Computational Complexity

Lecture 5

in which we relate space and time,

and see the essence of PSPACE (TQBF)

SPACE and TIME

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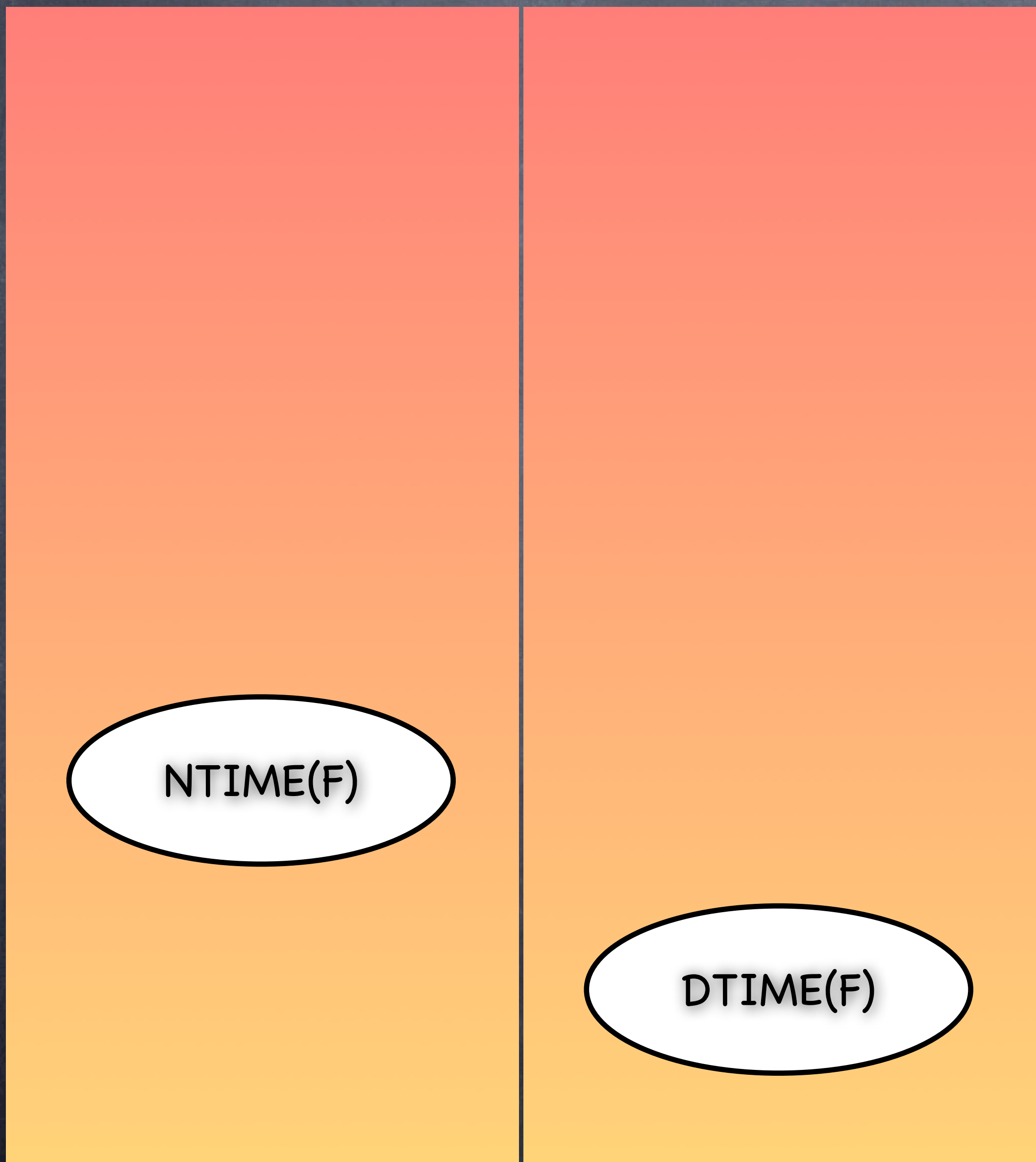

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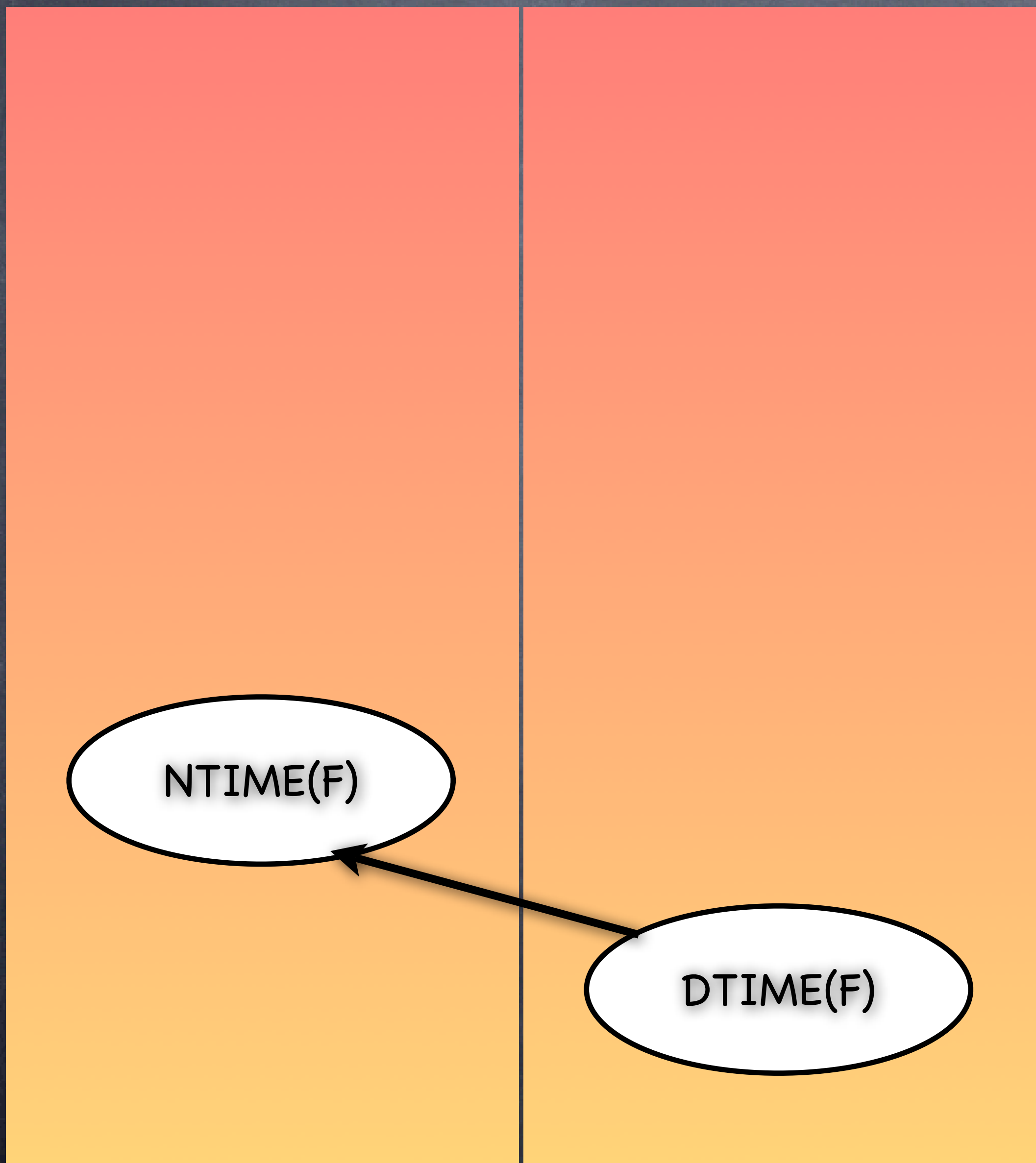

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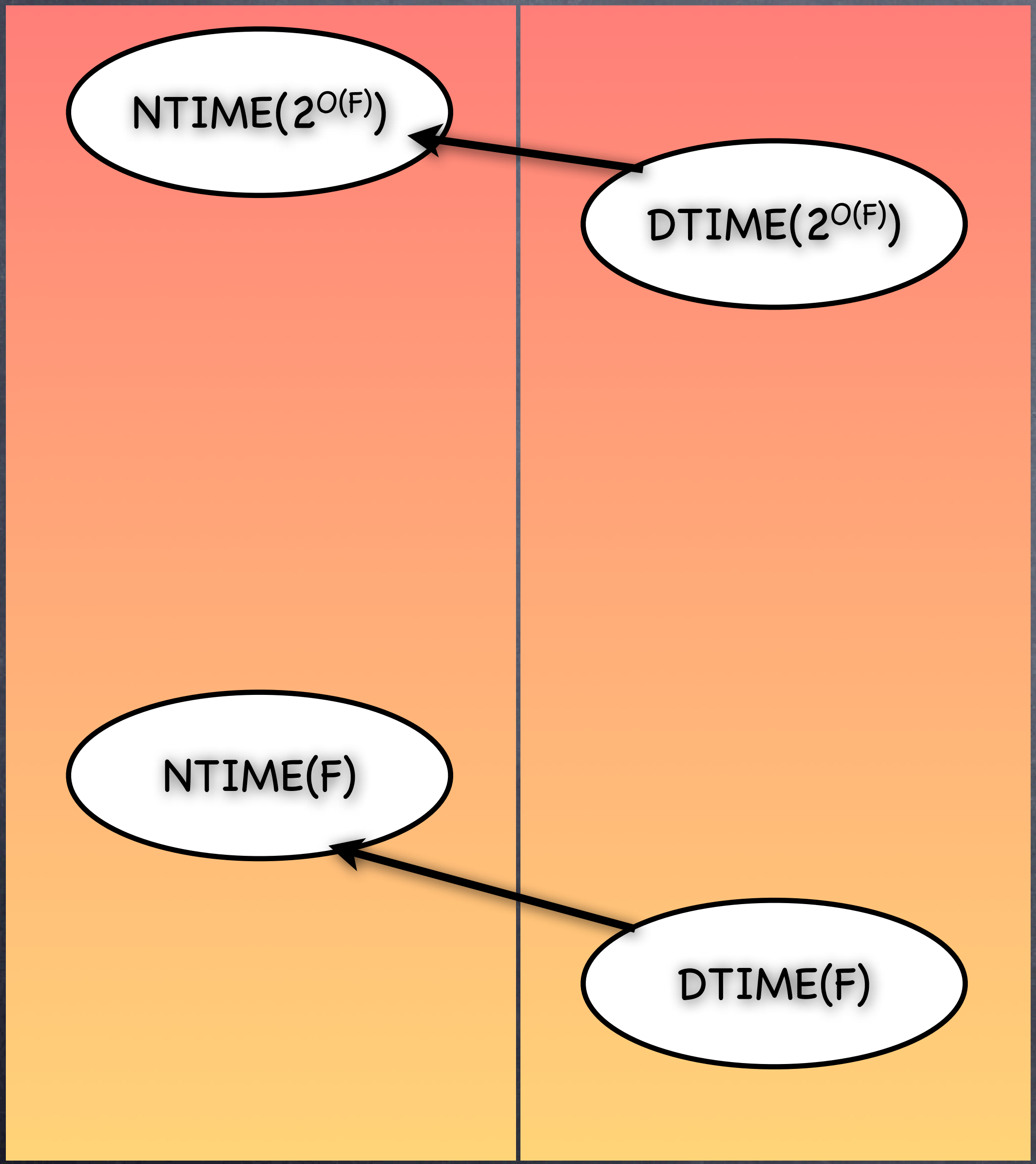

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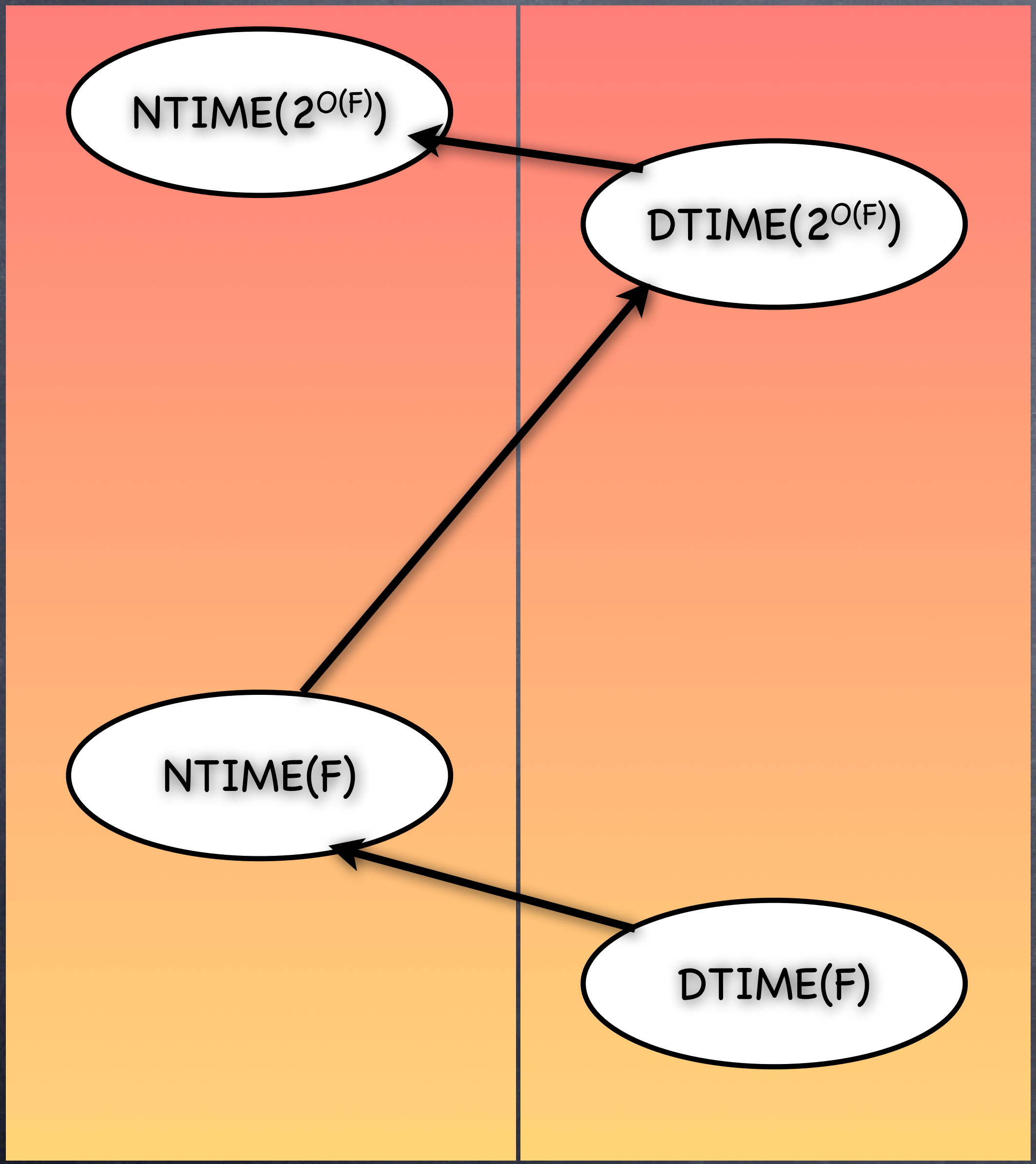

