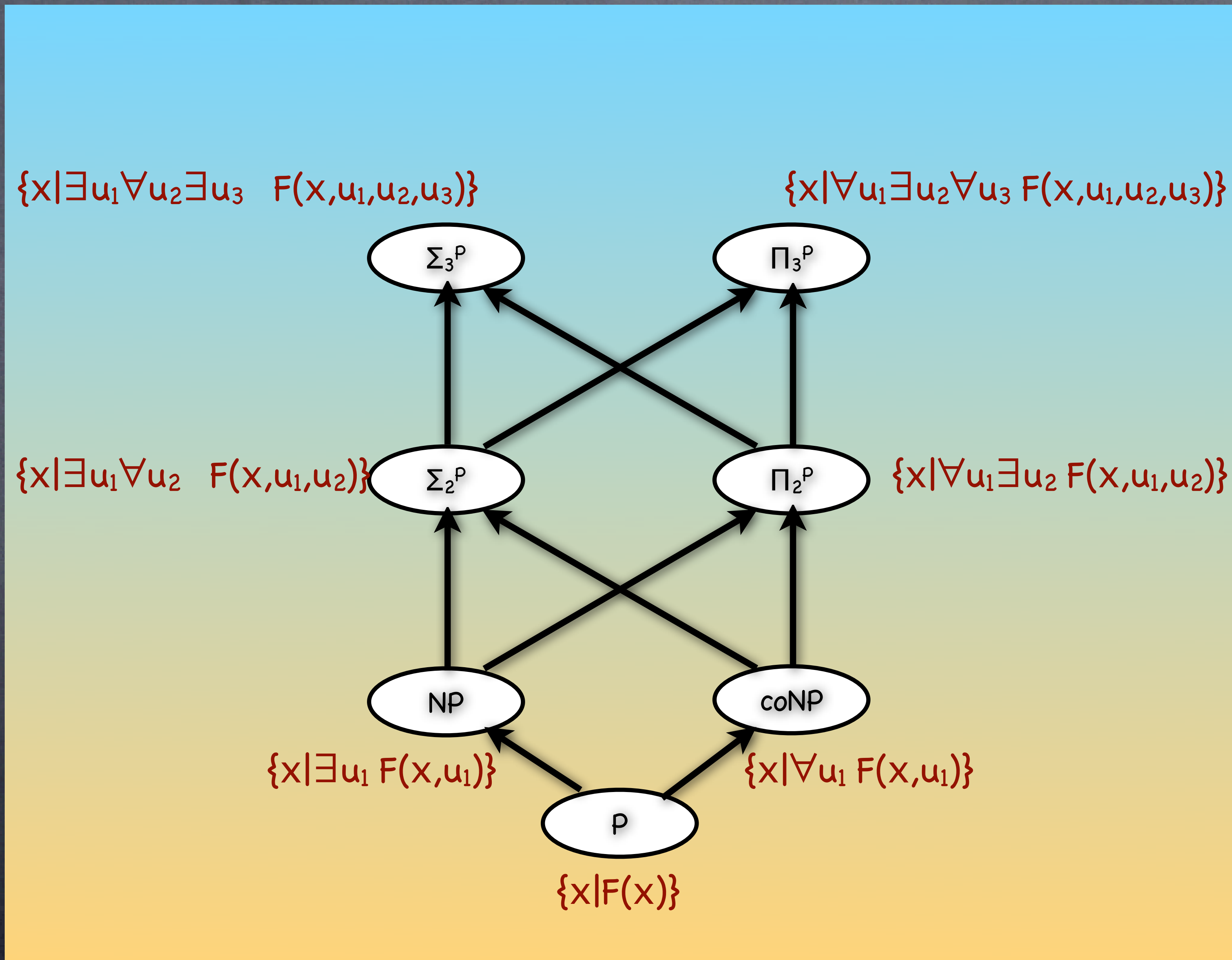
Computational Complexity

Lecture 8

More of the Polynomial Hierarchy

Oracle-based Definition

Recall PH



{x|∃u1∀u2∃u3 F(x,u1,u2,u3)}
{x|∀u1∃u2∀u3 F(x,u1,u2,u3)}
Σ3P
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Σ2P
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{x|∀u1∃u2 F(x,u1,u2)}
NP
coNP
{x|∃u1 F(x,u1)}
{x|∀u1 F(x,u1)}
P
{x|F(x)}

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  - Language oracle: answer is a single bit
    - This is what we consider

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  - $NP \cup co\text{-}NP \subseteq NP^{SAT}$
    - Can we better characterize NP^{SAT}?

