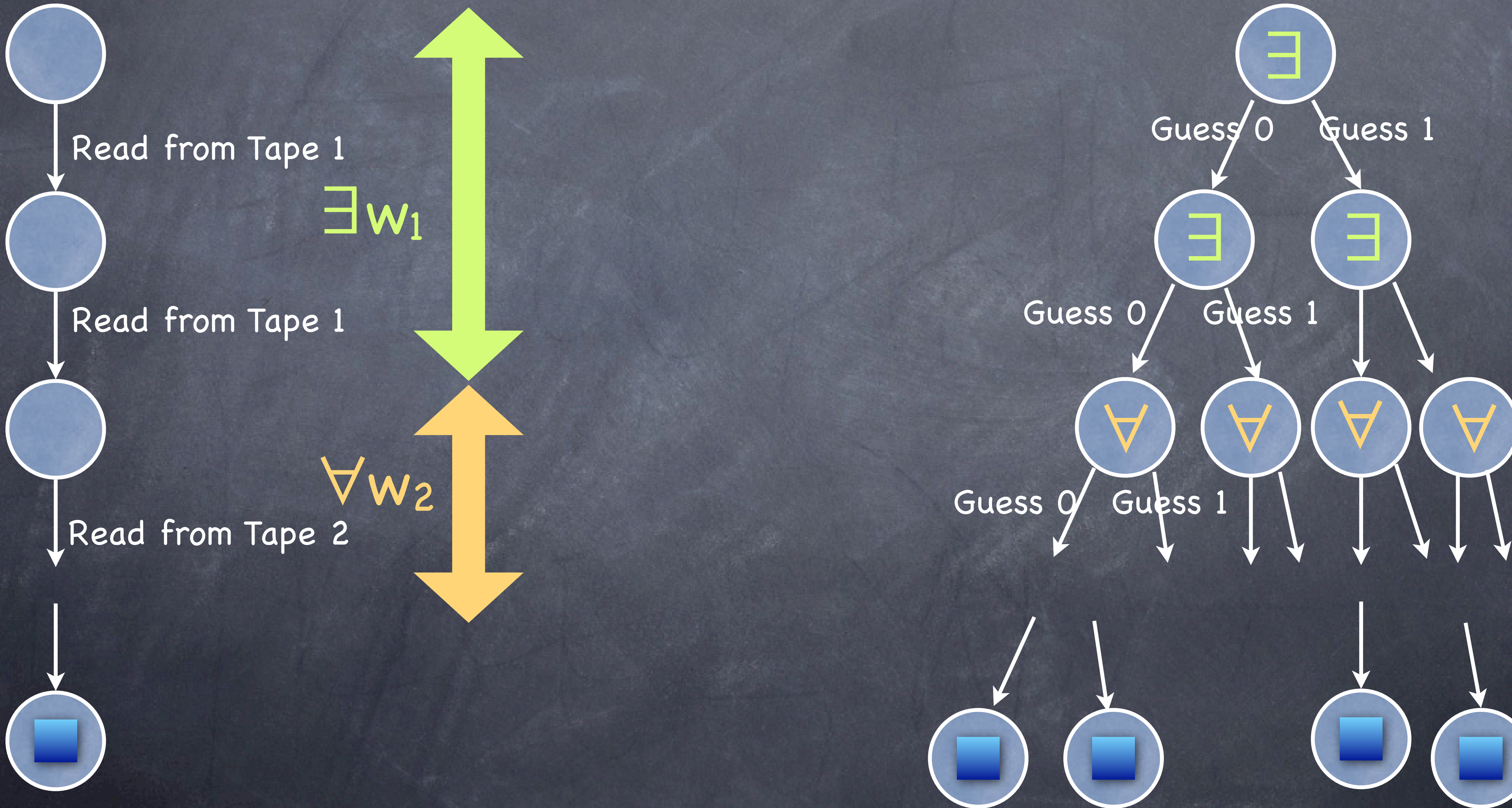
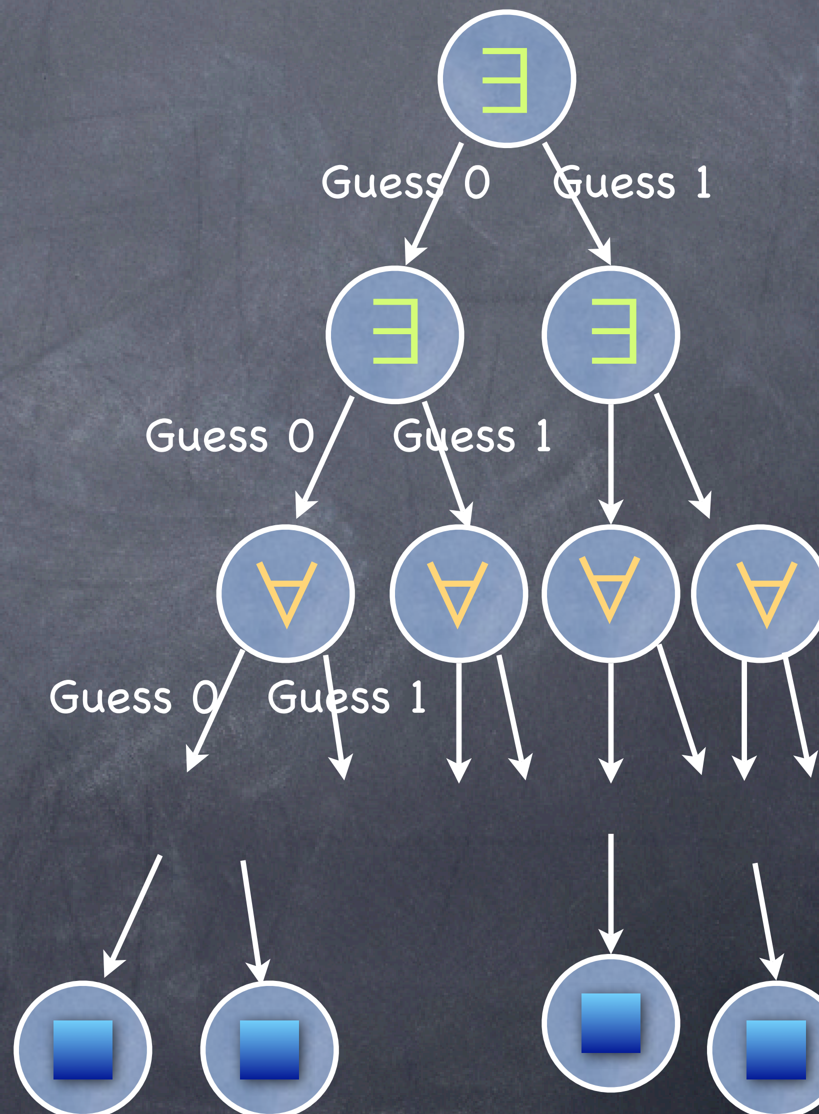
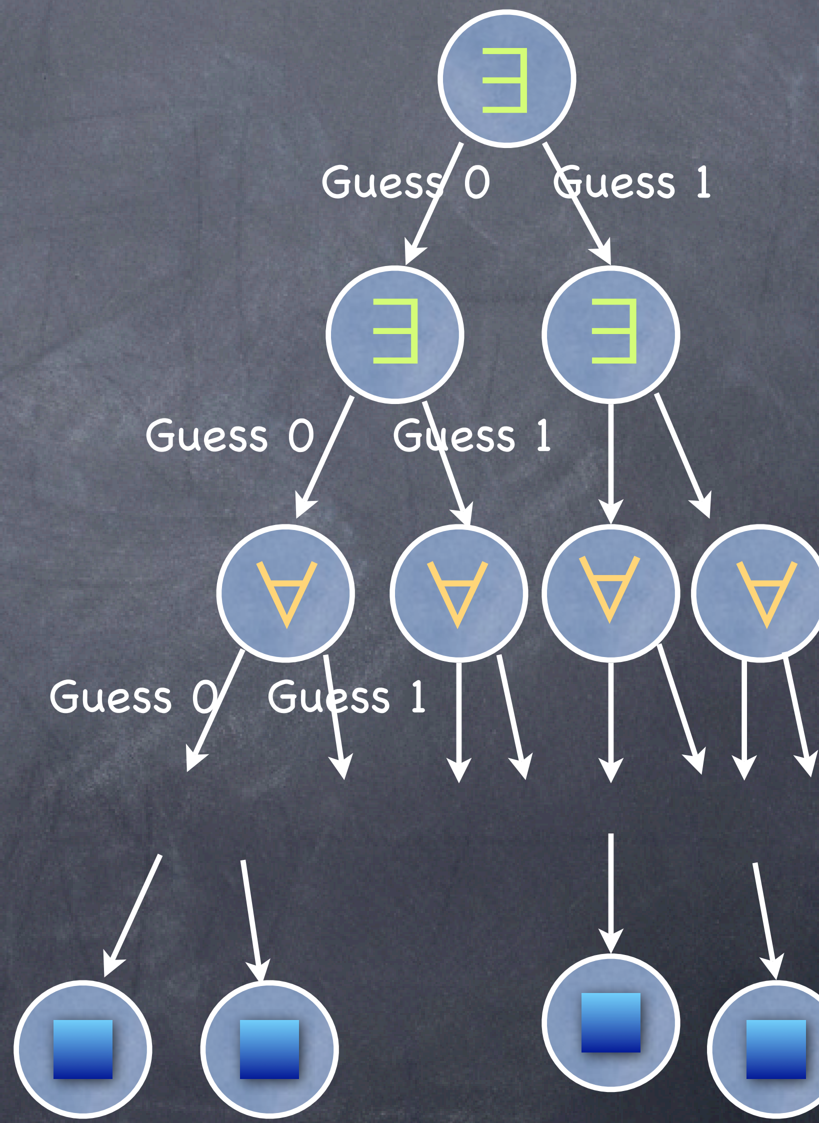