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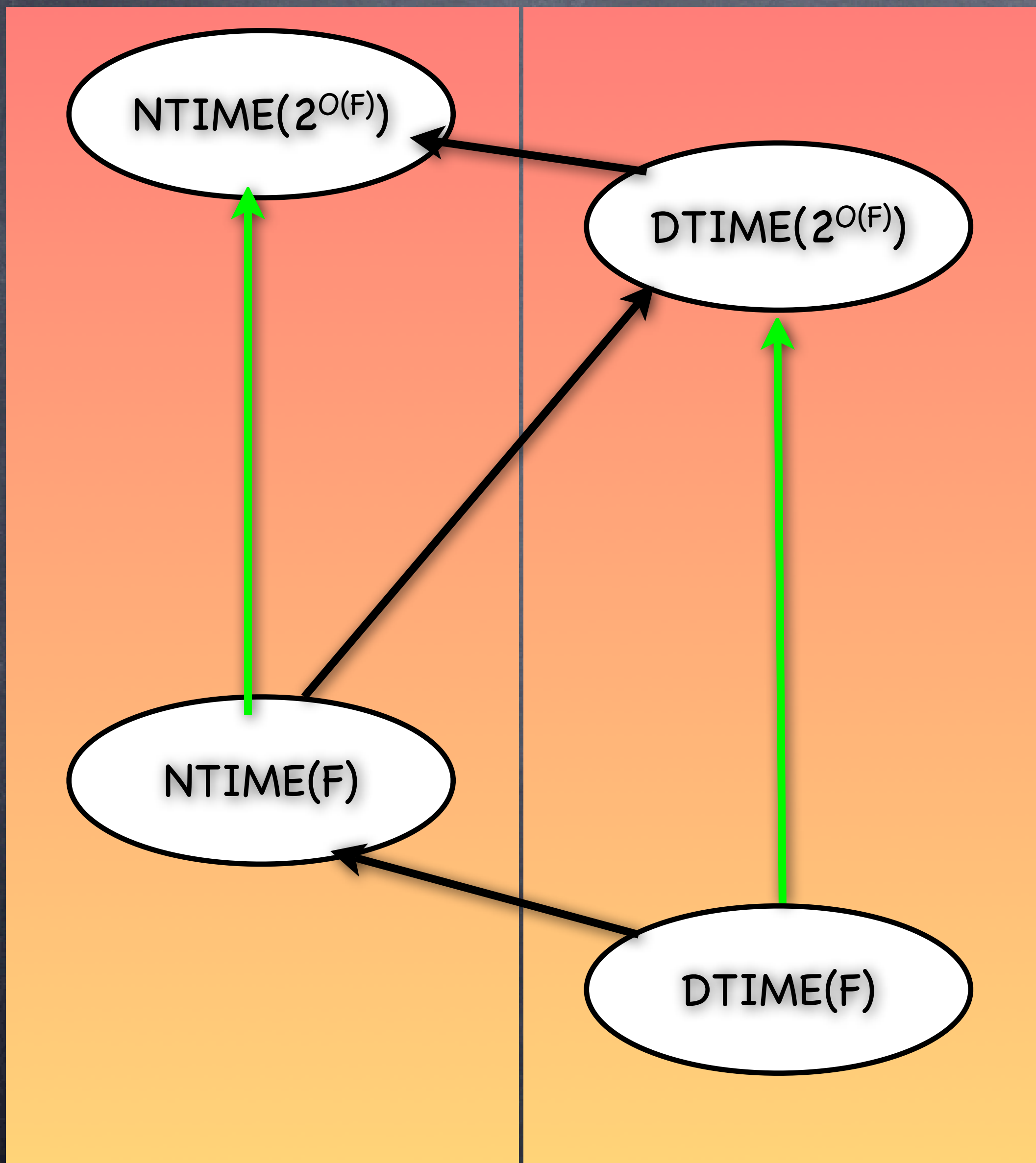

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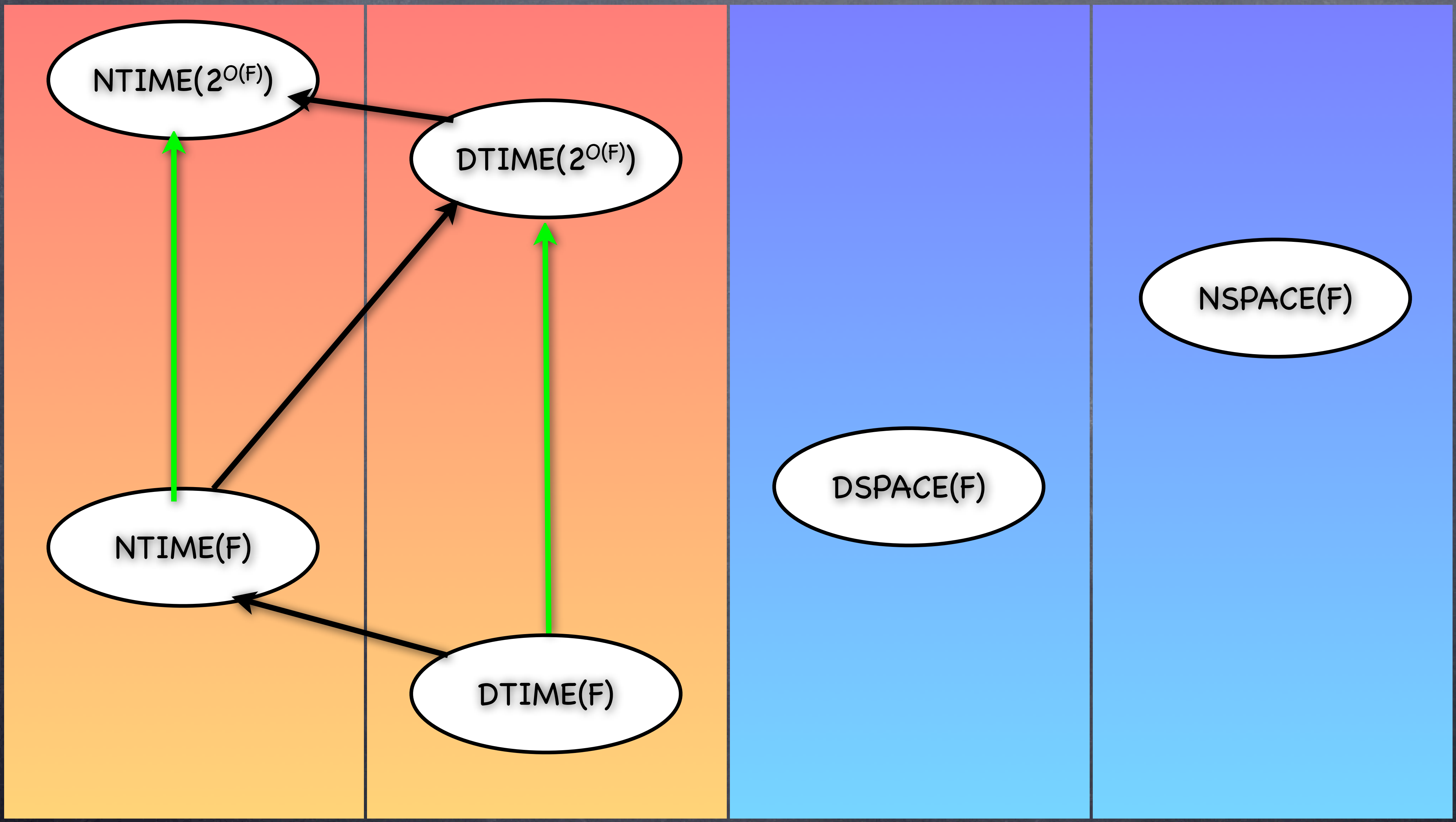

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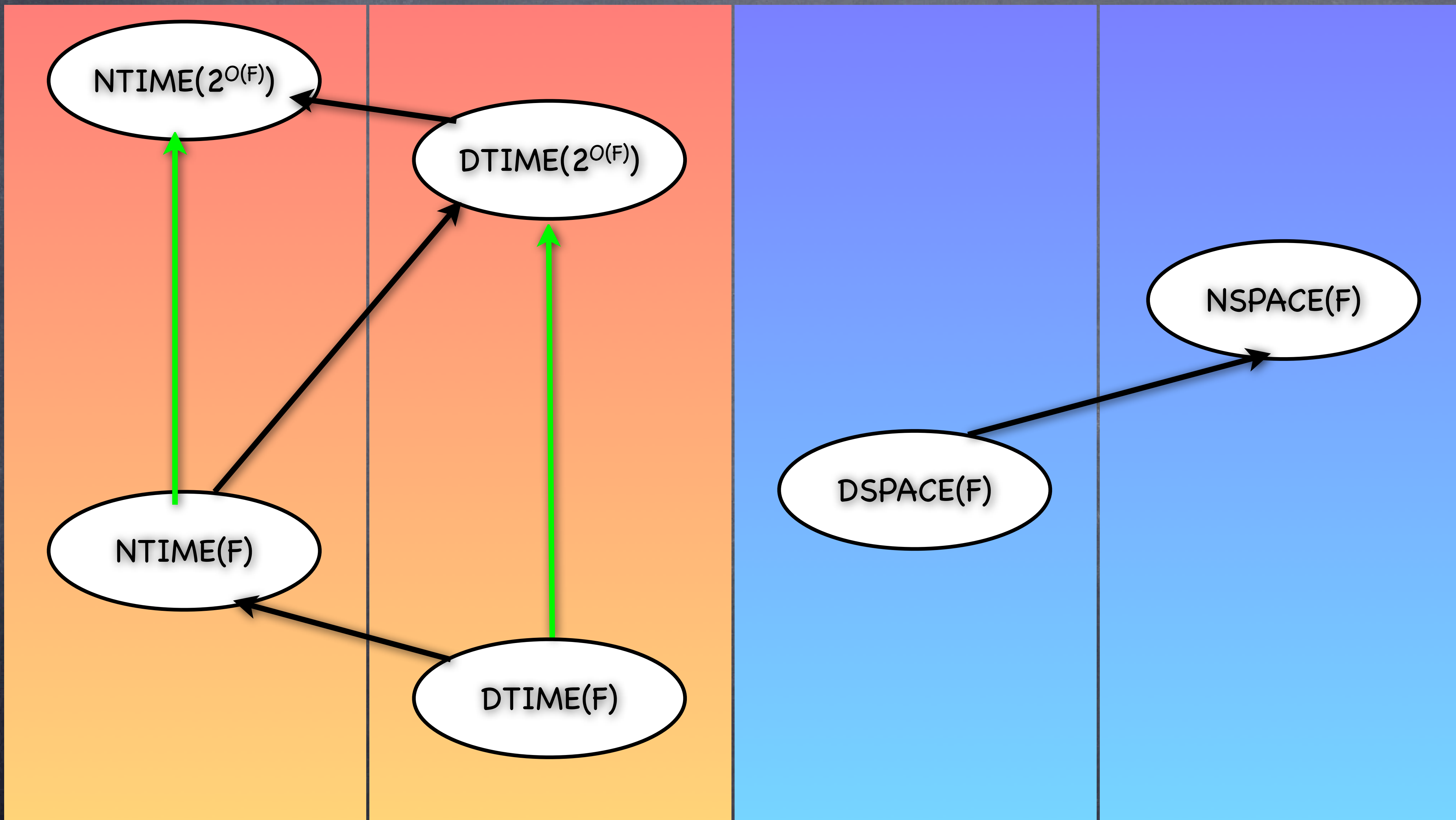