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    - But $NP^{L'} \subseteq NP^{\Pi_k} = NP^{\Sigma_k}$
- So $\Sigma_{k+1}^P \subseteq NP^{\Sigma_k}$
- Now to show $NP^{\Sigma_k} \subseteq \Sigma_{k+1}^P$

$$NP^{\Sigma_k} \subseteq \Sigma_{k+1}$$

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- To show $NP^A \subseteq \Sigma_{k+1}{}^P$ if A in $\Sigma_k{}^P$
  - For $B \in NP^A$ poly-time TM M s.t. $B = \{ x |\ \exists w\ M^A(x,w)=1\}$

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    - Then B also in Σ_{k+1}^P

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  - $C=\{(x,w,ans) \mid \exists u_1 \forall u_2 v_1 \exists u_3 v_2 \ldots Q_k u_k Q'_k v_k \quad M^{<ans>}(x,w)=1 \wedge [(ans=1 \wedge F(z,u_1,\ldots)=1)$ or $(ans=0 \wedge F(z,v_1,\ldots)=0)]\ \}$
  - Changes for 2 queries: $z=Z(x,w) \rightarrow (z^{(1)},z^{(2)}) = Z(x,w,ans)$, $u_i \rightarrow u_i^{(1)},u_i^{(2)}$, $v_i \rightarrow v_i^{(1)},v_i^{(2)}$, and use conjunction of two checks (for j=1 and j=2) of the form $[\ (ans^{(j)}=1 \wedge F(z^{(j)},u_1^{(j)},\ldots)=1)$ or $(ans^{(j)}=0 \wedge F(z^{(j)},v_1^{(j)},\ldots)=0)\ ]$

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- **$\Sigma_{k+1}{}^P$ = NP^{Σ_k}** (with $\Sigma_0{}^P = P$)

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- $\Sigma_{k+1}^P = NP^{\Sigma_k}$ (with $\Sigma_0^P = P$)
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Oracle Version

- $\Sigma_{k+1}{}^{P} = NP^{\Sigma_k}$ (with $\Sigma_0{}^{P} = P$)
- $\Pi_{k+1}{}^{P} = \text{co-}NP^{\Pi_k}$ (with $\Pi_0{}^{P} = P$)
  - $\Pi_{k+1}{}^{P} = \text{co-}(NP^{\Sigma_k}) = \text{co-}NP^{\Sigma_k} = \text{co-}NP^{\Pi_k}$

Δ_k^P

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  - $P^{\Sigma_k} \subseteq NP^{\Sigma_k} \cap coNP^{\Sigma_k}$

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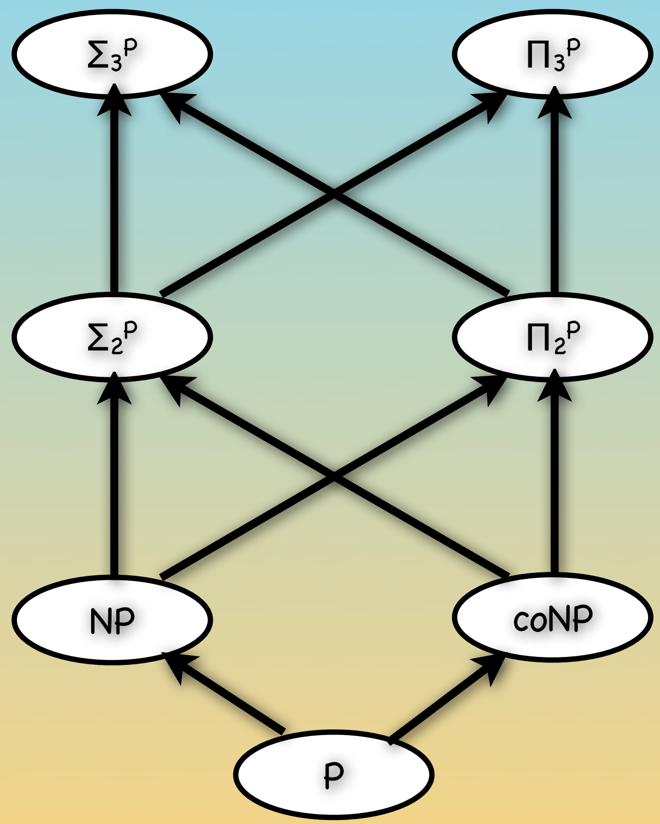