ATM



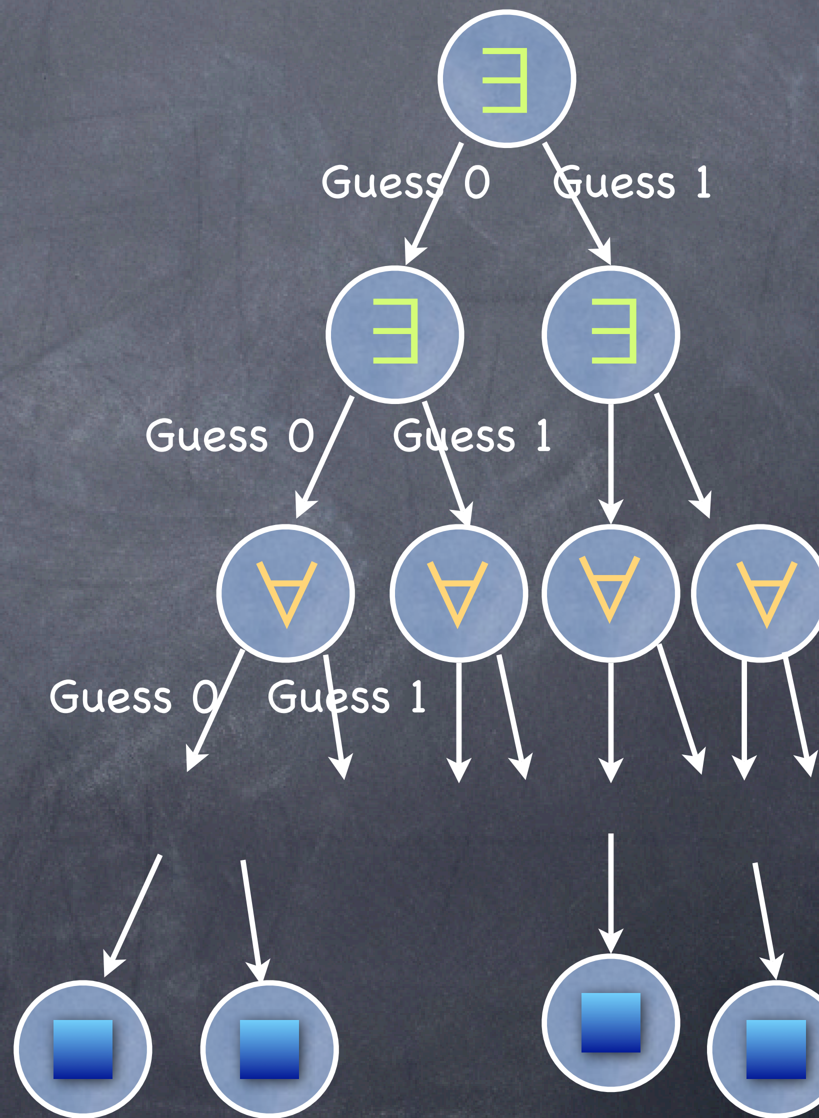
ATM

- Alternating Turing Machine



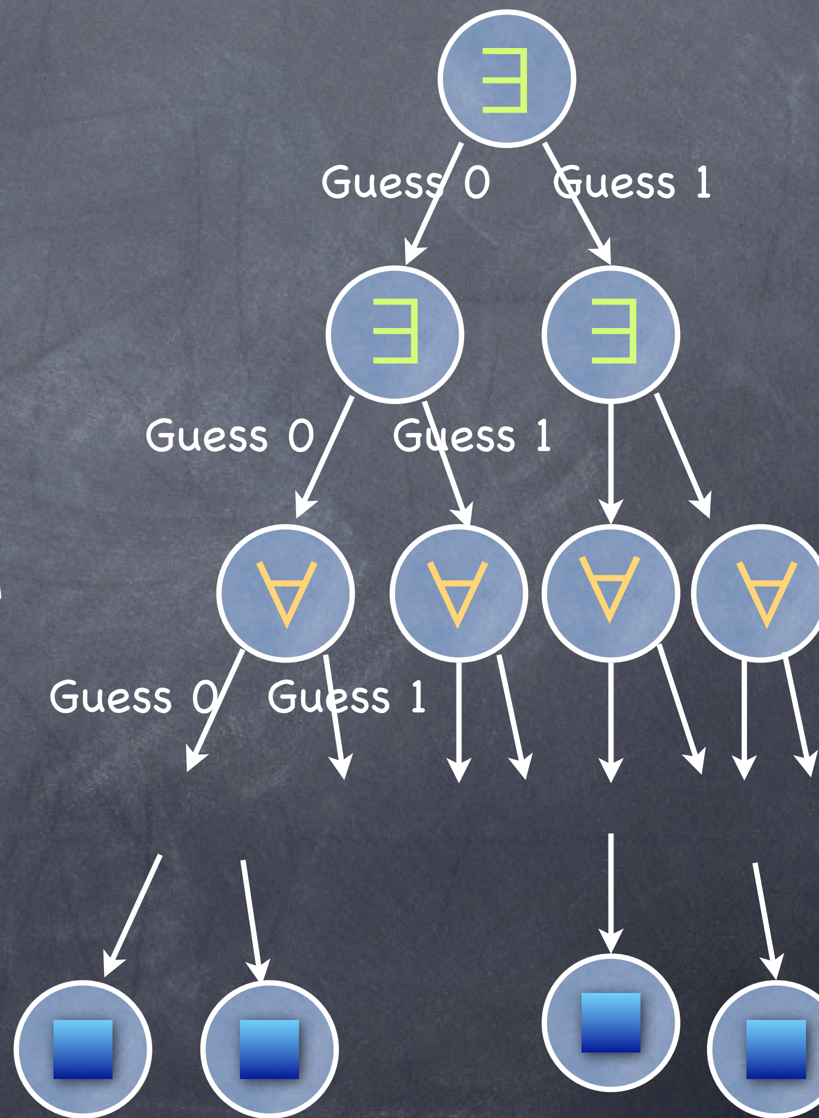
ATM

- Alternating Turing Machine
  - At each step, execution can fork into two



ATM

- Alternating Turing Machine
  - At each step, execution can fork into two
    - Exactly like an NTM or co-NTM



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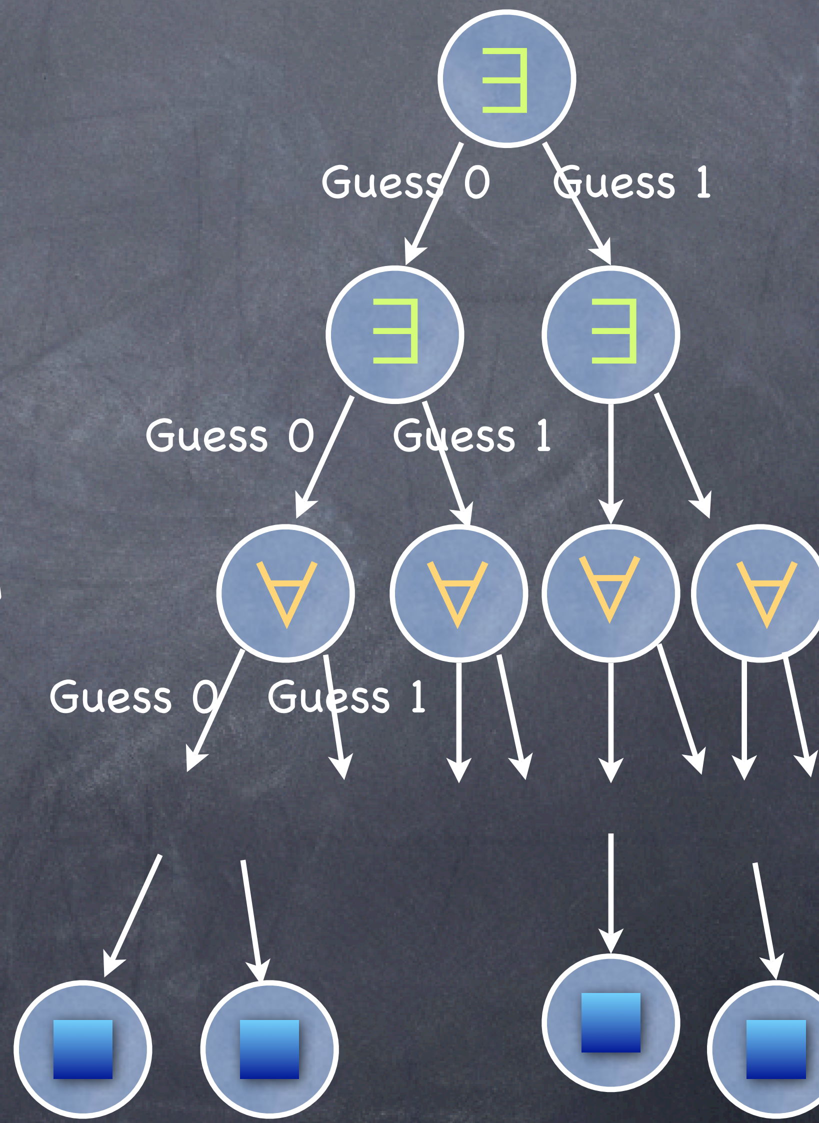
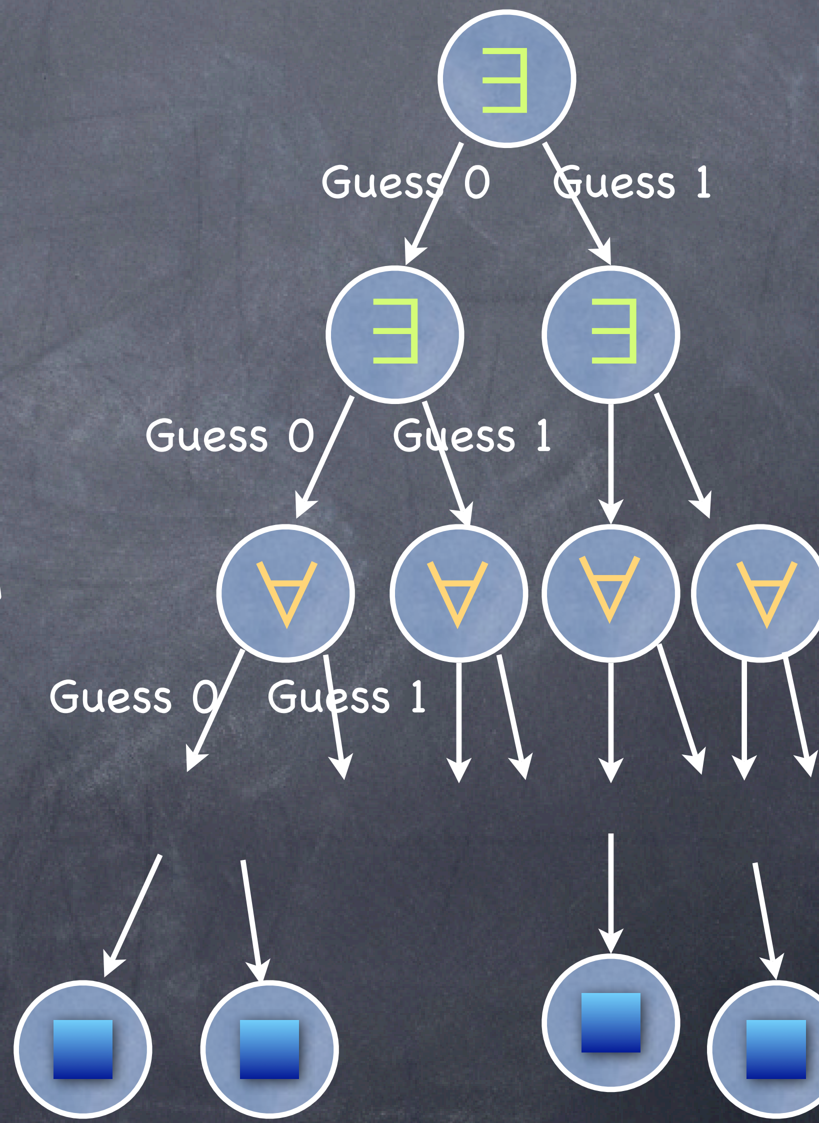