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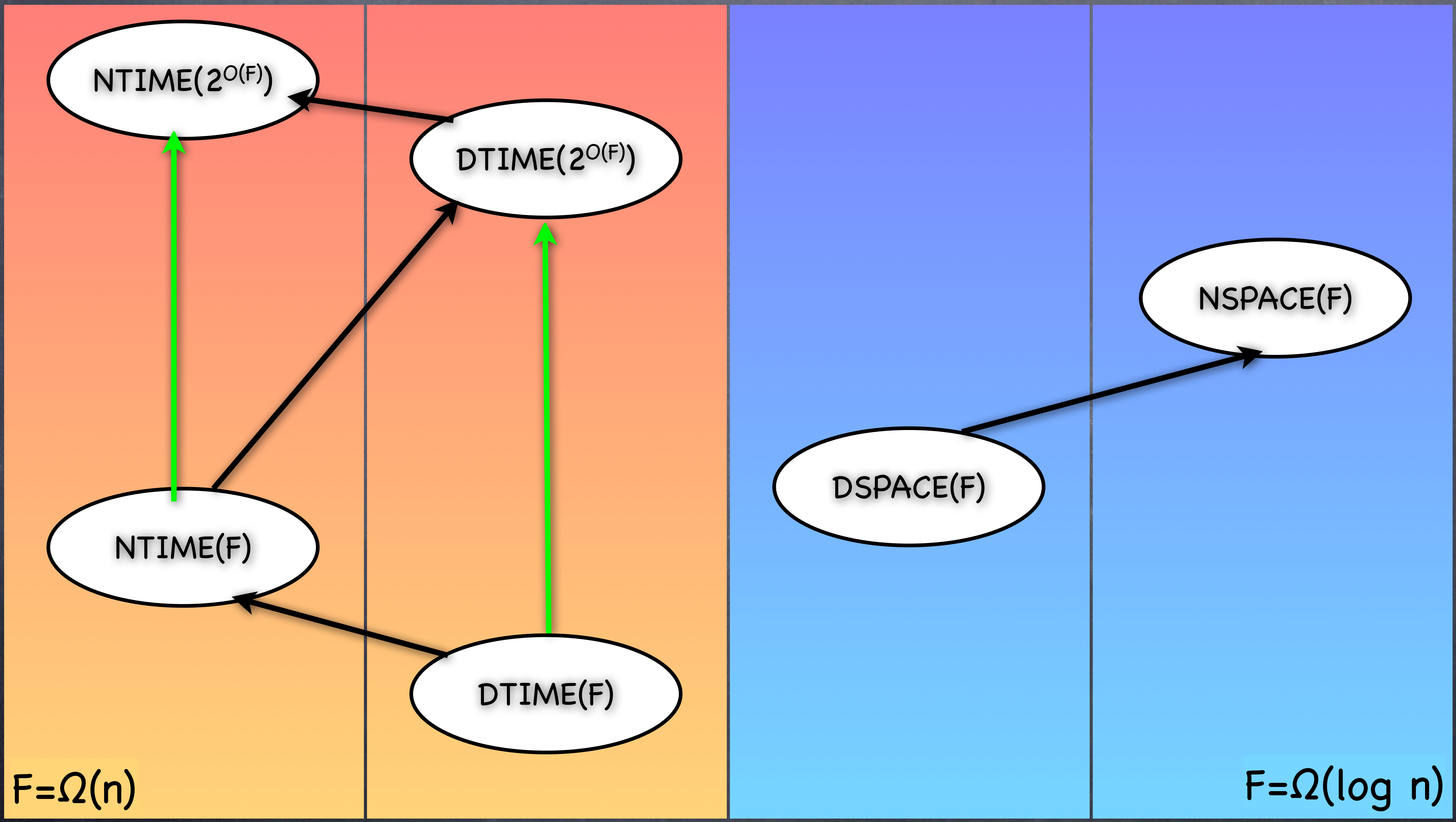

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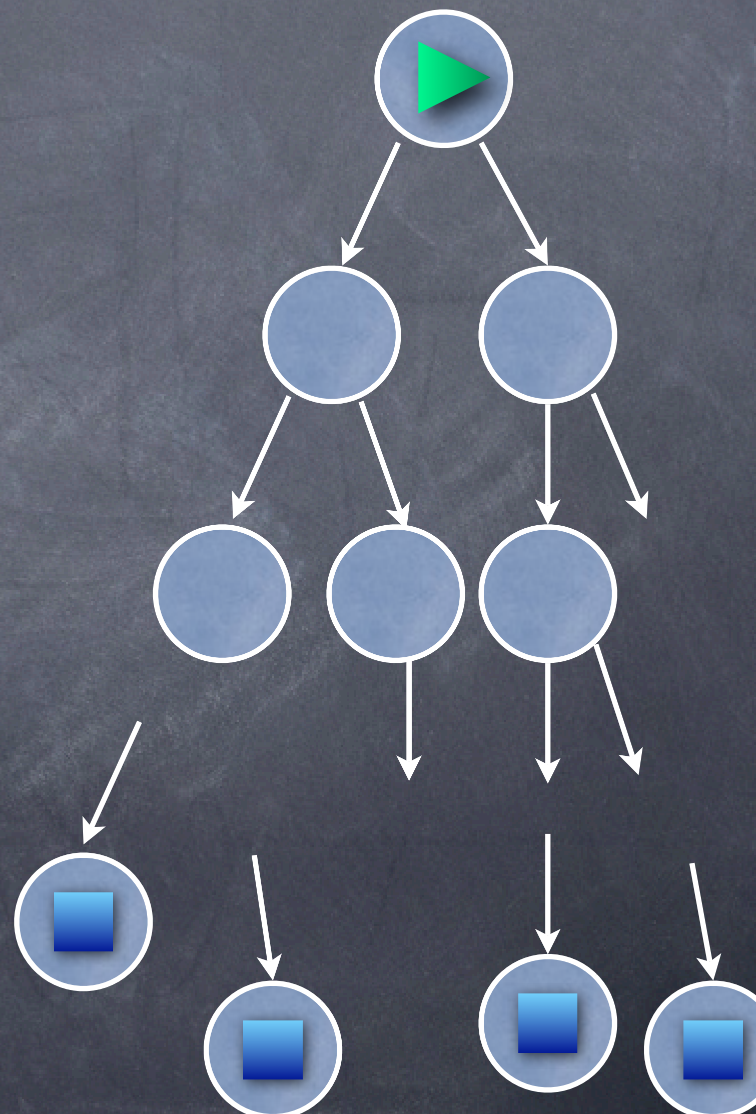

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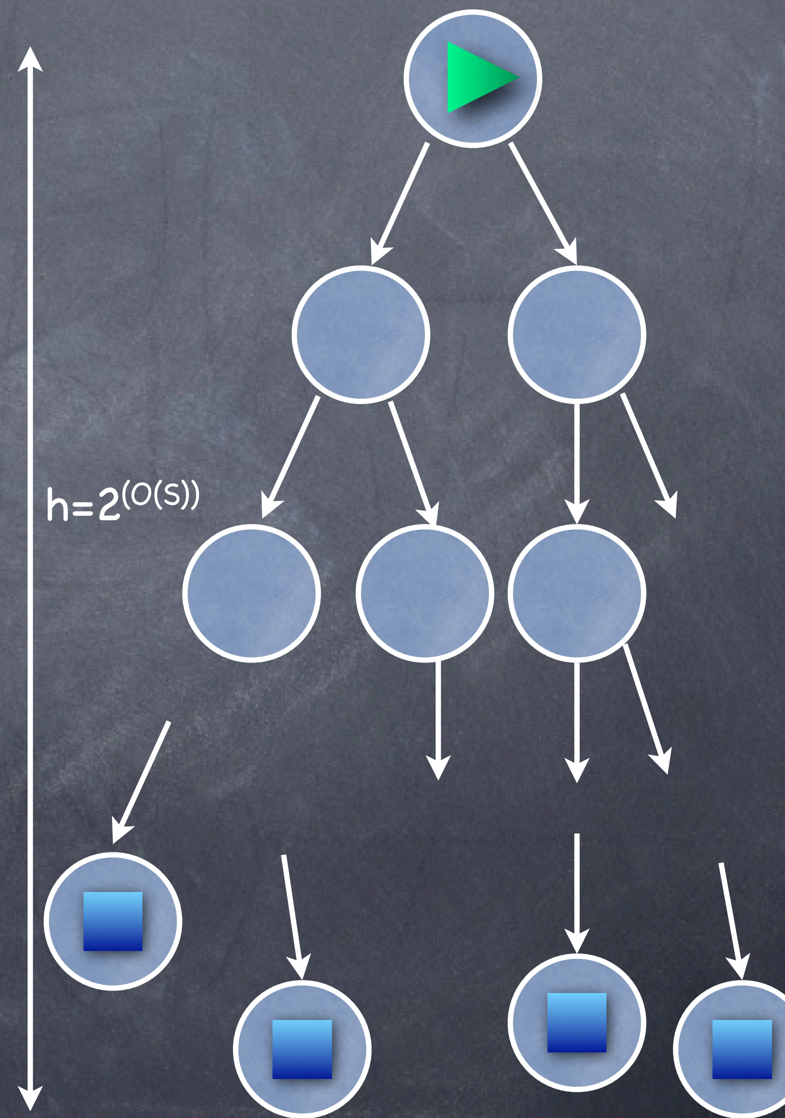

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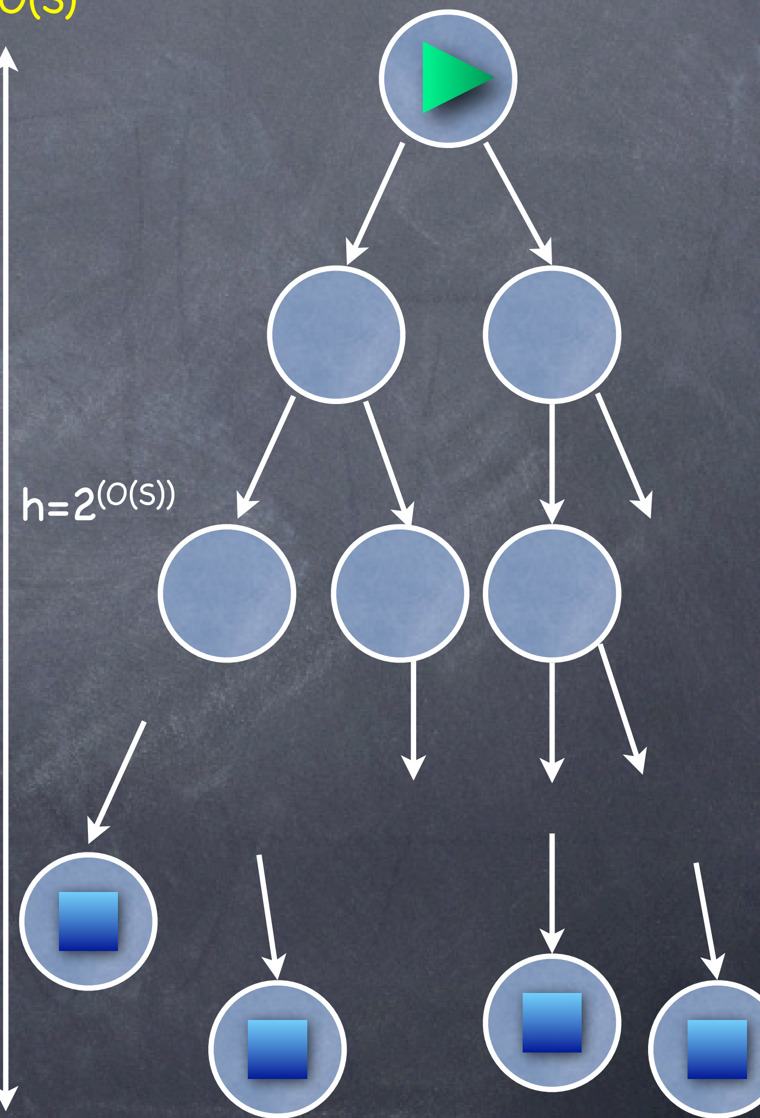

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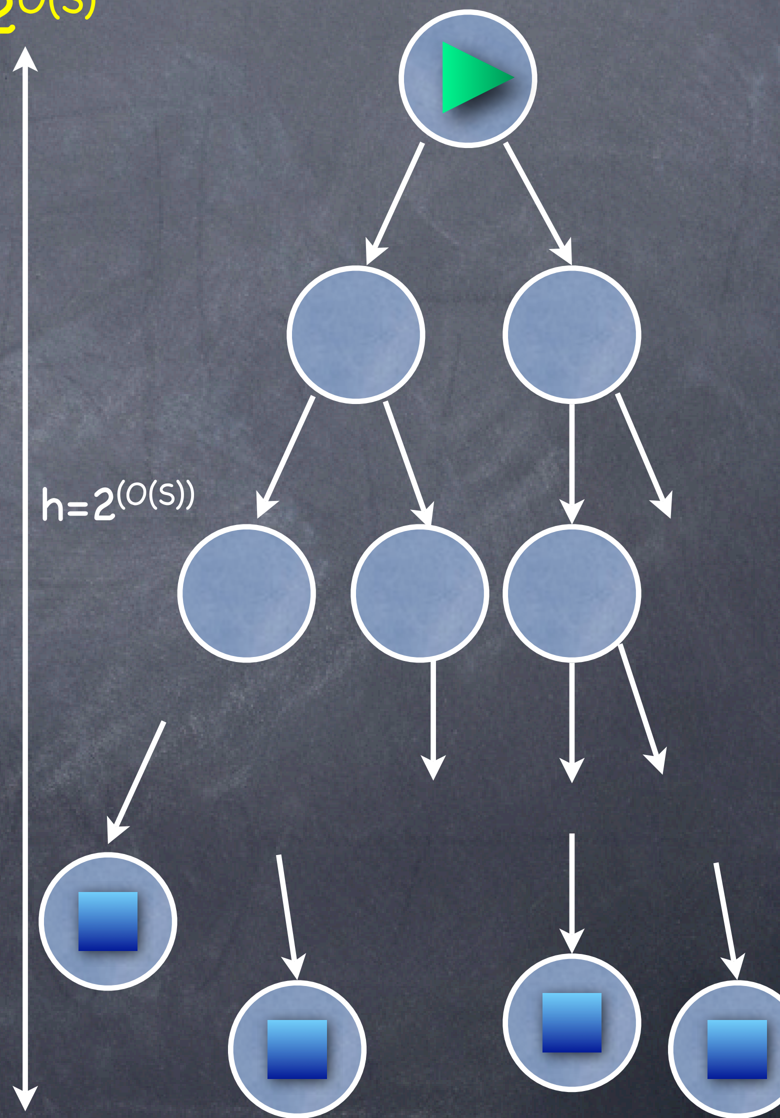