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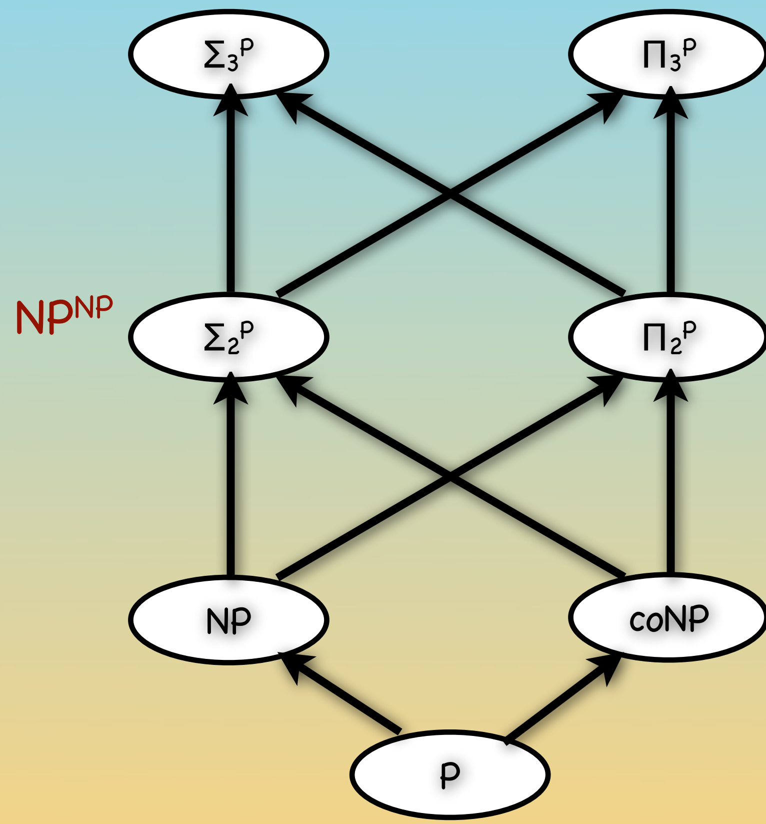

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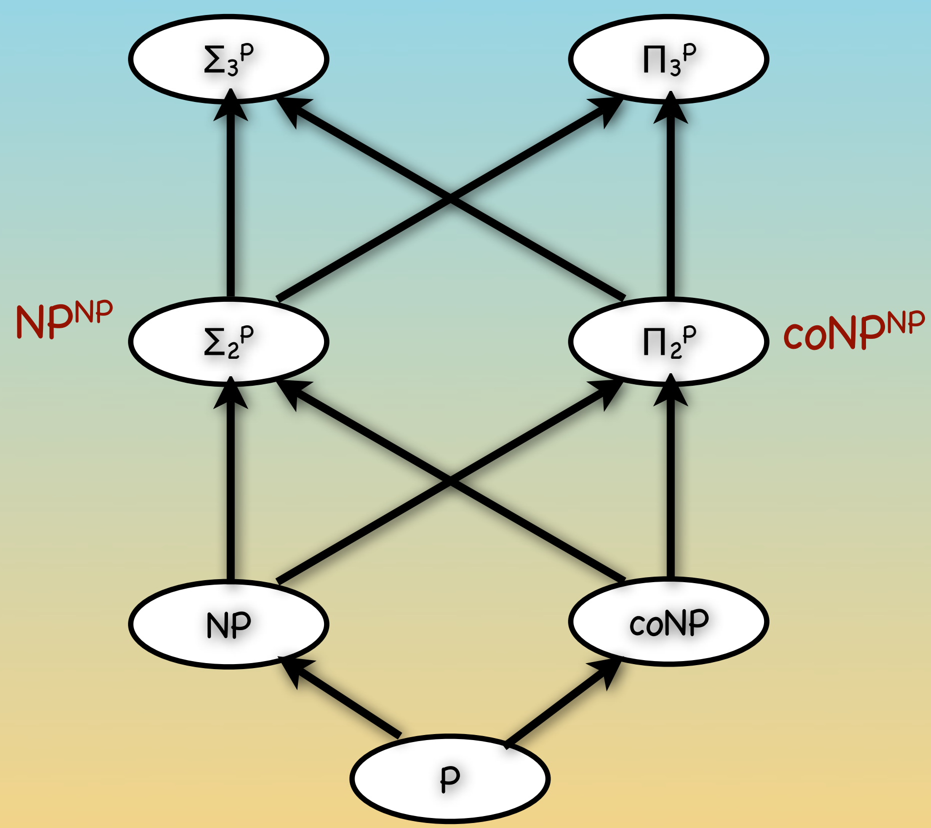