ATM

- Alternating Turing Machine
  - At each step, execution can fork into two
    - Exactly like an NTM or co-NTM
  - Accepting rule is more complex
    - Like in the game tree for QBF



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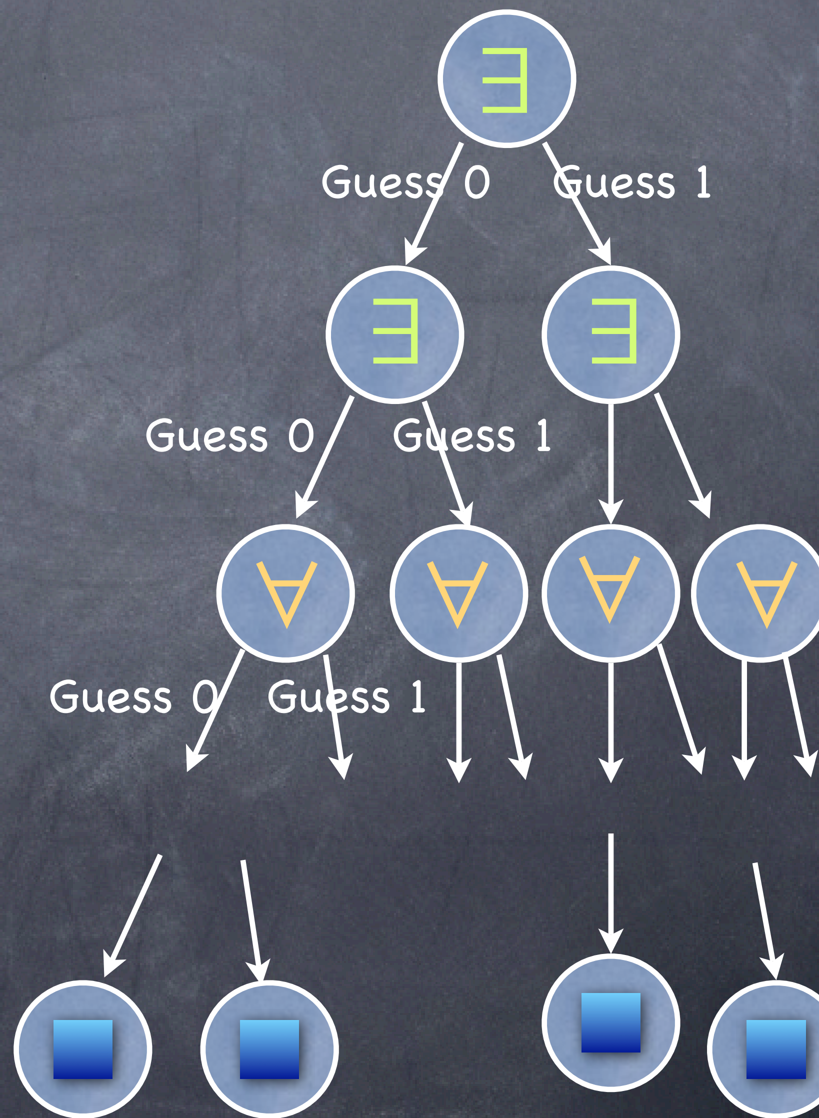
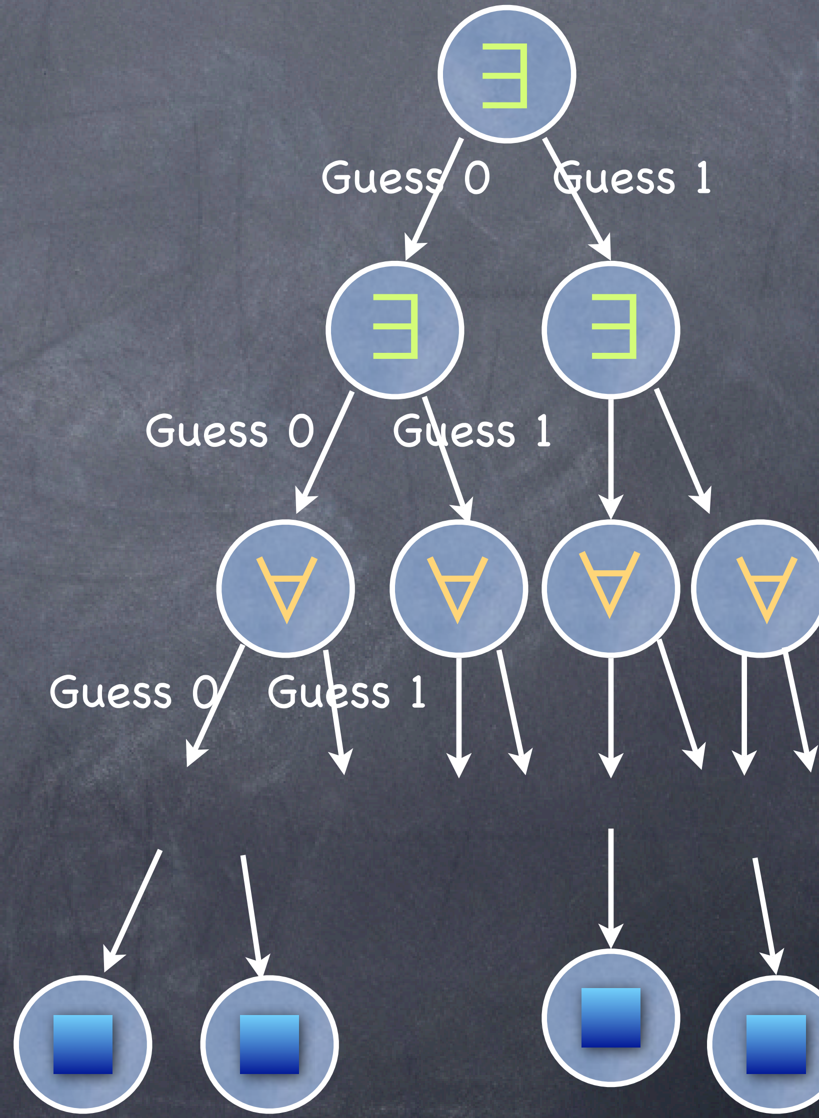


ATM

- Two kinds of configurations: ∃ and ∀



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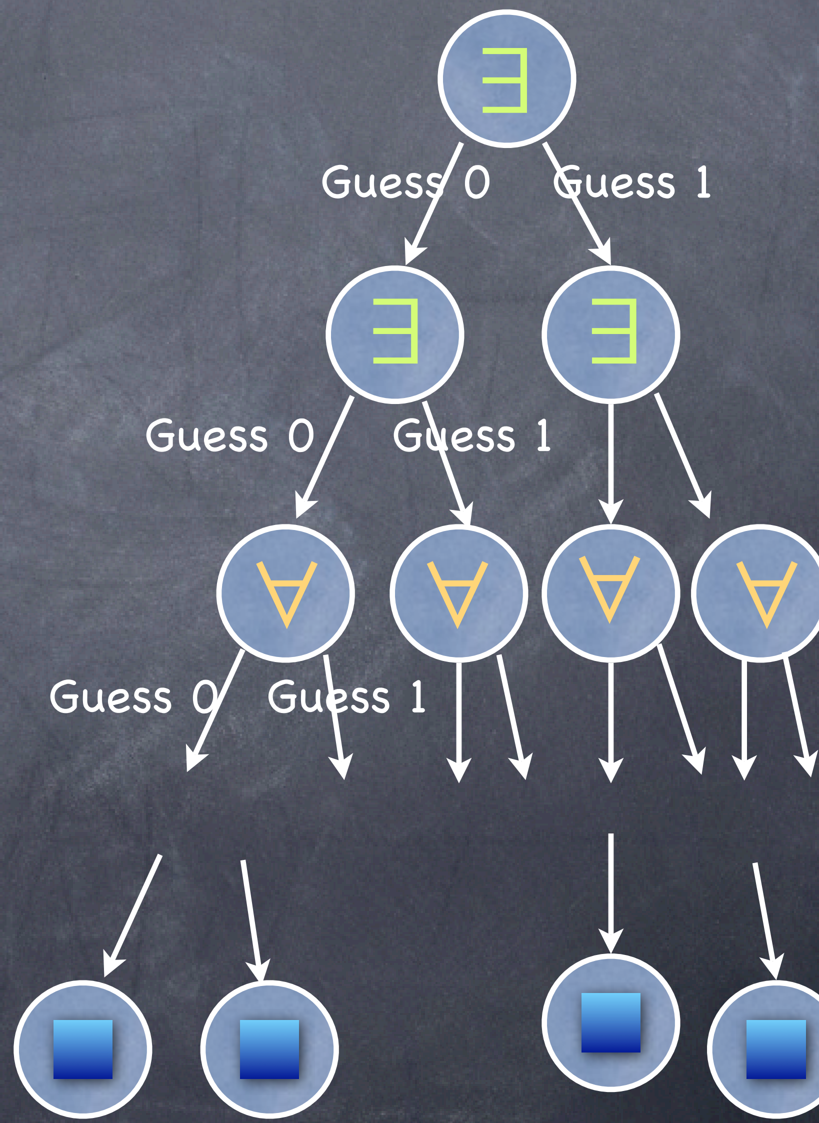