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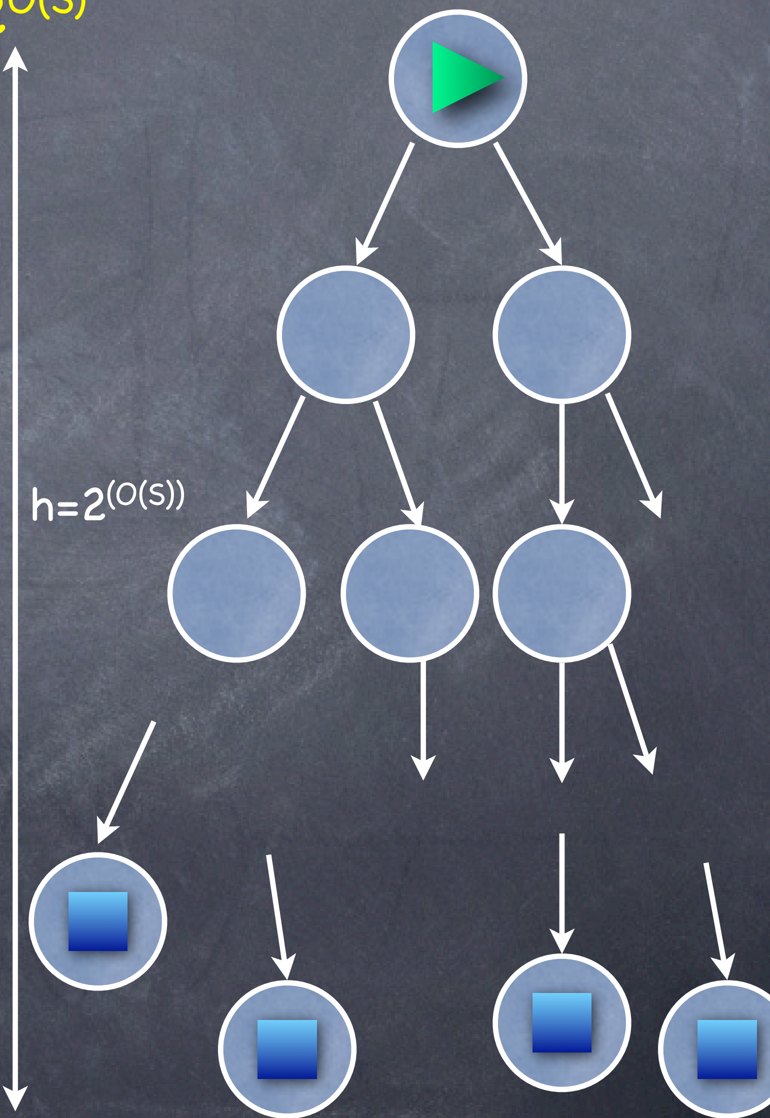

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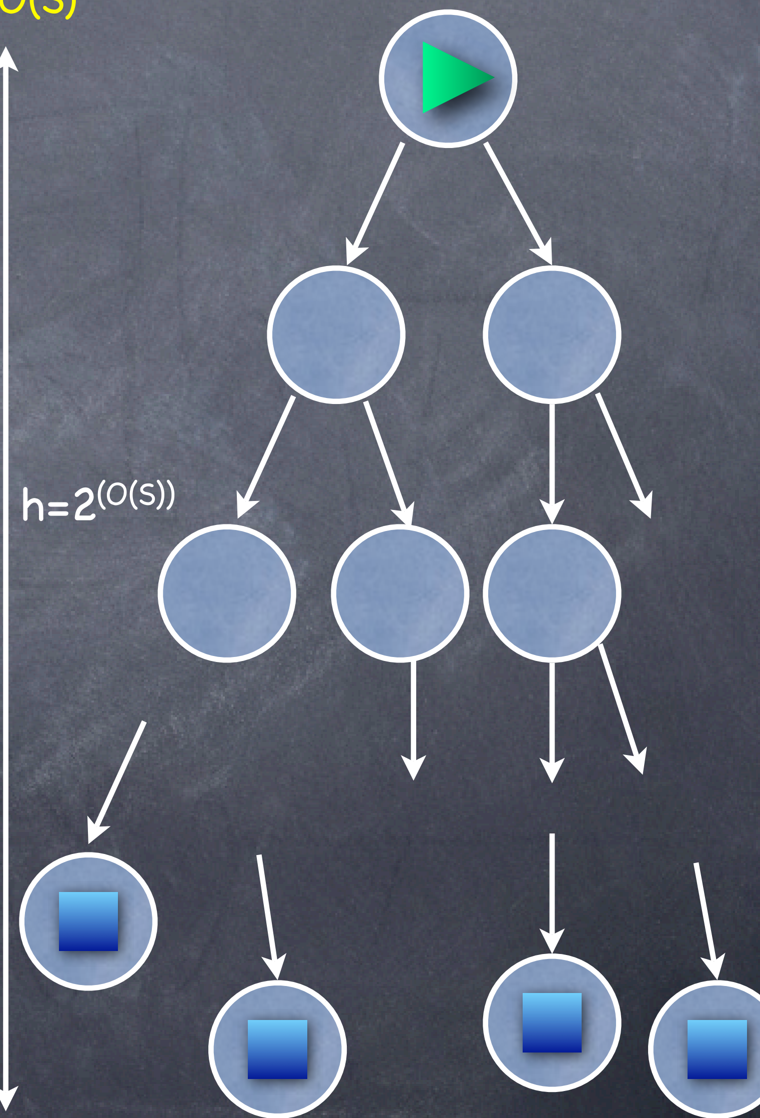

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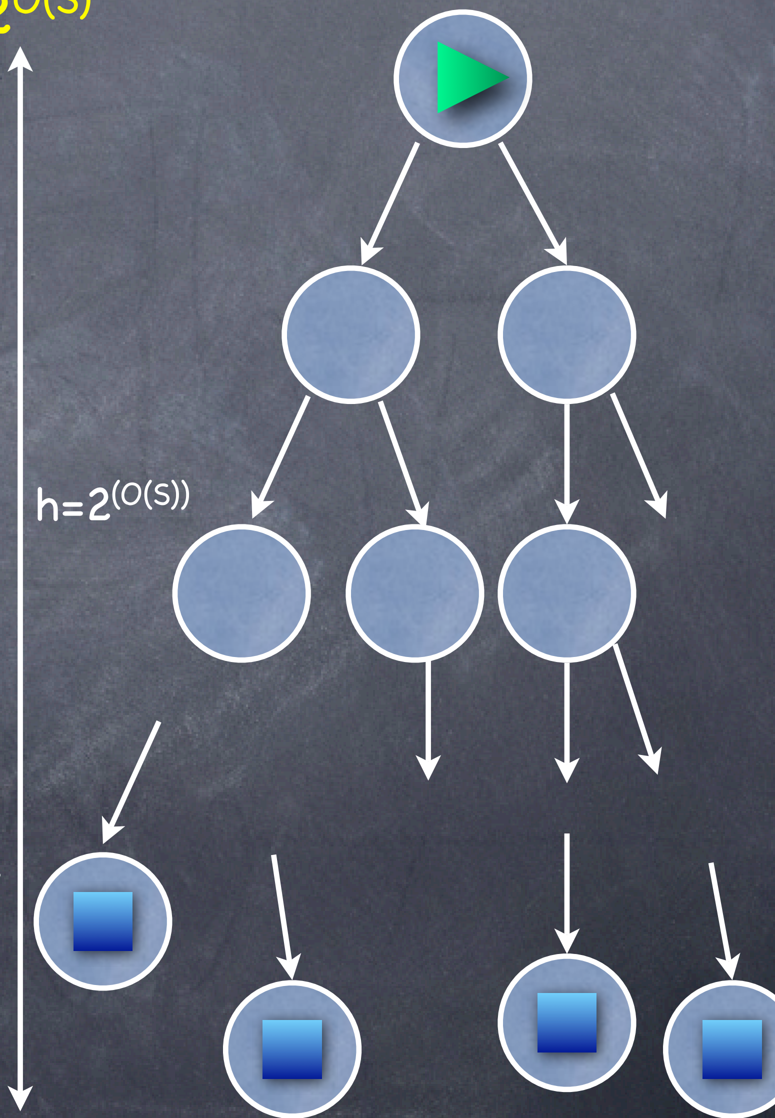
    - $\mathrm{poly}(2^{O(S)}) = 2^{O(S)}$



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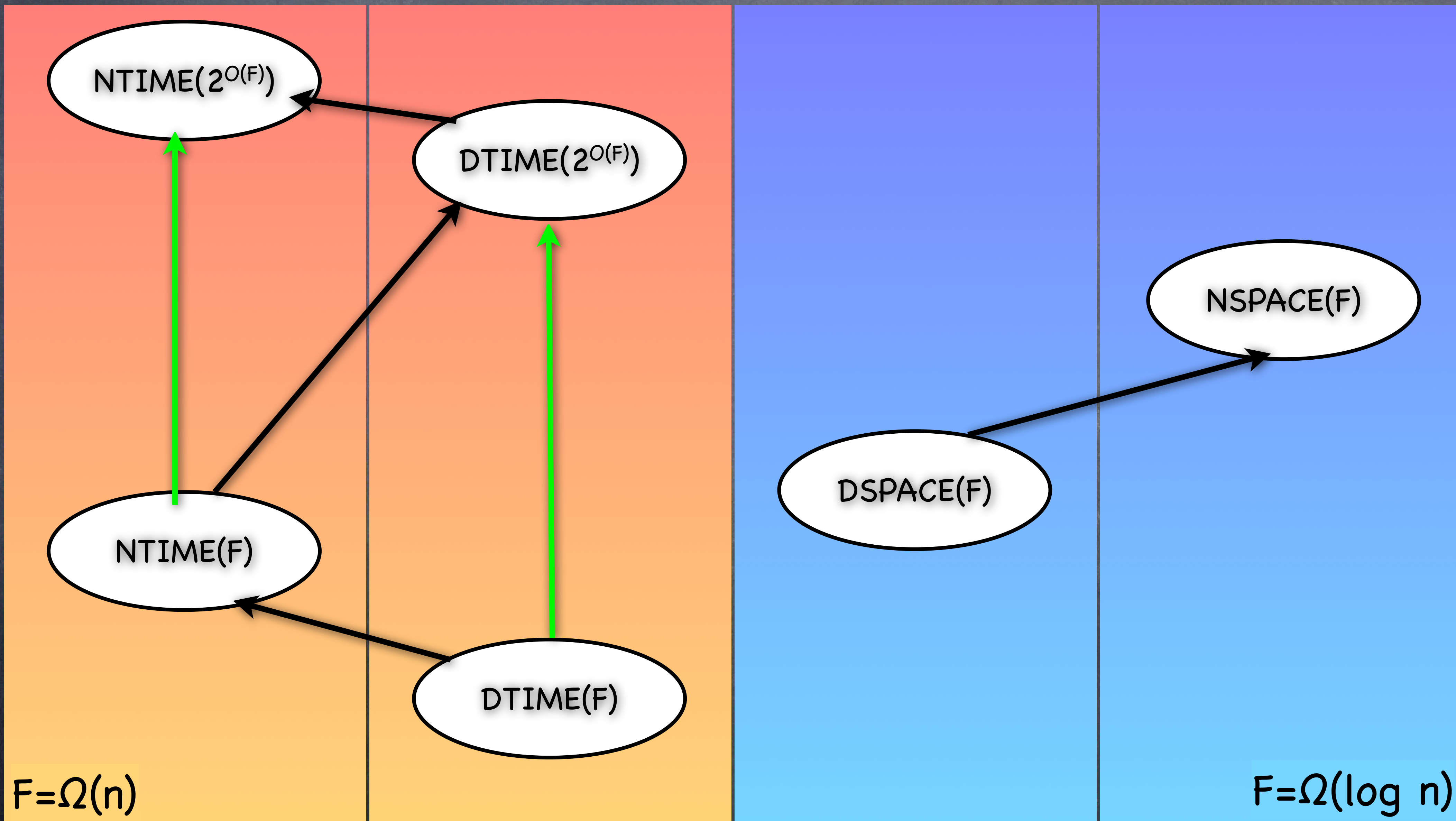