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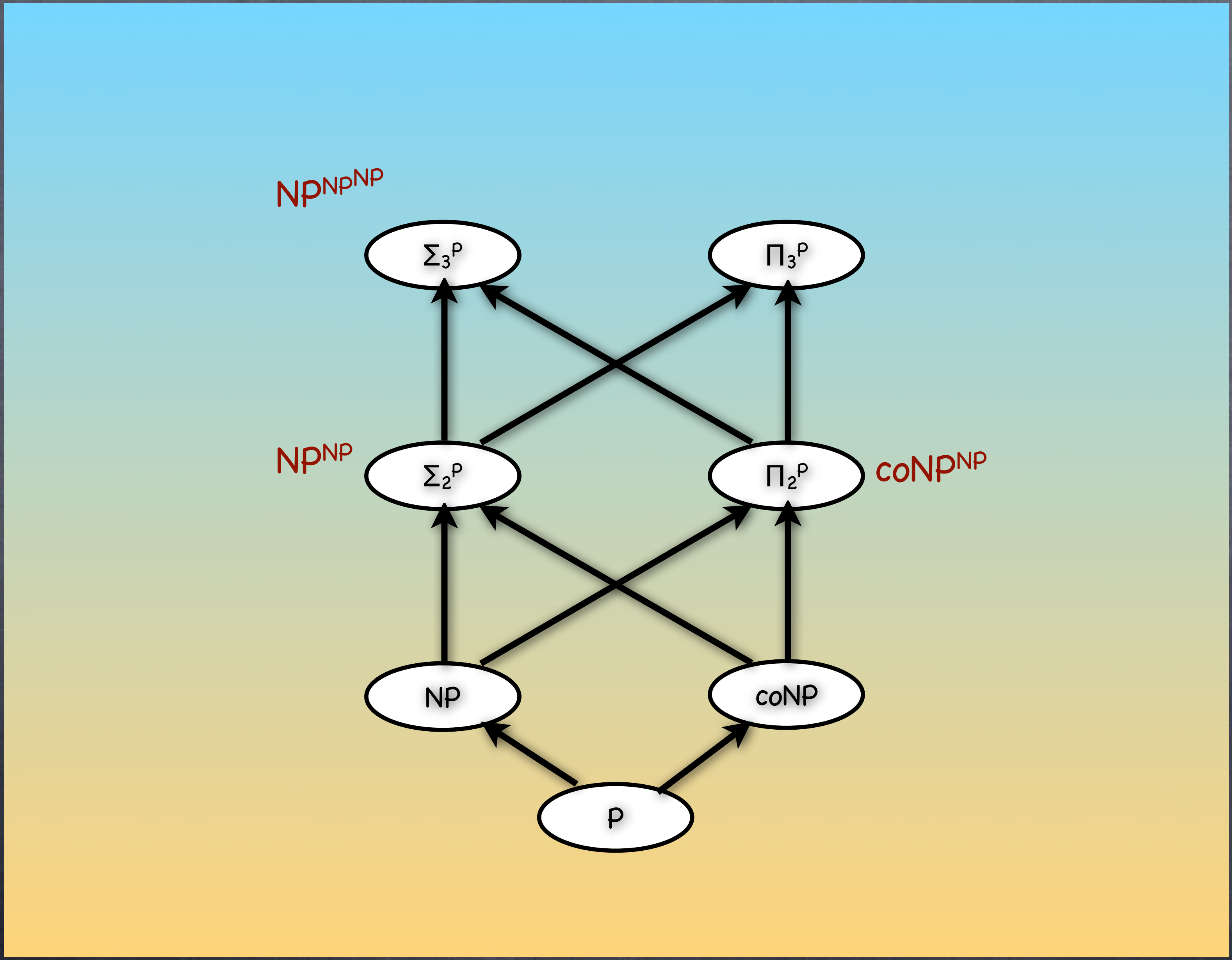