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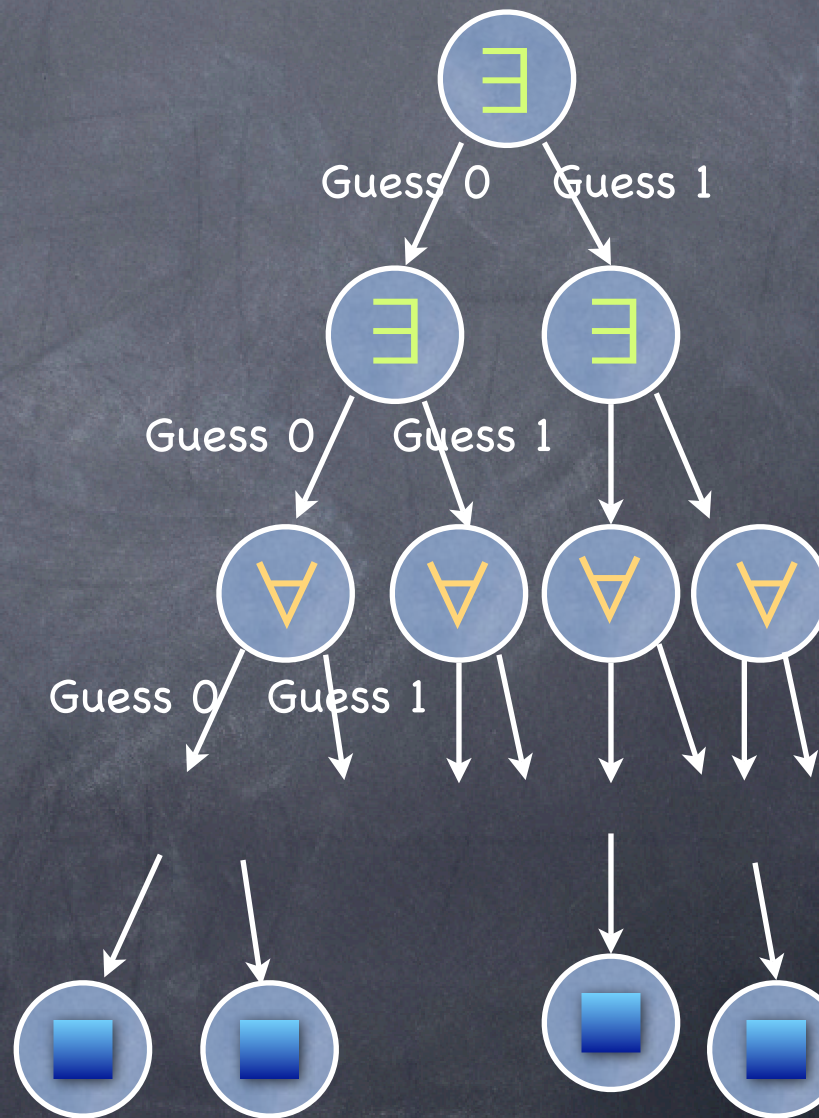
  - Depending on the state
  - A ∃ configuration is accepting if either child is accepting



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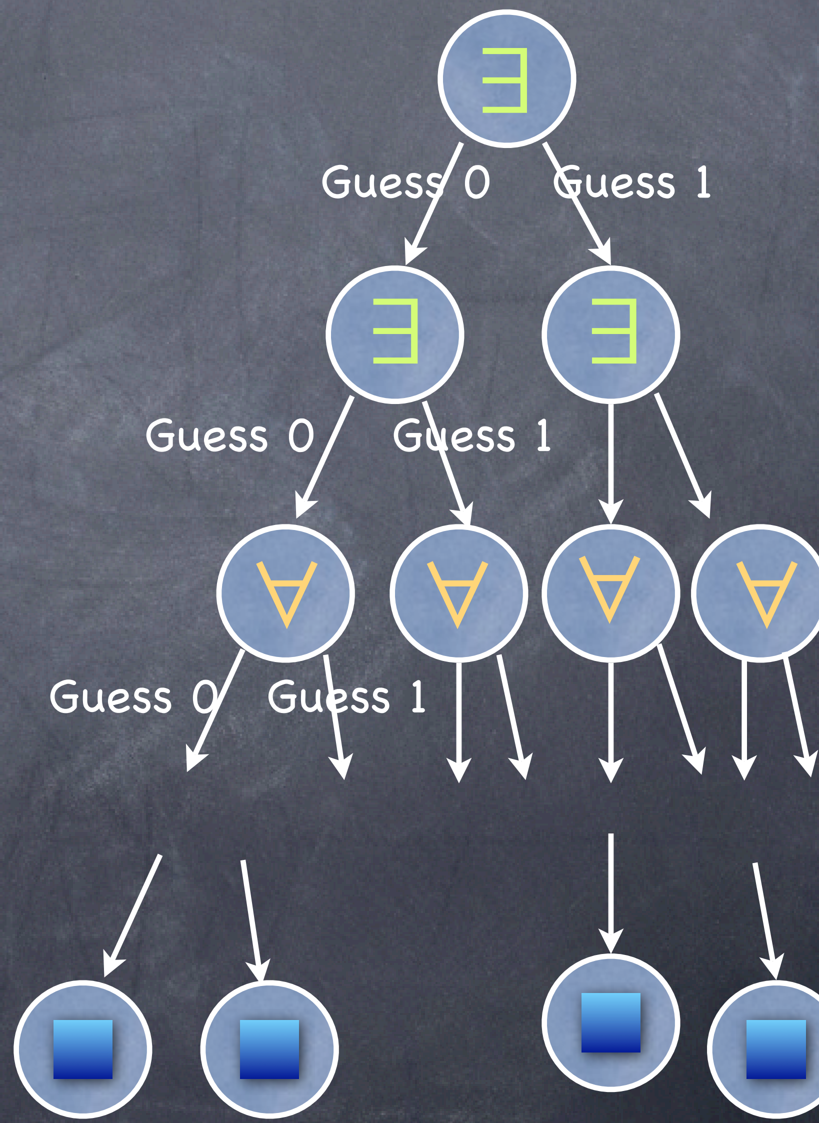

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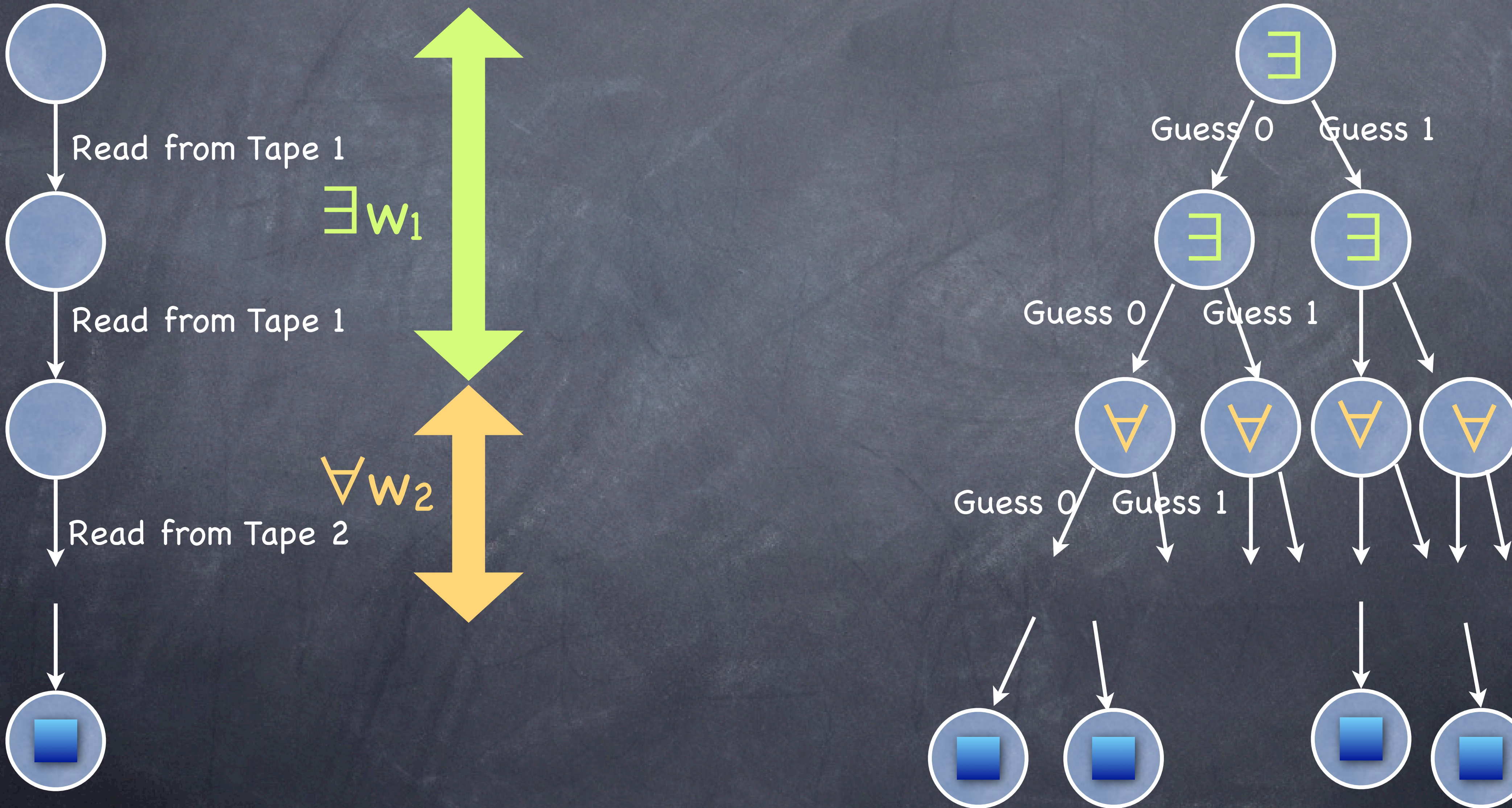