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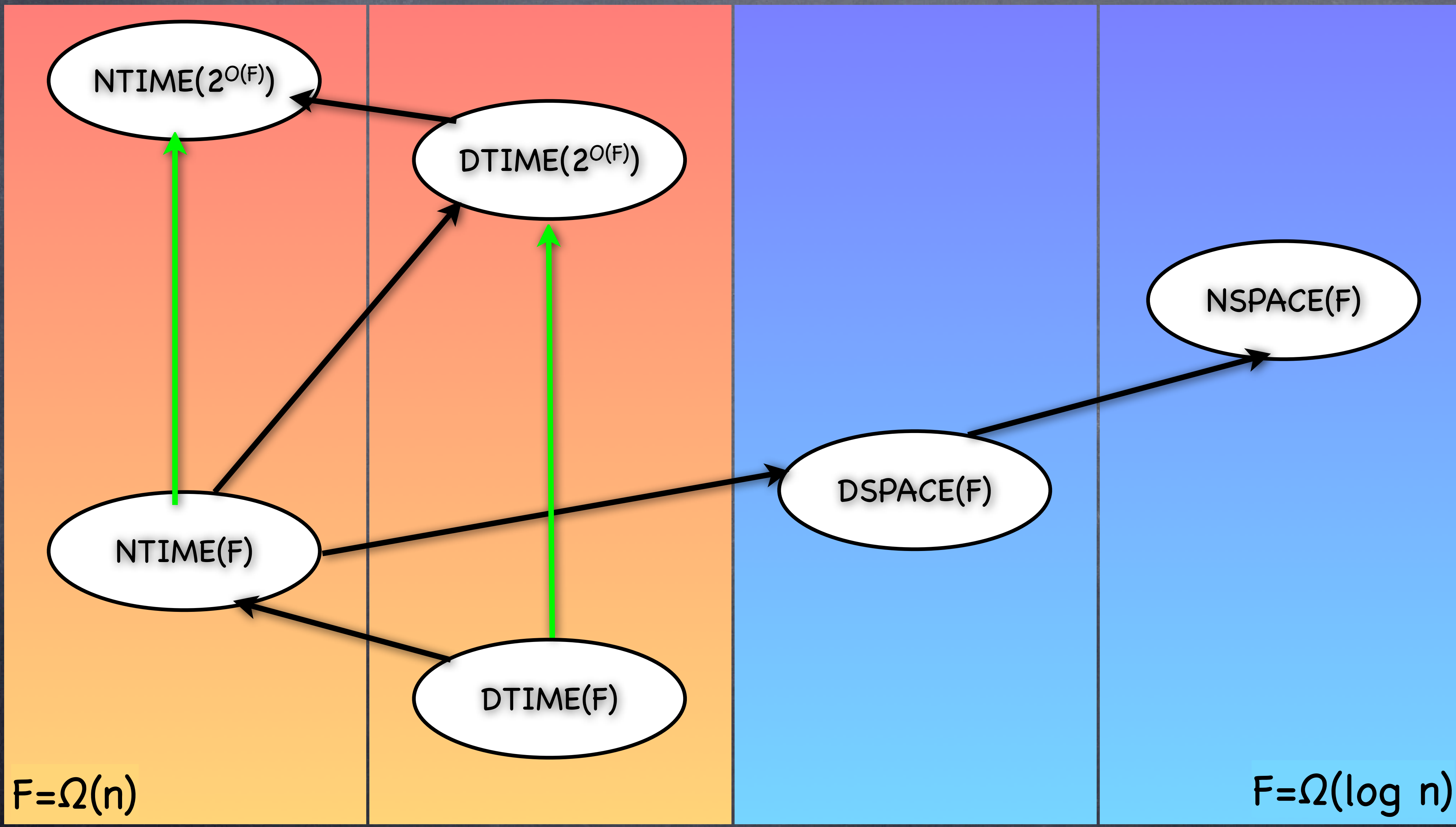

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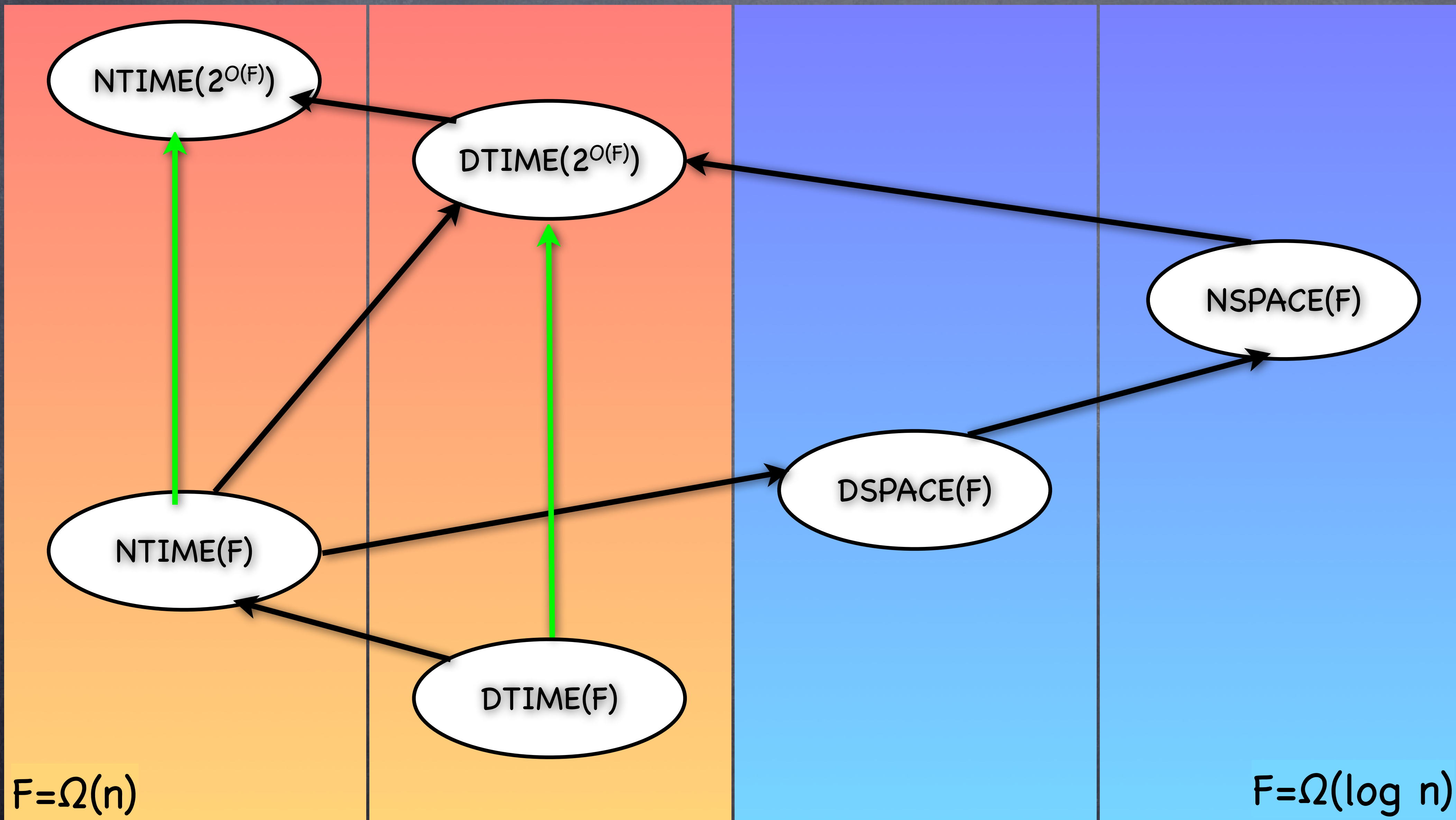

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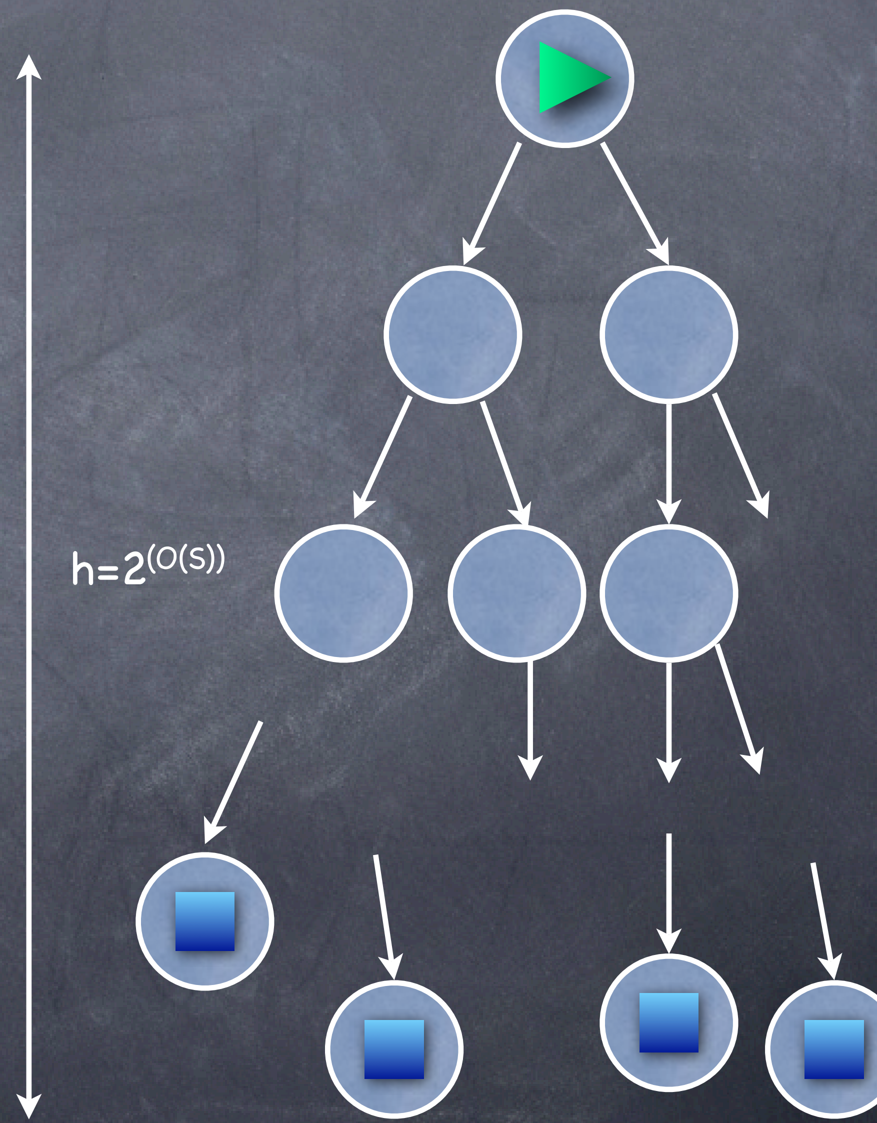

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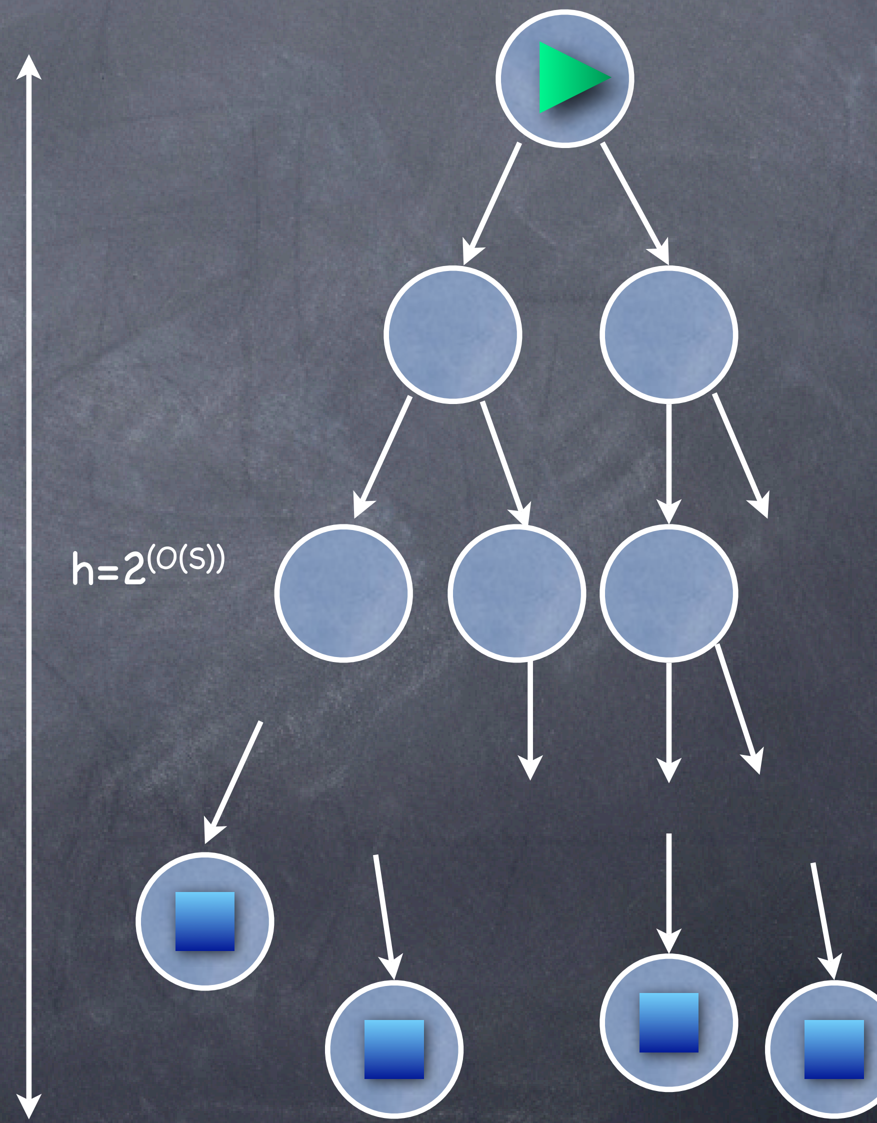