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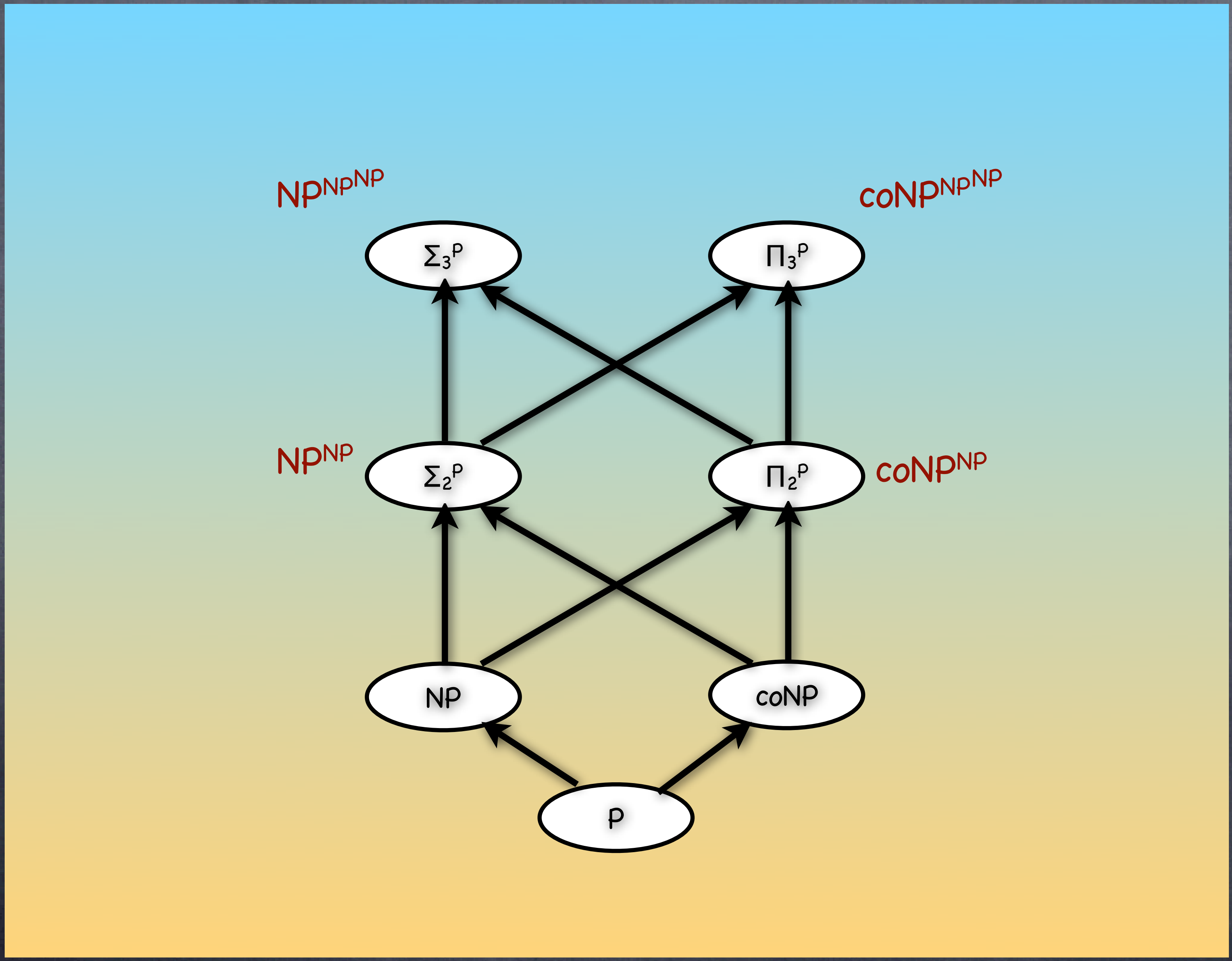