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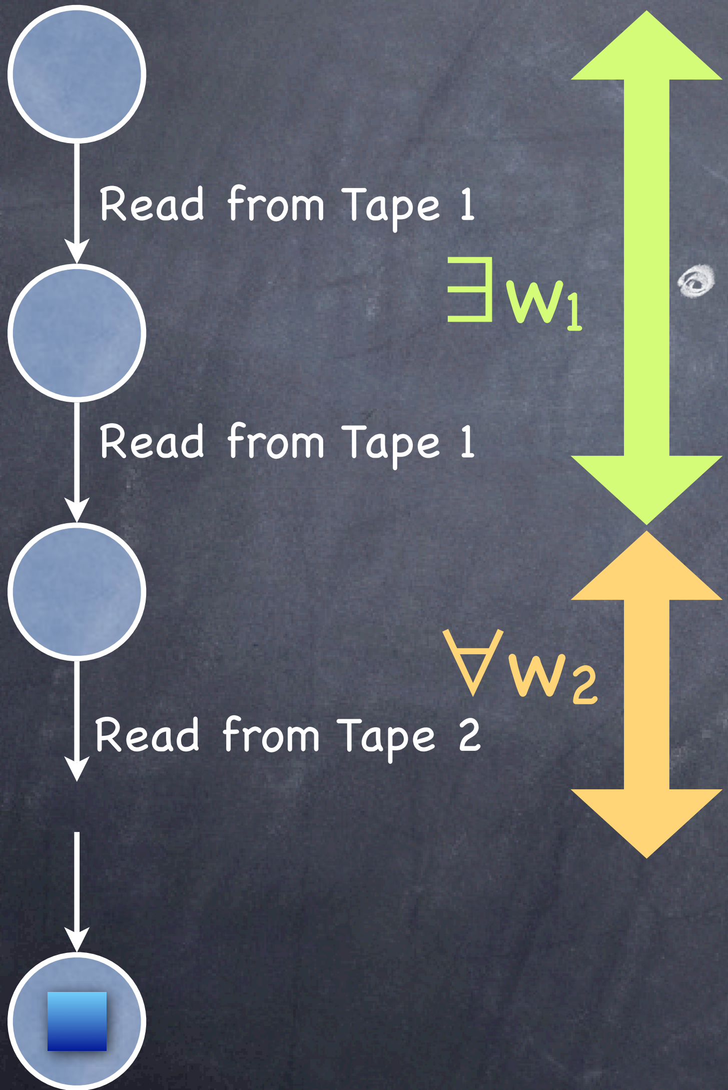
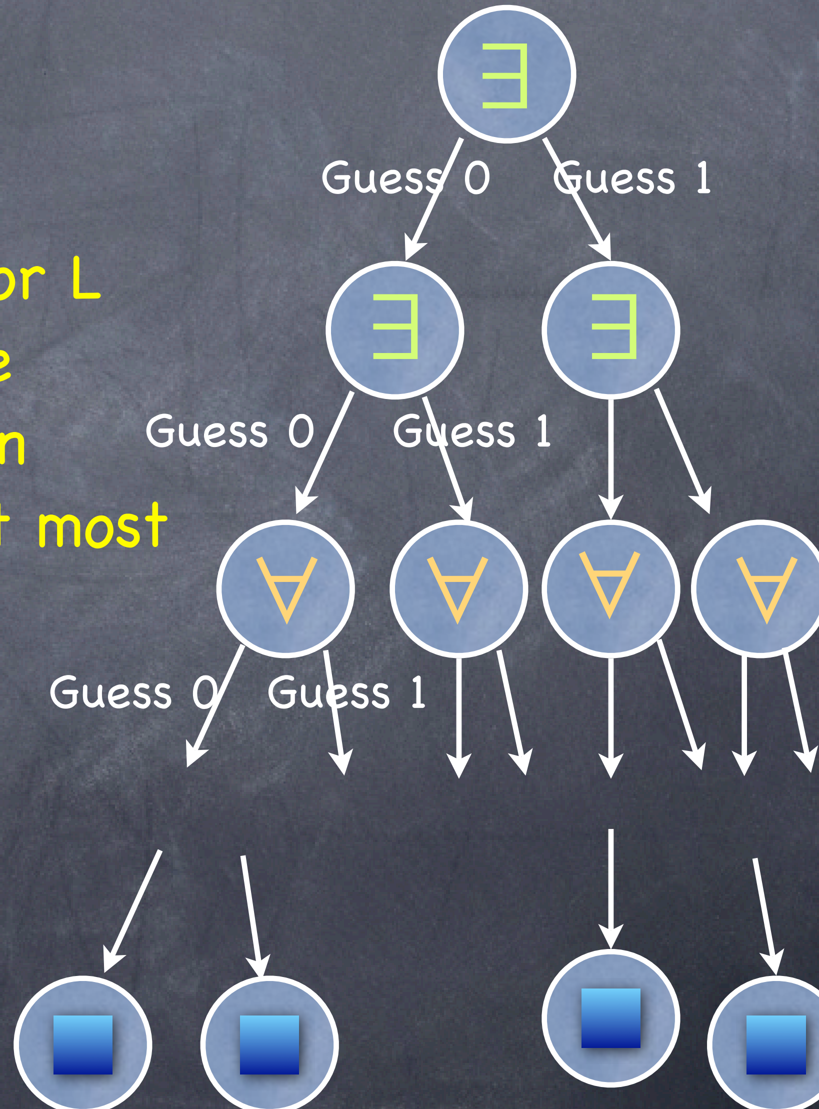


- Given a verifier for L using k certificate tapes, can build an ATM for L with at most k alternations



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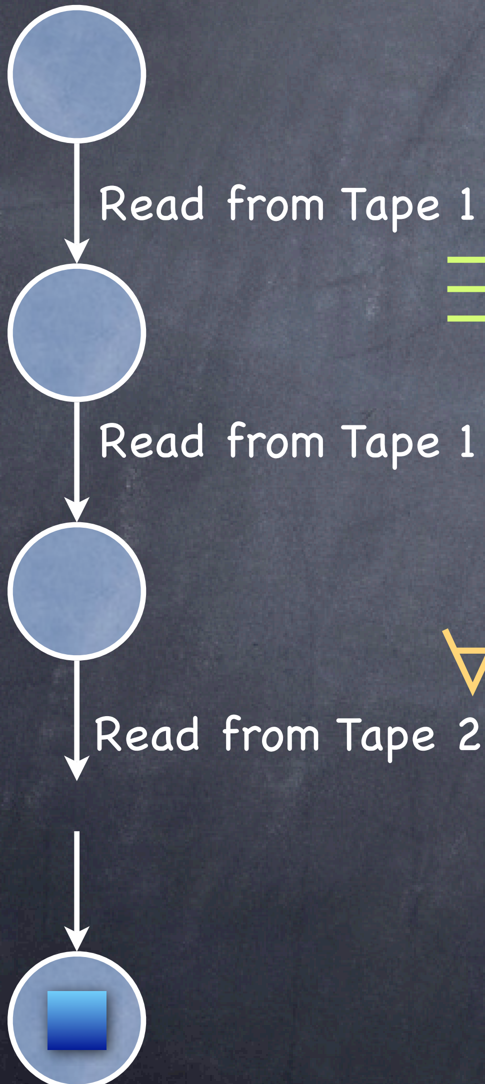
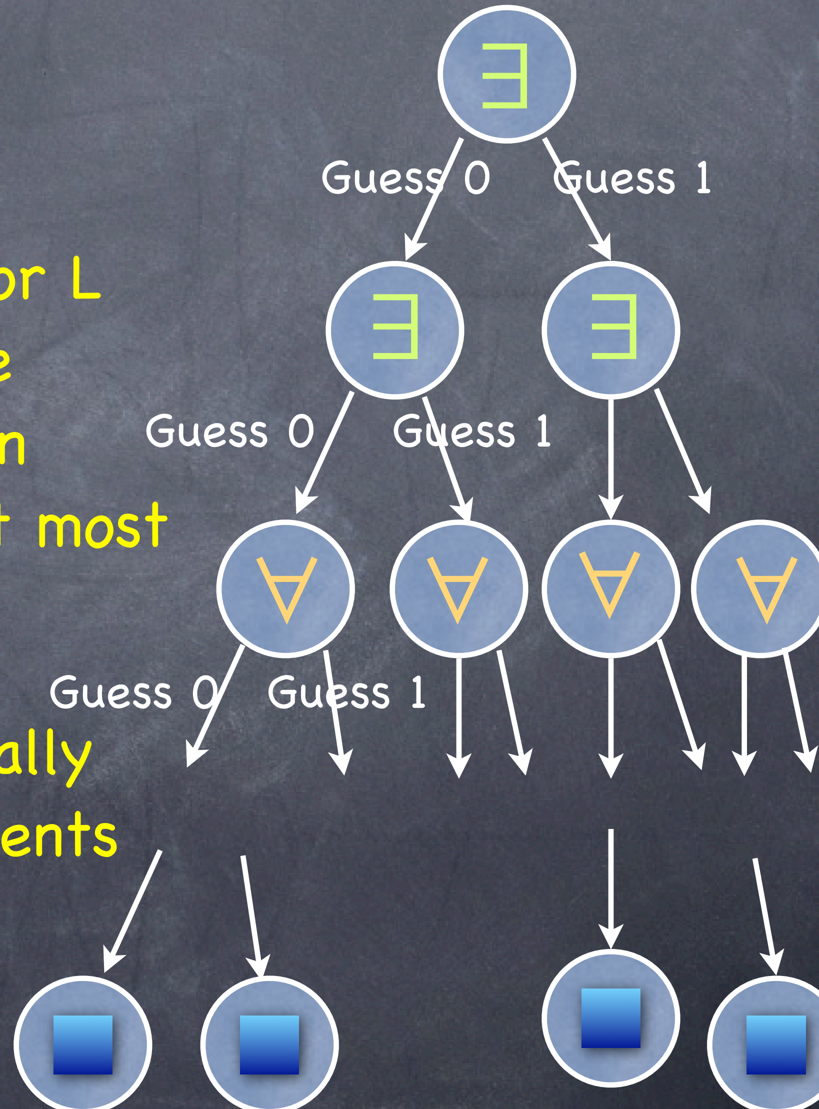