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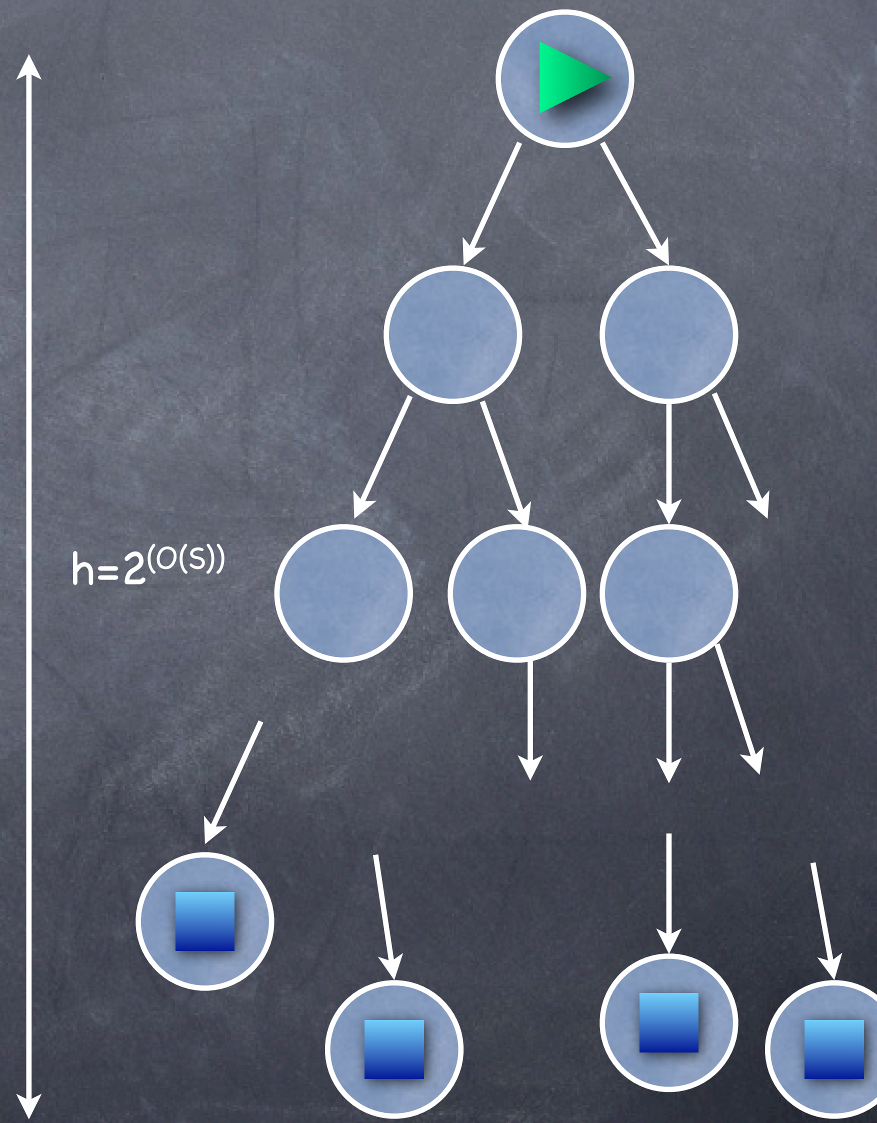

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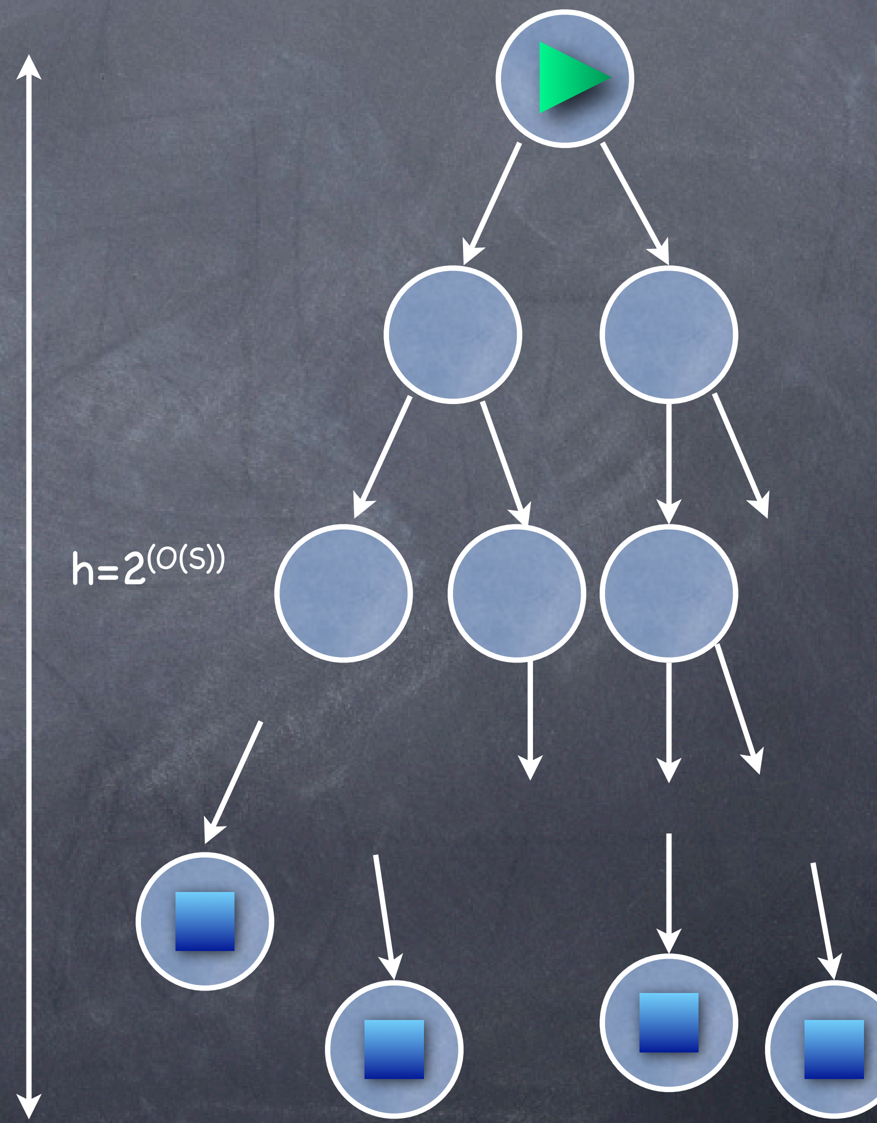

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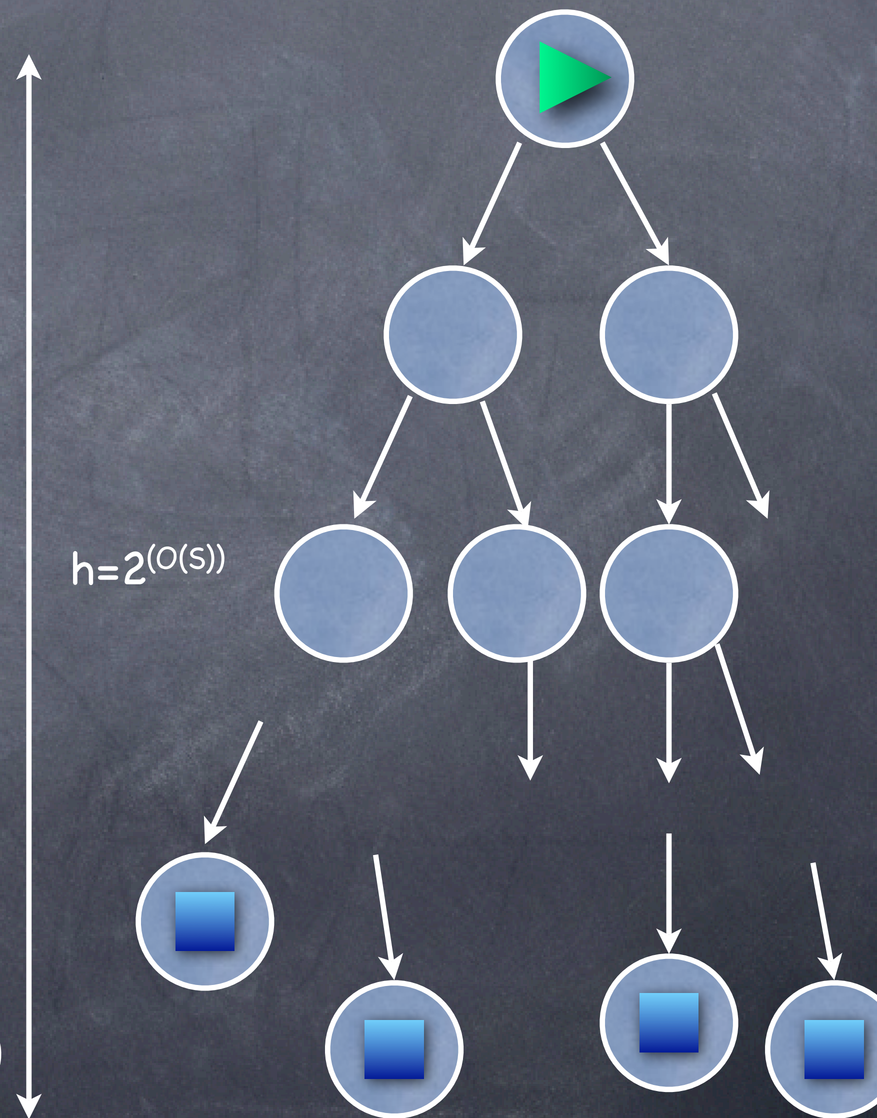
  - Space needed = O(log h)*O(S) = O(S^2)



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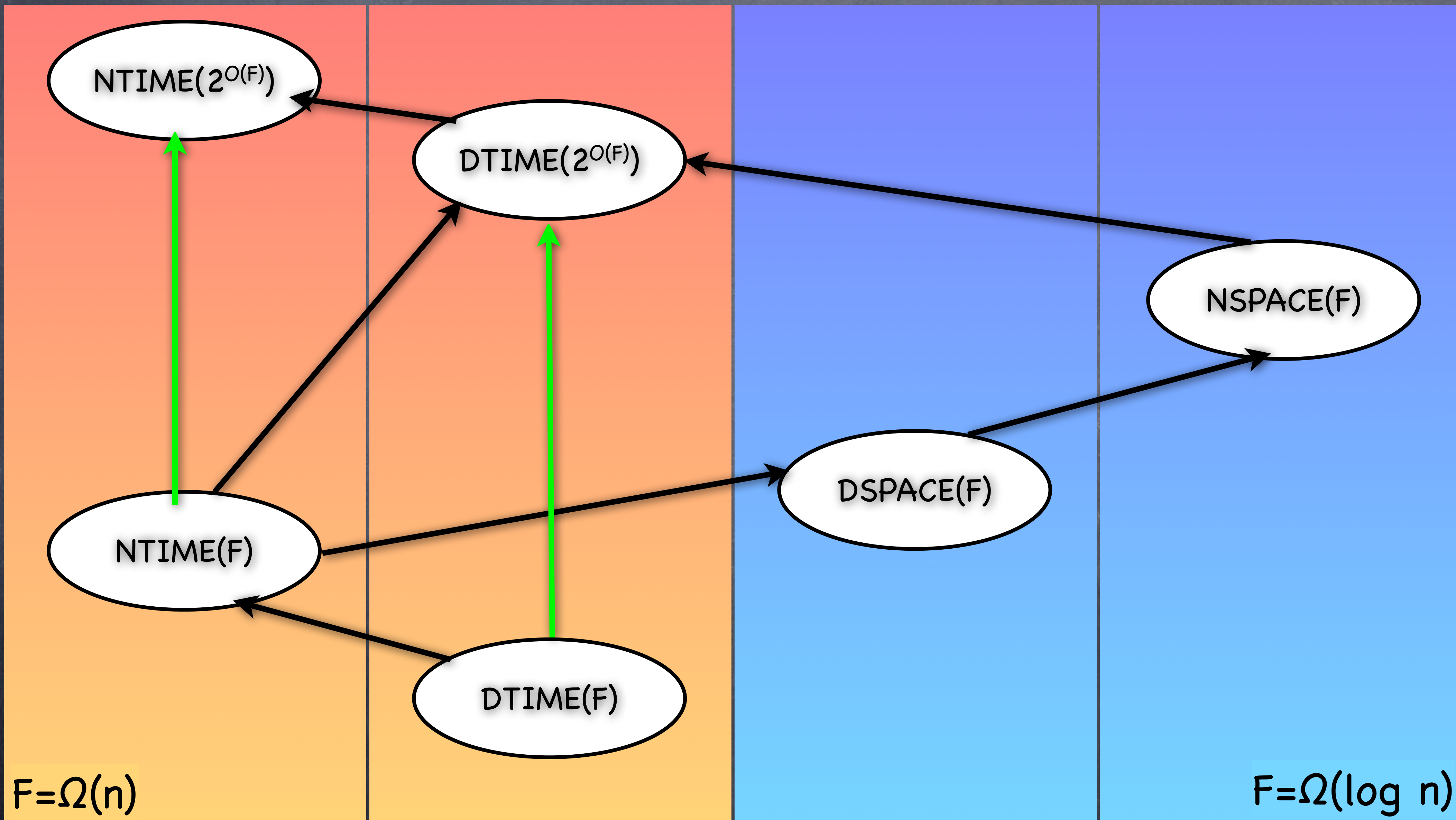