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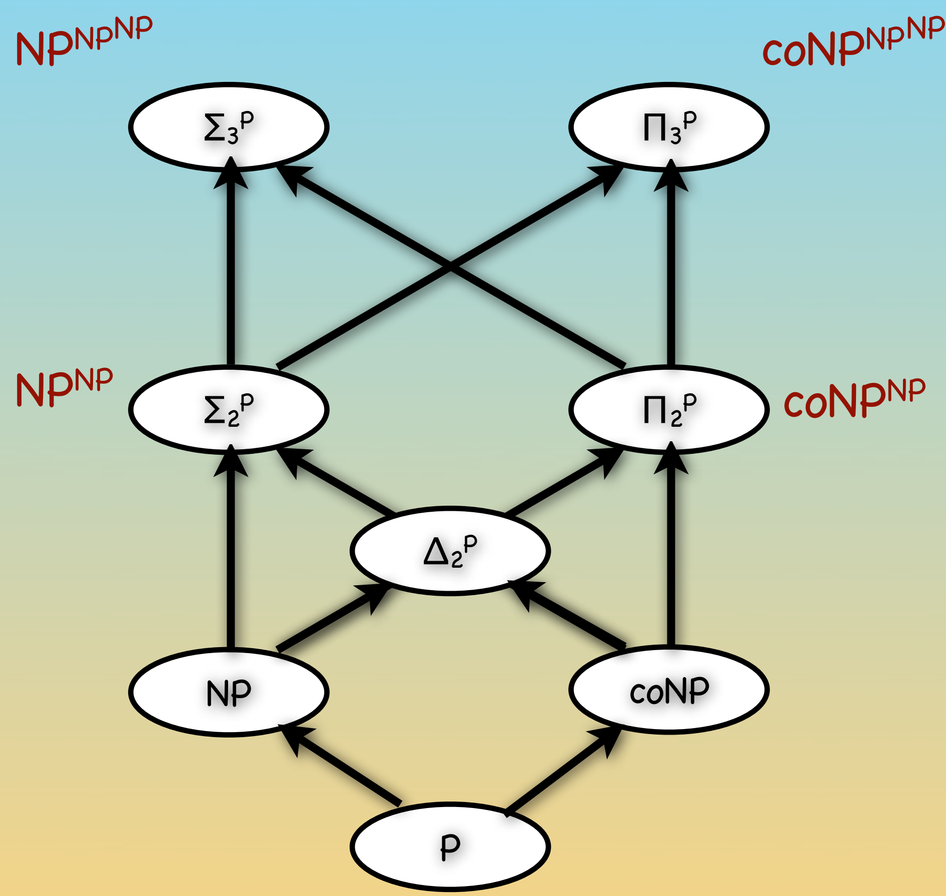