- Given a verifier for L using k certificate tapes, can build an ATM for L with at most k alternations
- Non-deterministically guesses tape contents and runs verifier



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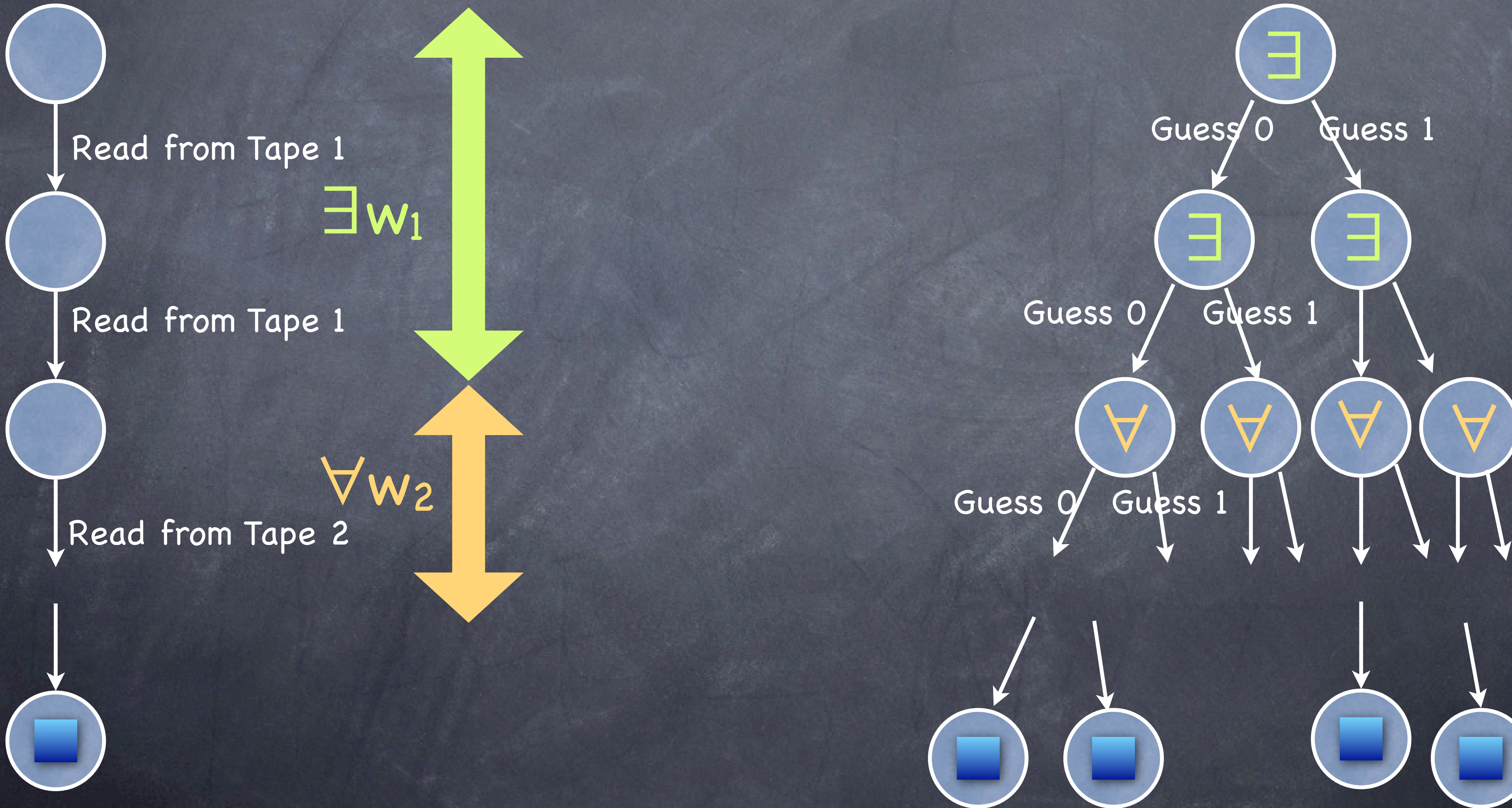

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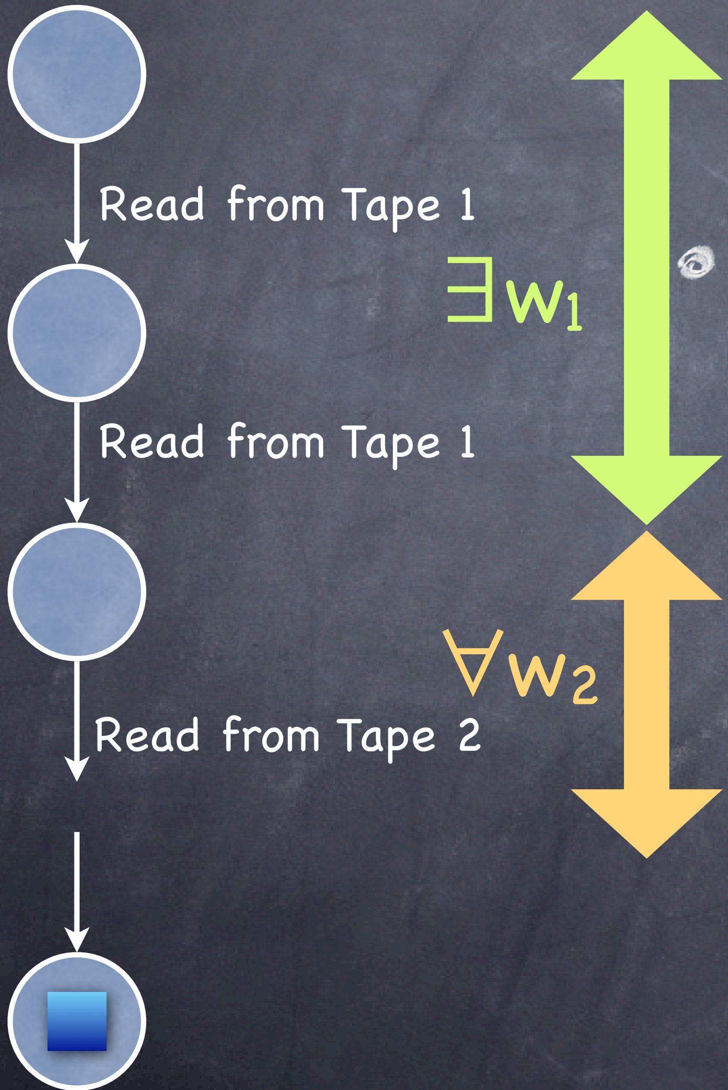

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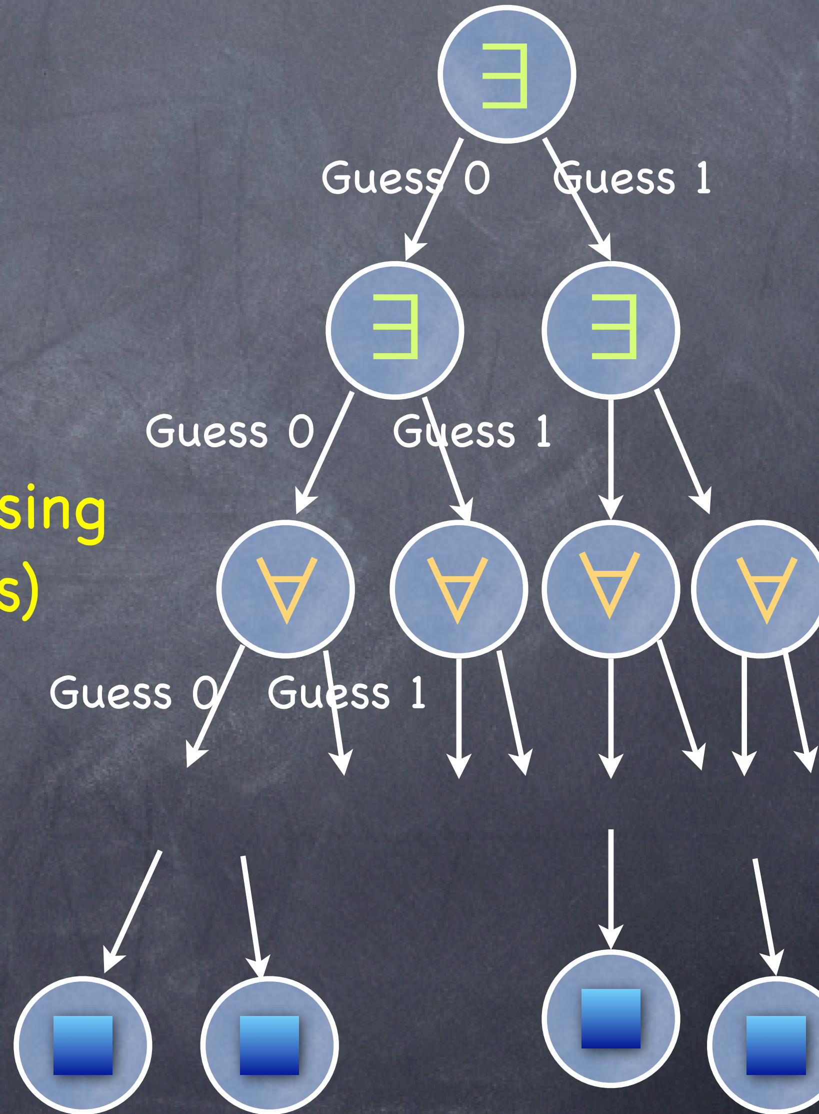