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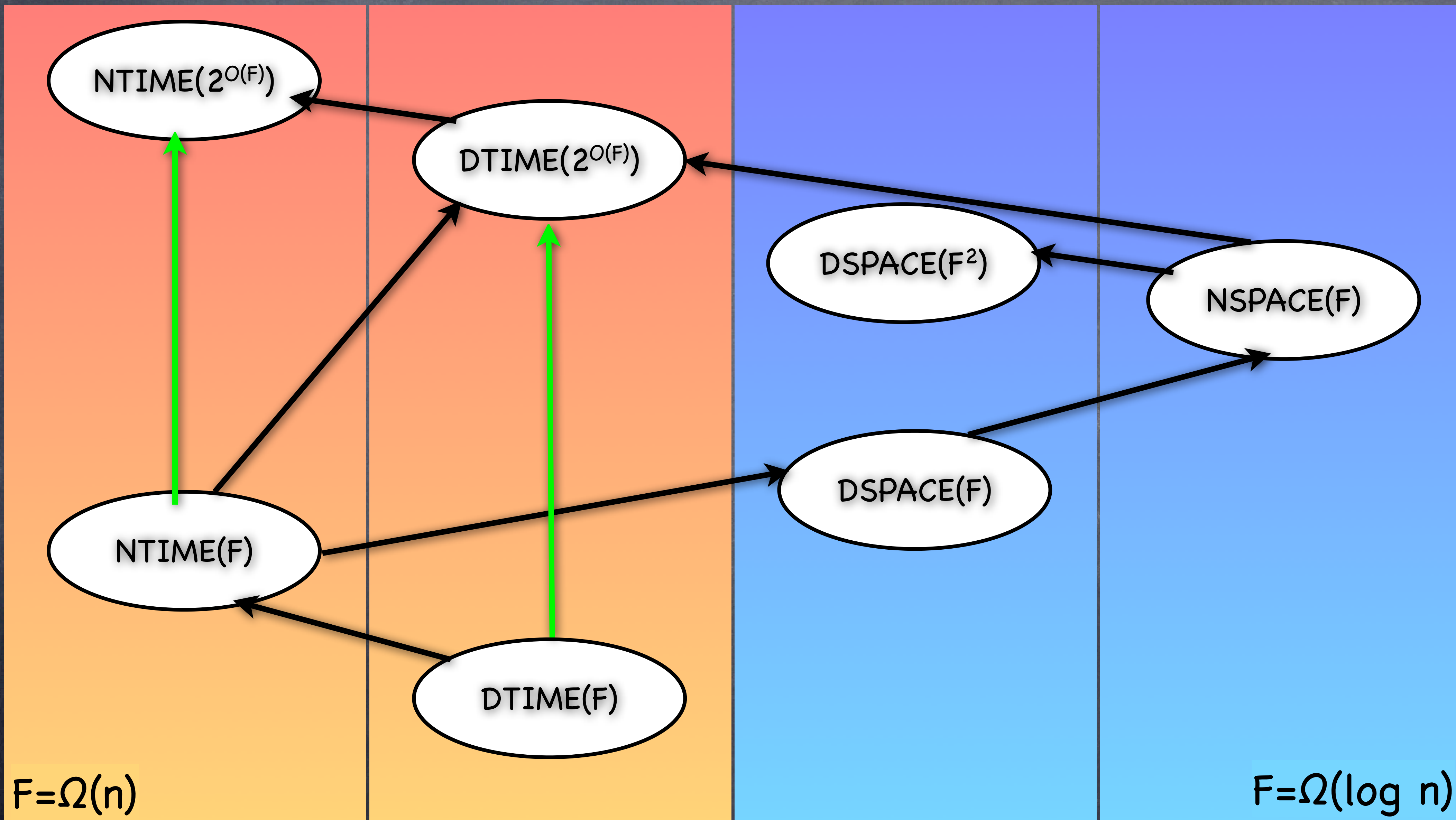

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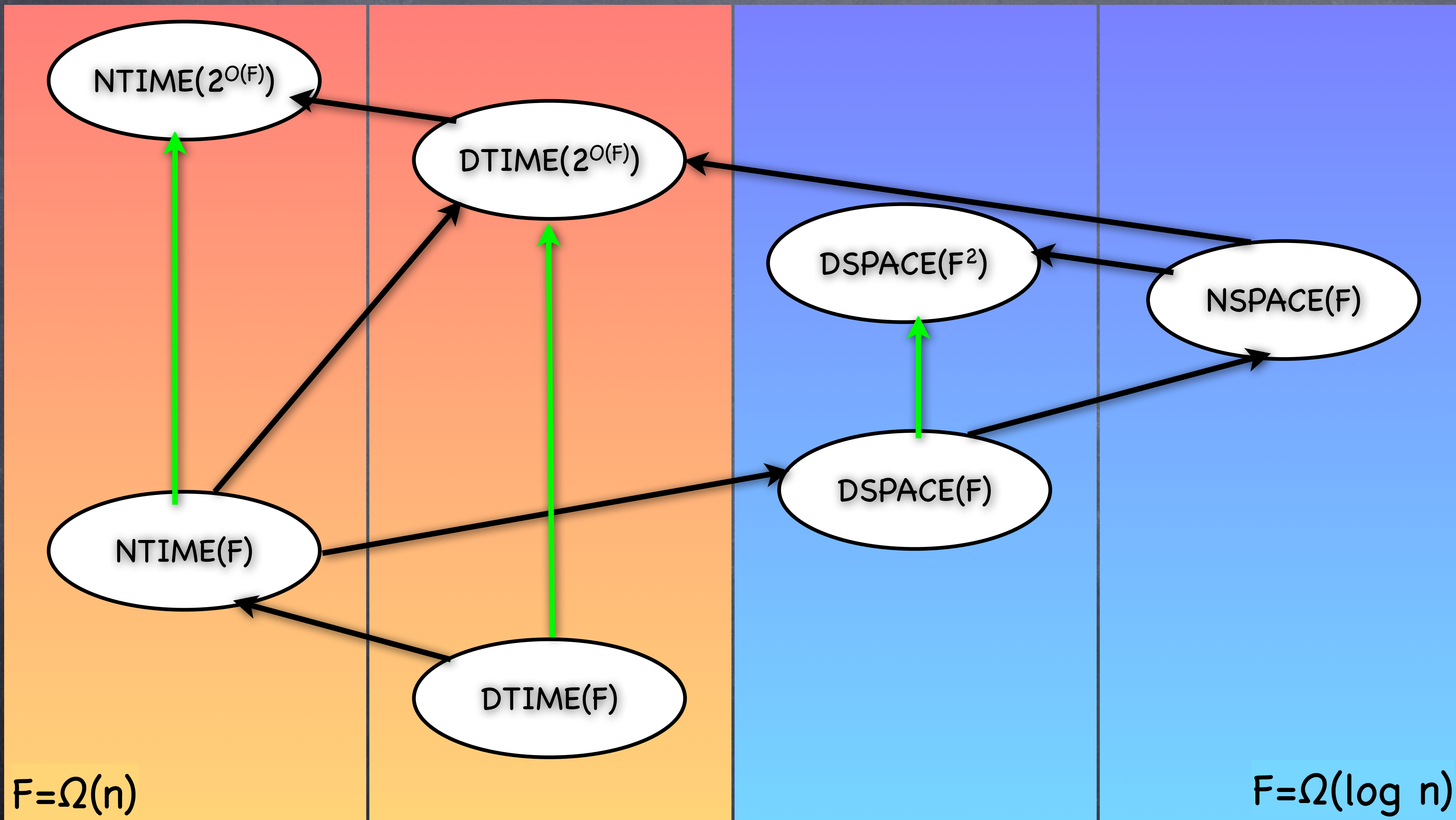

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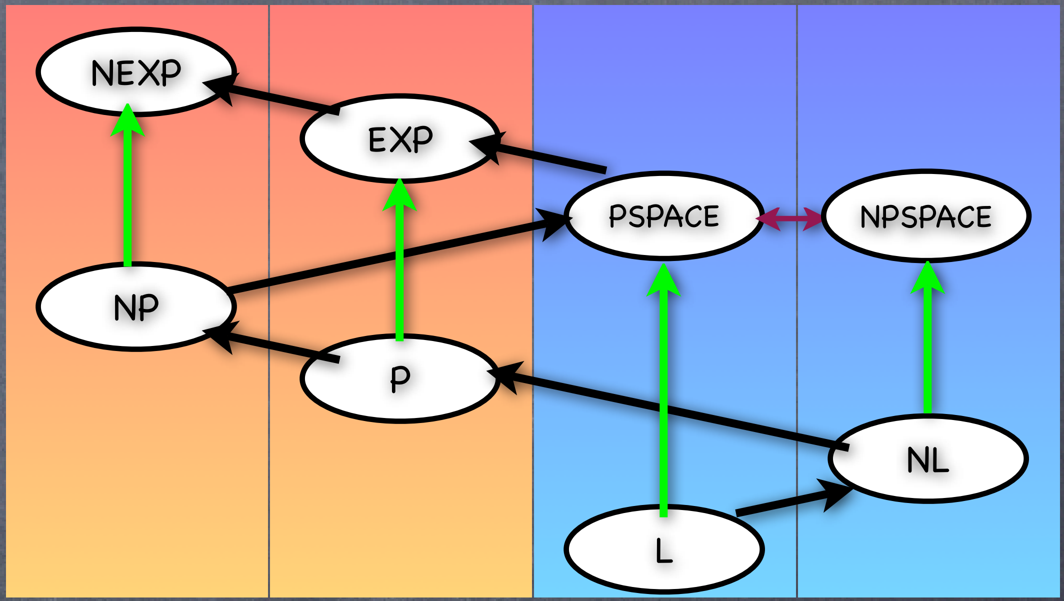

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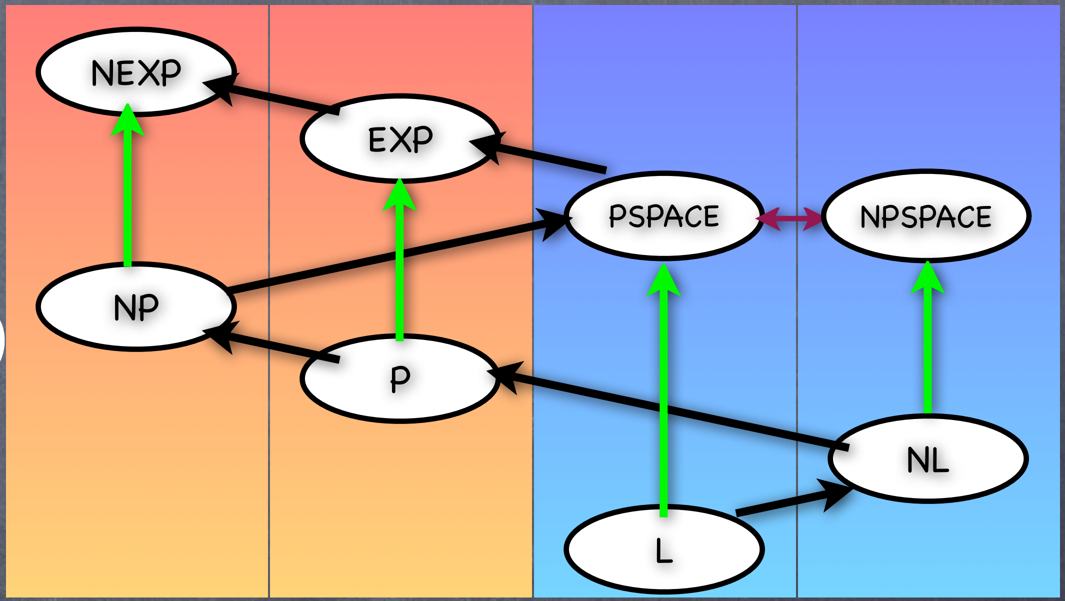

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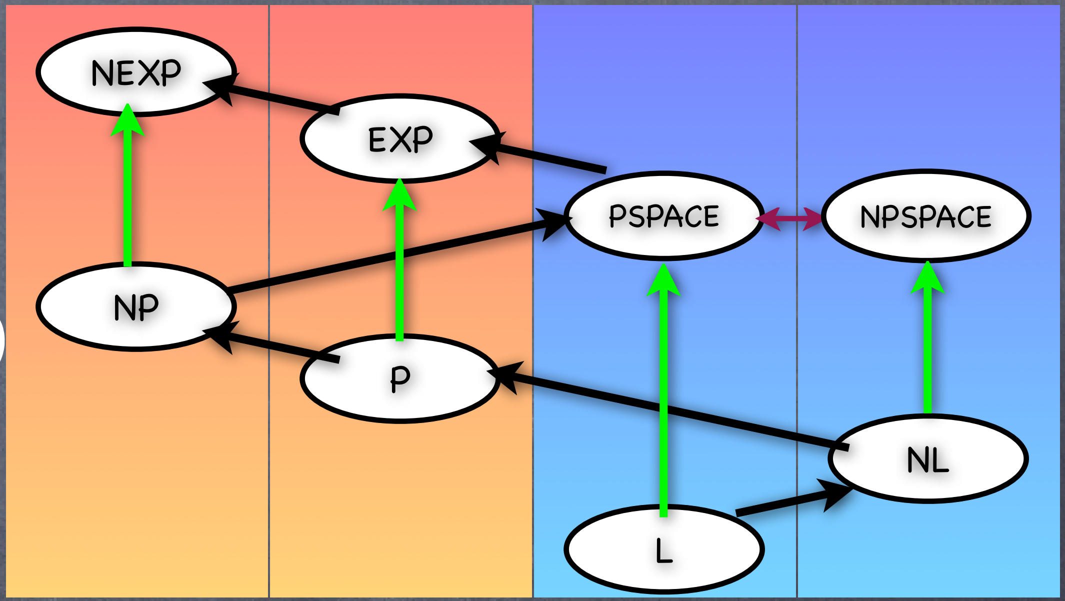
- Coming up:
  - PSPACE-completeness



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- (An) essence of PSPACE: Understanding 2-player games
  - Can the first/second player always win?