$NP^{NP^{NP}}$
$coNP^{NP^{NP}}$
Σ_3^P
Π_3^P
NP^{NP}
Σ_2^P
Π_2^P
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Δ_2^P
NP
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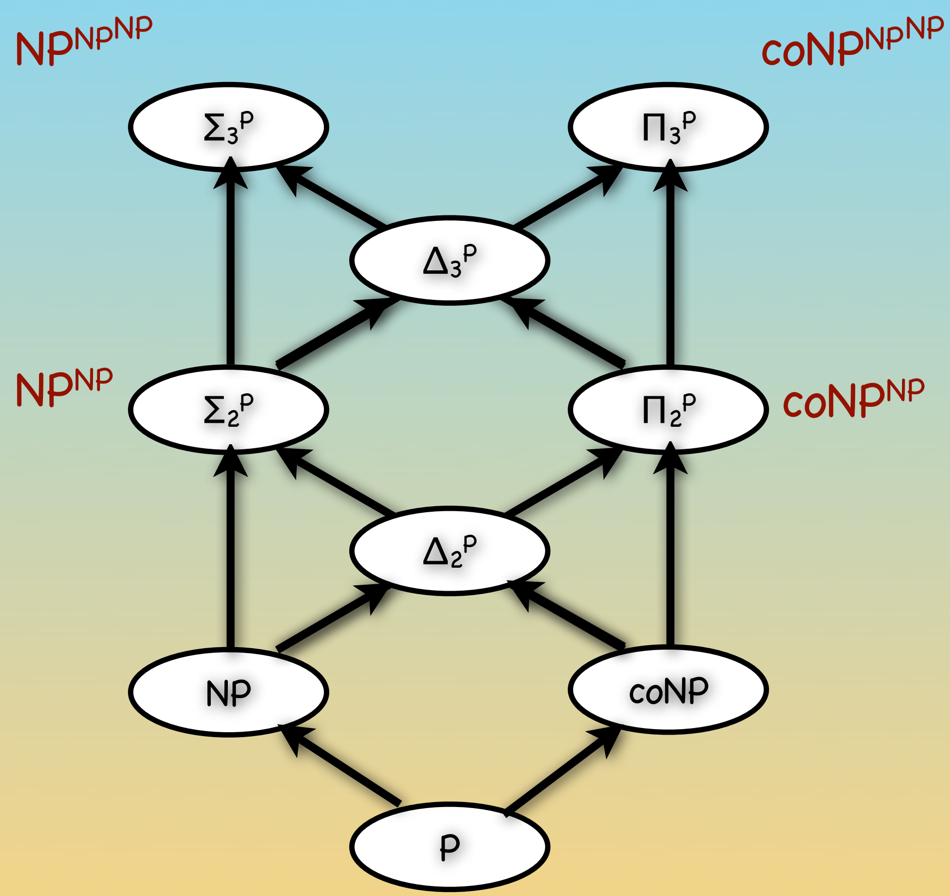

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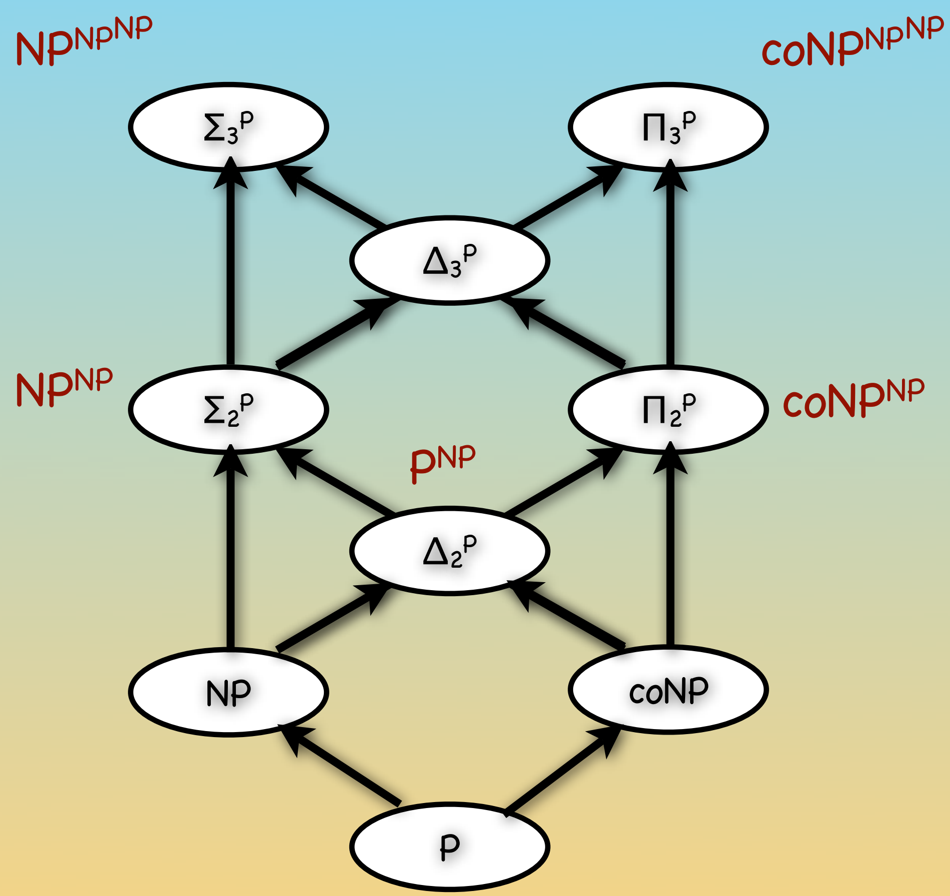