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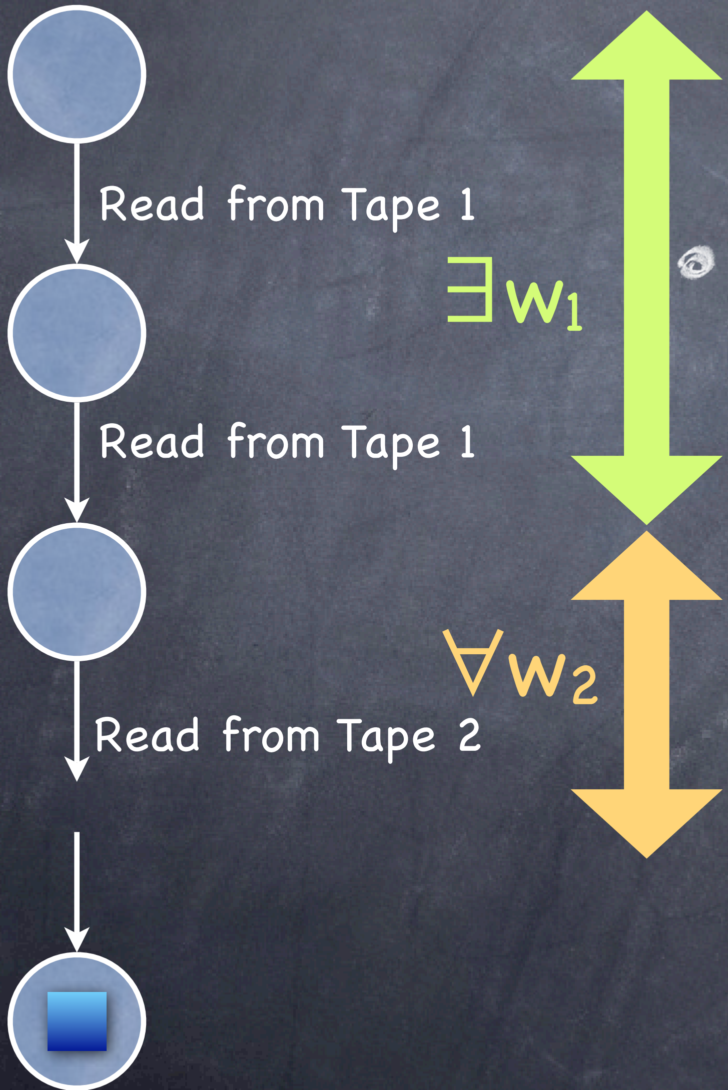

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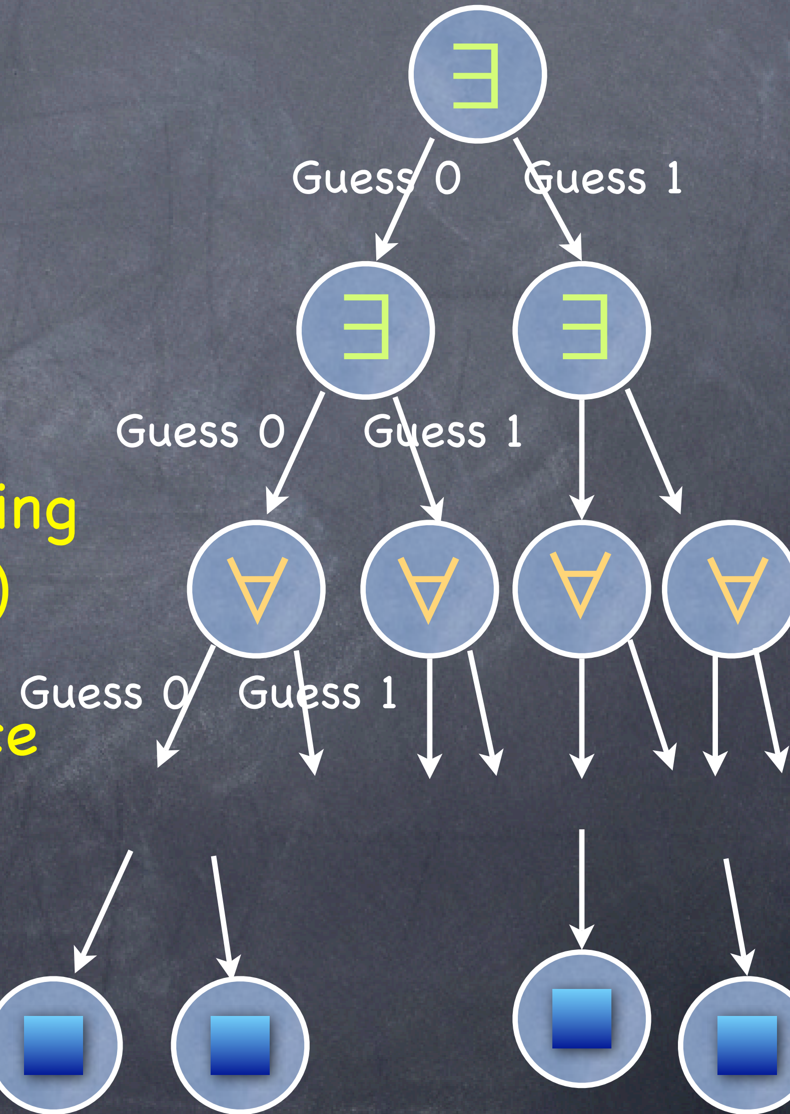

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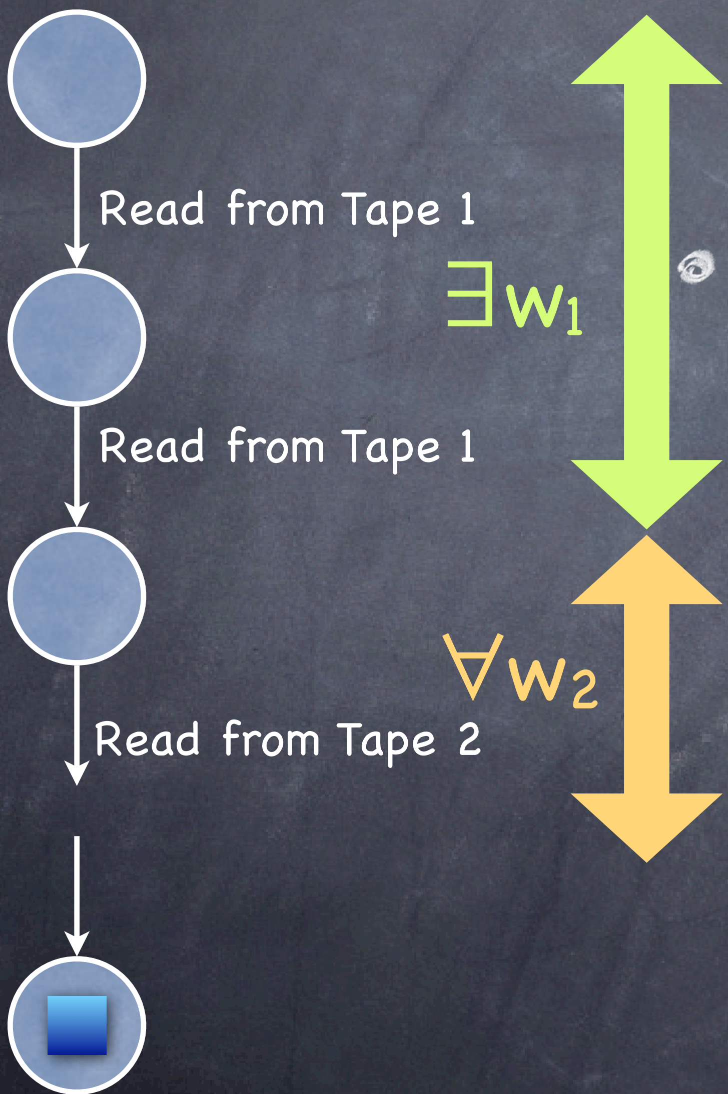

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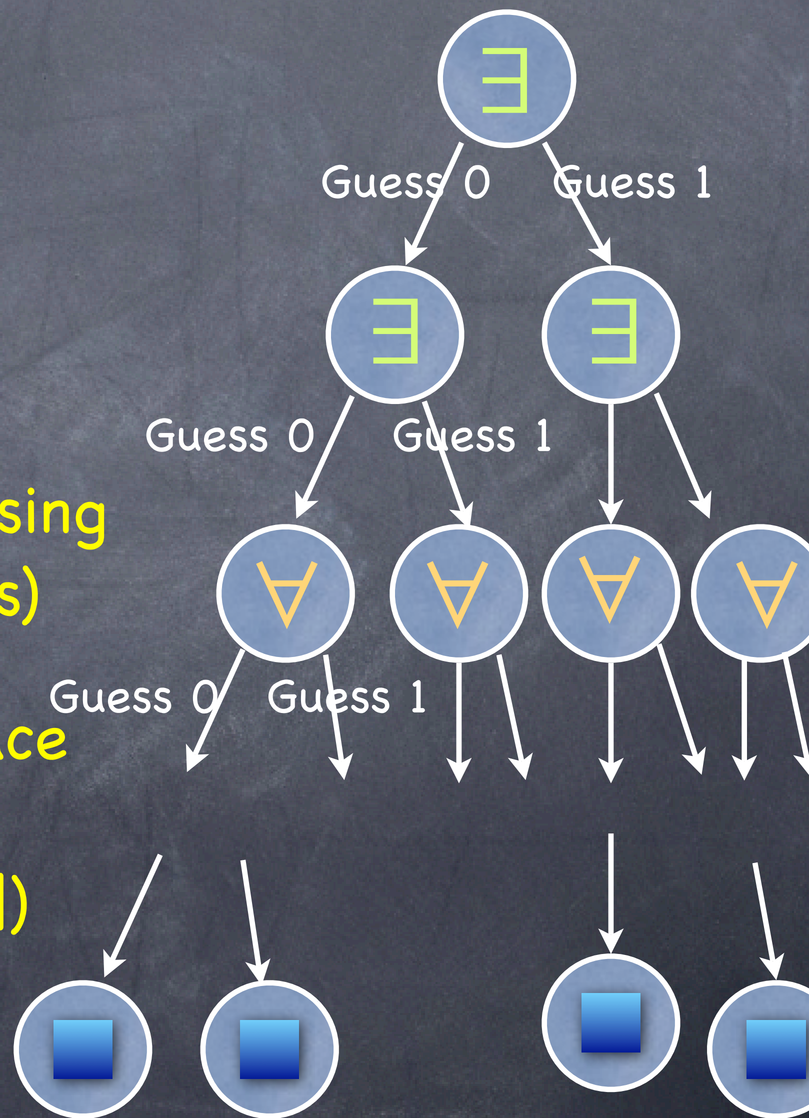
- Same time/space requirements (in terms of $|x|$)
- $|w_i|$ = #choices