NP^NP^NP
coNP^NP^NP
Σ3P
Π3P
Δ3P
NP^NP
Σ2P
Π2P
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P^NP
Δ2P
NP
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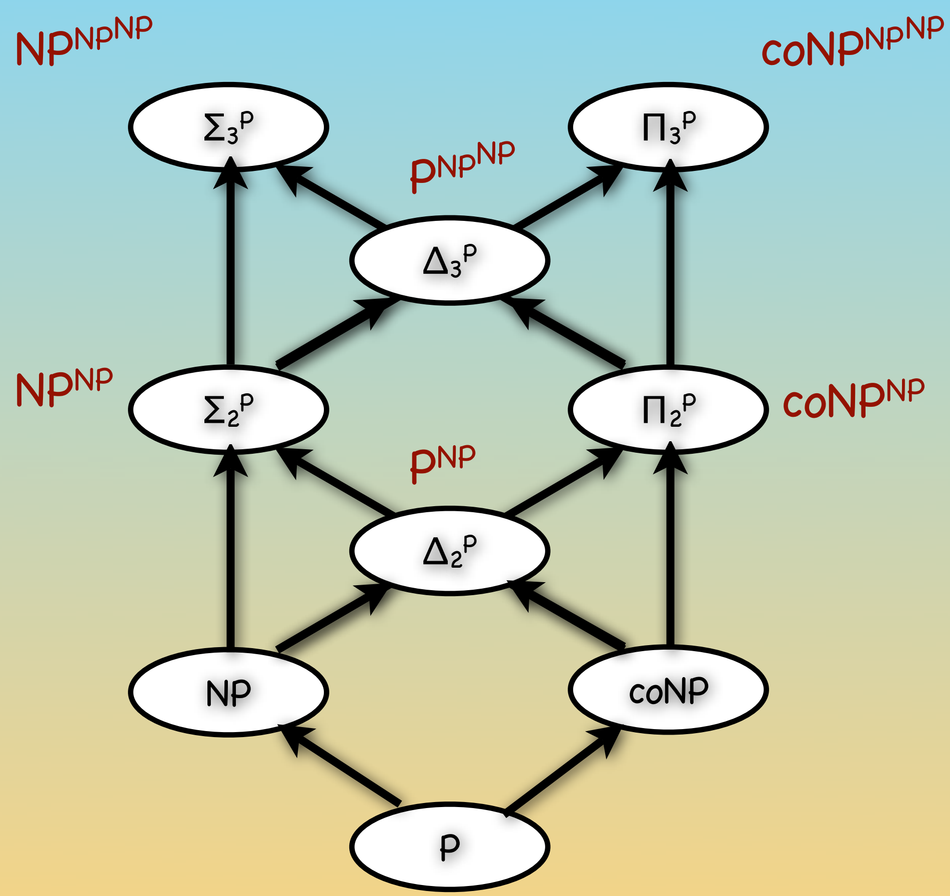

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Today

Today

- Today, more PH

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- Today, more PH
  - Oracle-based definitions (in particular $NP^{NP} = \Sigma_2^P$)

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- Next lecture, more PH

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- Next lecture, more PH
  - Alternating TM-based definitions

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  - Alternating TM-based definitions
  - Time-Space tradeoffs