Time, Space, Alternations

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Time, Space, Alternations

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  - Time: Maximum number of steps in any thread
  - Space: Maximum space in any configuration reached
  - Alternations: Maximum number of quantifier switches in any thread

ATIME

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ATIME vs. DSPACE

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- AP = PSPACE

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ASPACE vs. DTIME

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ASPACE vs. DTIME

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    - Instead of connectivity, can recursively label all accepting nodes (2 lookups per node: in poly(S) time). So ASPACE(S) ⊆ DTIME($2^{O(S)}$)

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    - Instead of connectivity, can recursively label all accepting nodes (2 lookups per node: in poly(S) time). So ASPACE(S) $\subseteq$ DTIME($2^{O(S)}$)
  - To show DTIME($2^{O(S)}$) $\subseteq$ ASPACE(S)

$\mathrm{DTIME}(2^{O(S)}) \subseteq \mathrm{ASPACE}(S)$

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- To decide, is configuration after t steps accepting

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    - Naive recursion: Extra $O(S)$ space at each level for $2^{O(S)}$ levels!

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    Gets the AND check for free. No need to use a stack.
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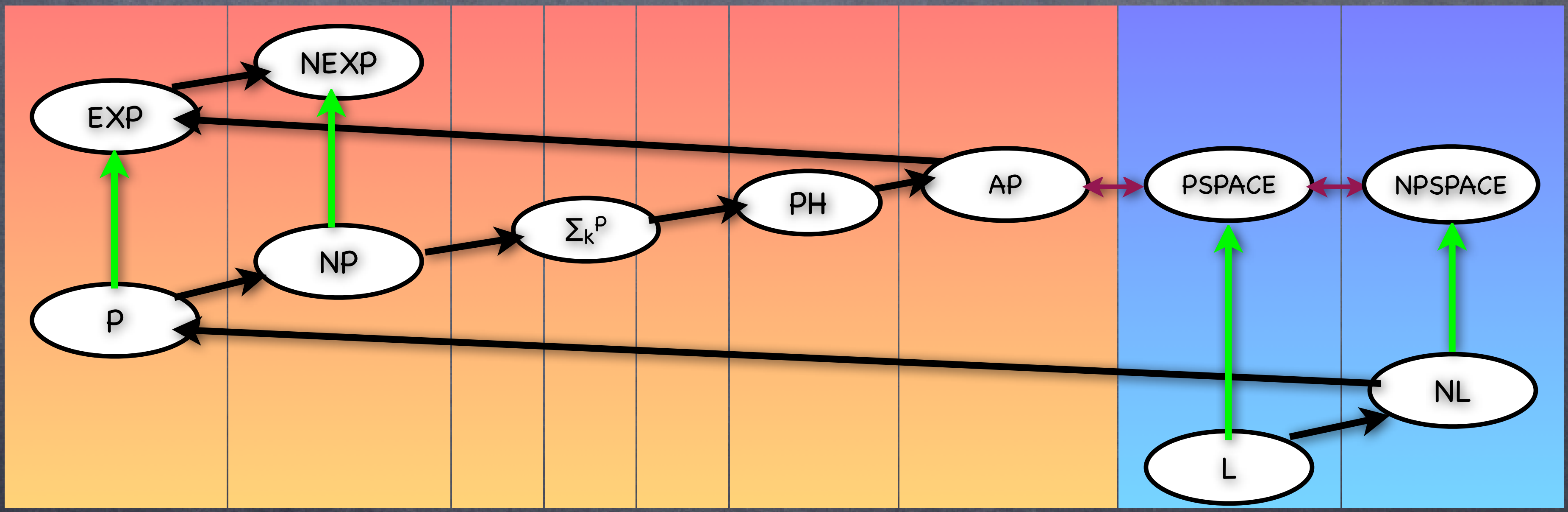
Zoo



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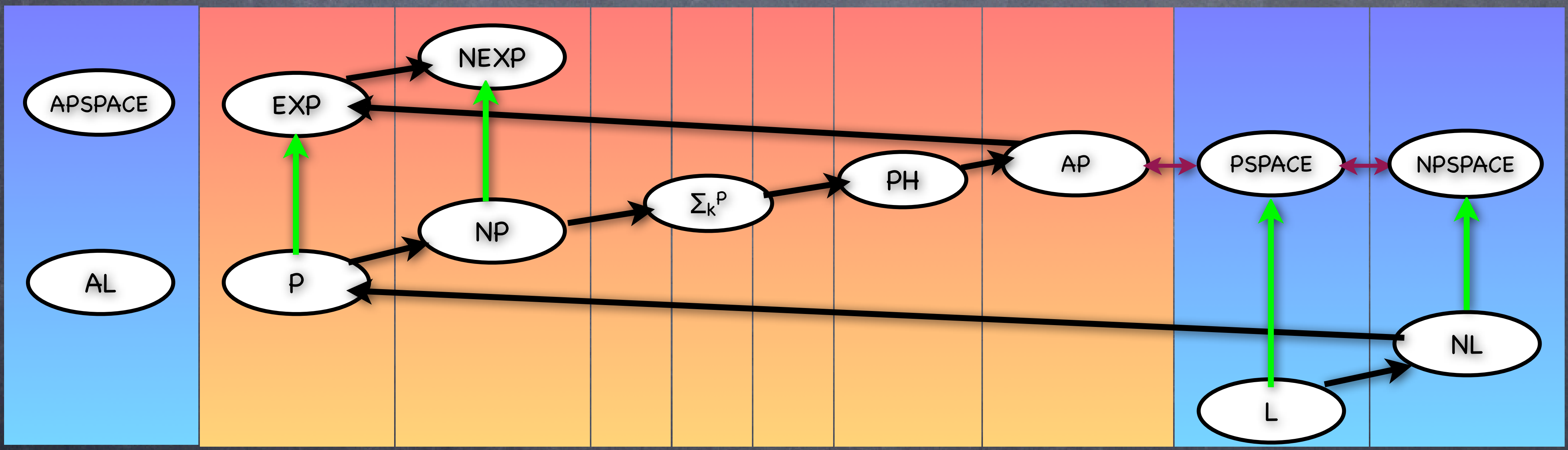

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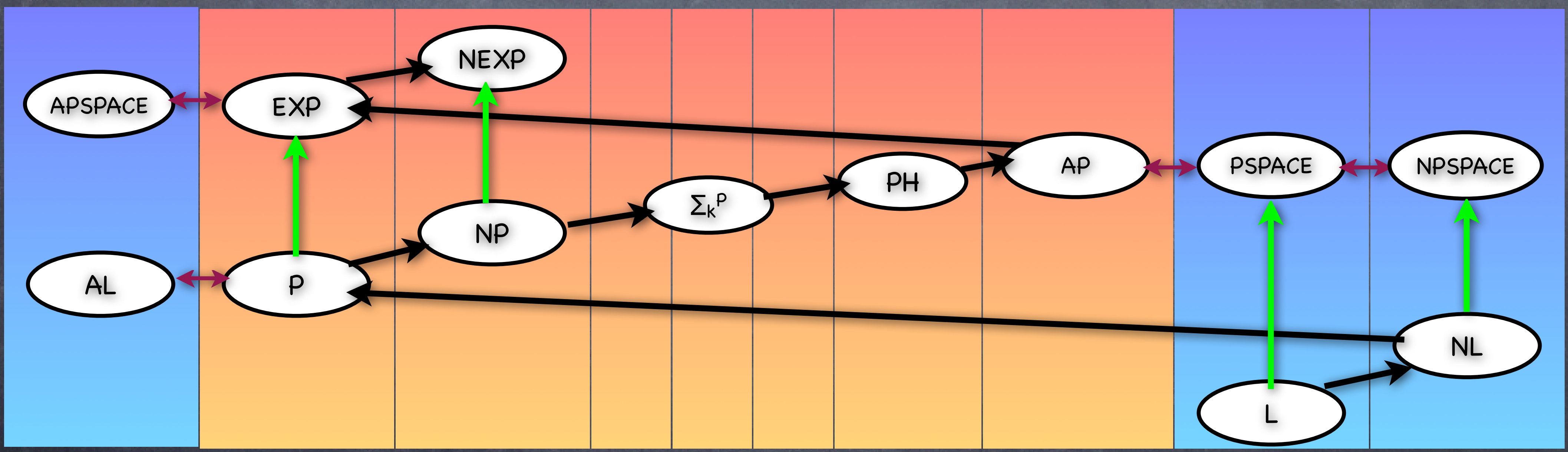

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    - NTIME(n) $\subseteq$ DTISP($n^{1+\epsilon}$,n^{δ}) $\Rightarrow$ NTIME(n^{t}) $\subseteq$ NTIME($n^{t(1/2+\epsilon')}$) !

Today

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