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- Given a QBF game does Alice have a sure-to-win strategy

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  - Else adversary has a winning strategy

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- e.g. ψ_1: $\exists x \forall y\ (x=y)$, ψ_2: $\forall y \exists x\ (x=y)$

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Game-Tree

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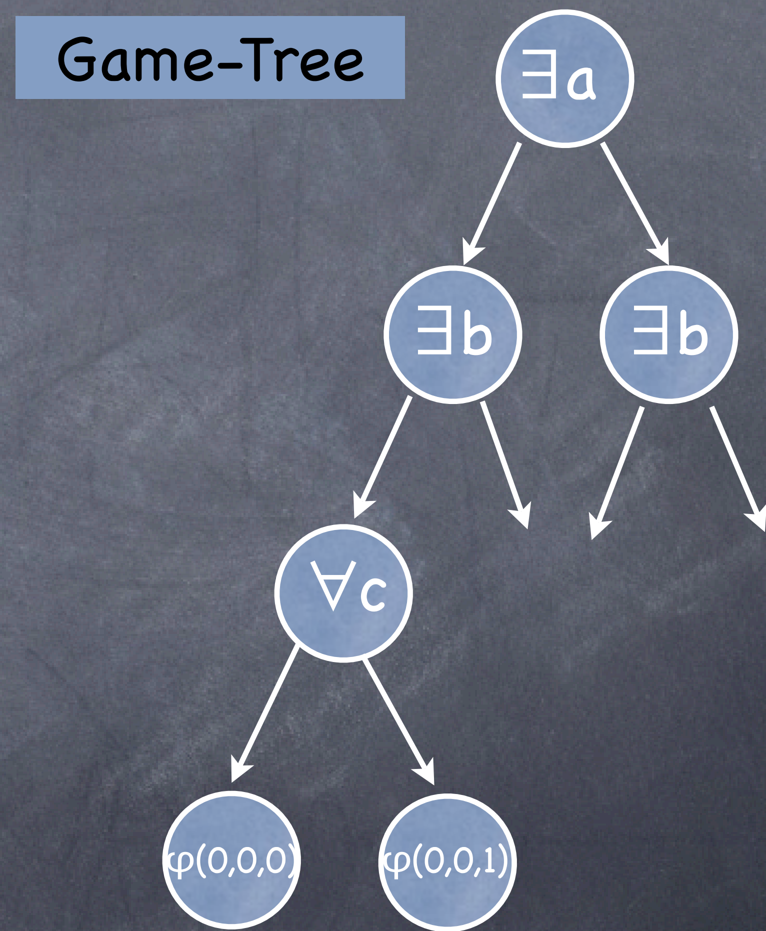

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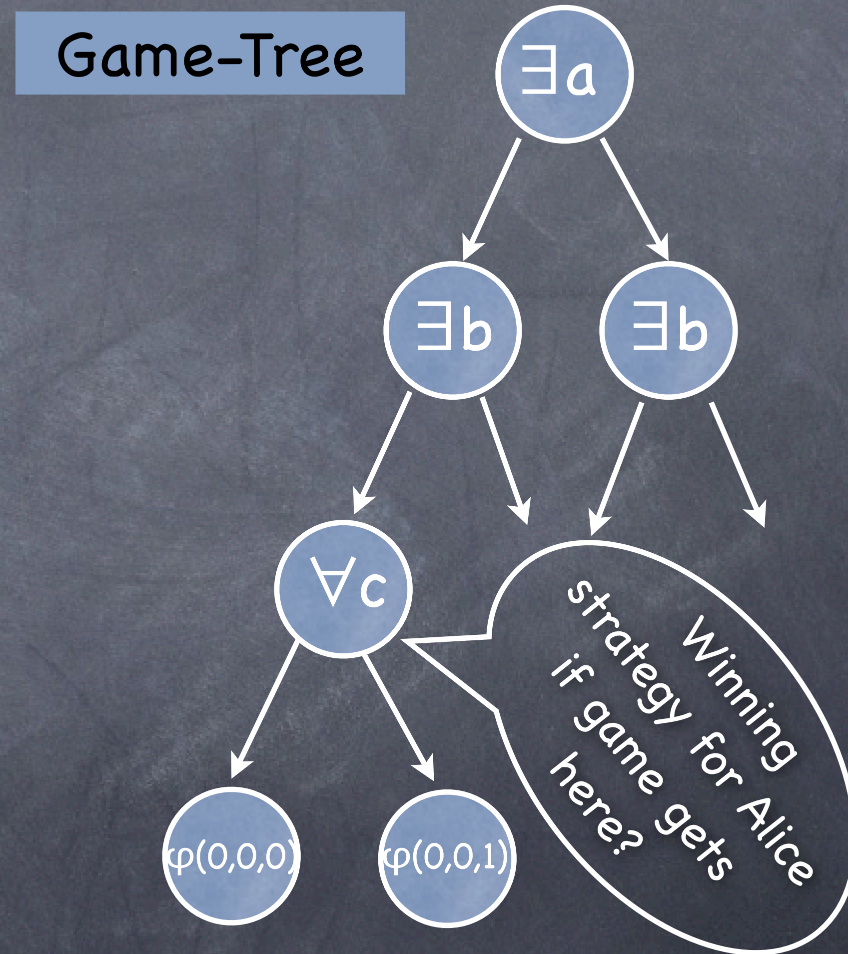

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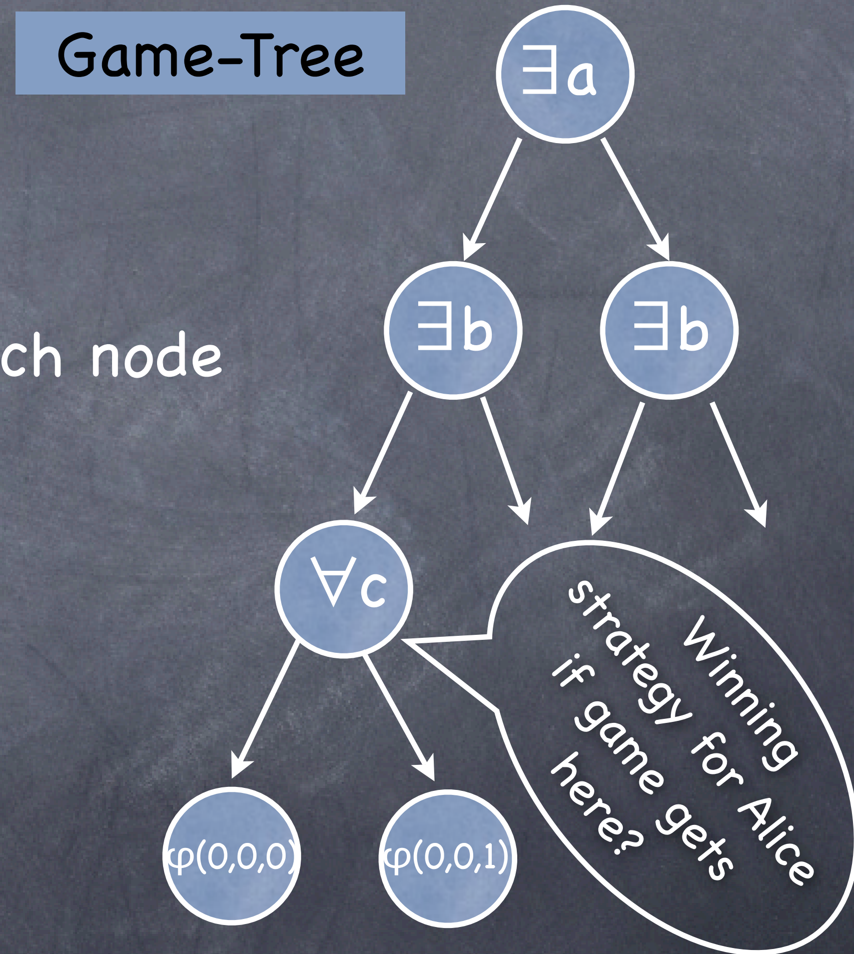

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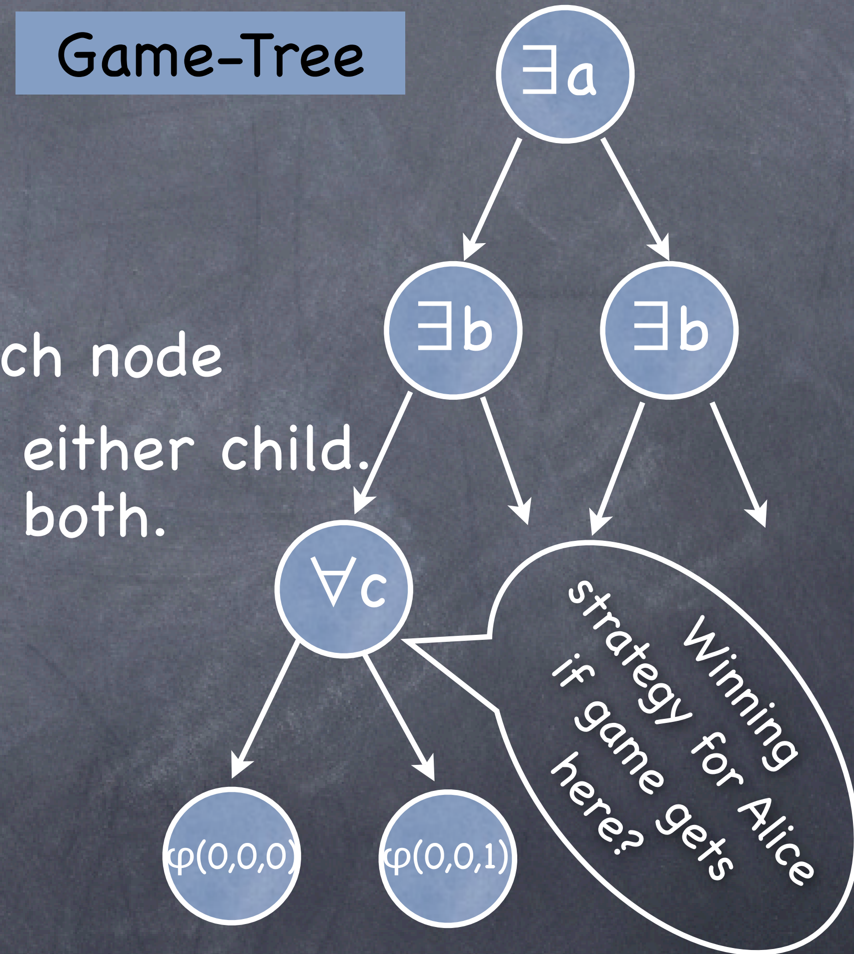

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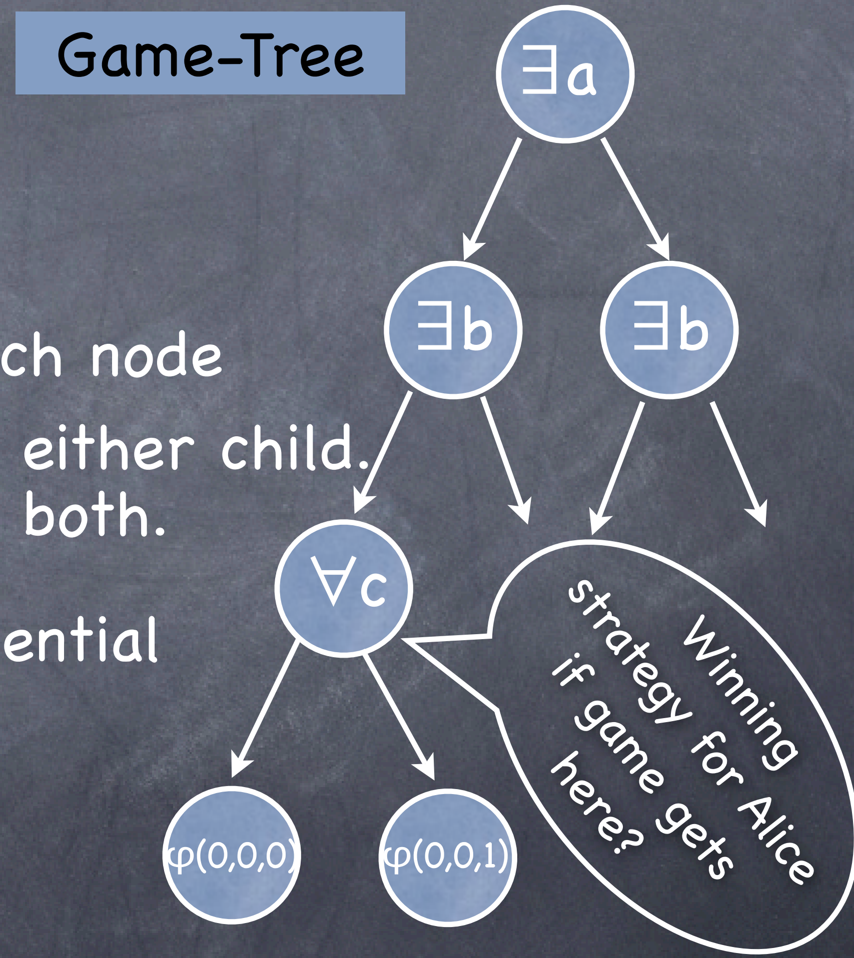

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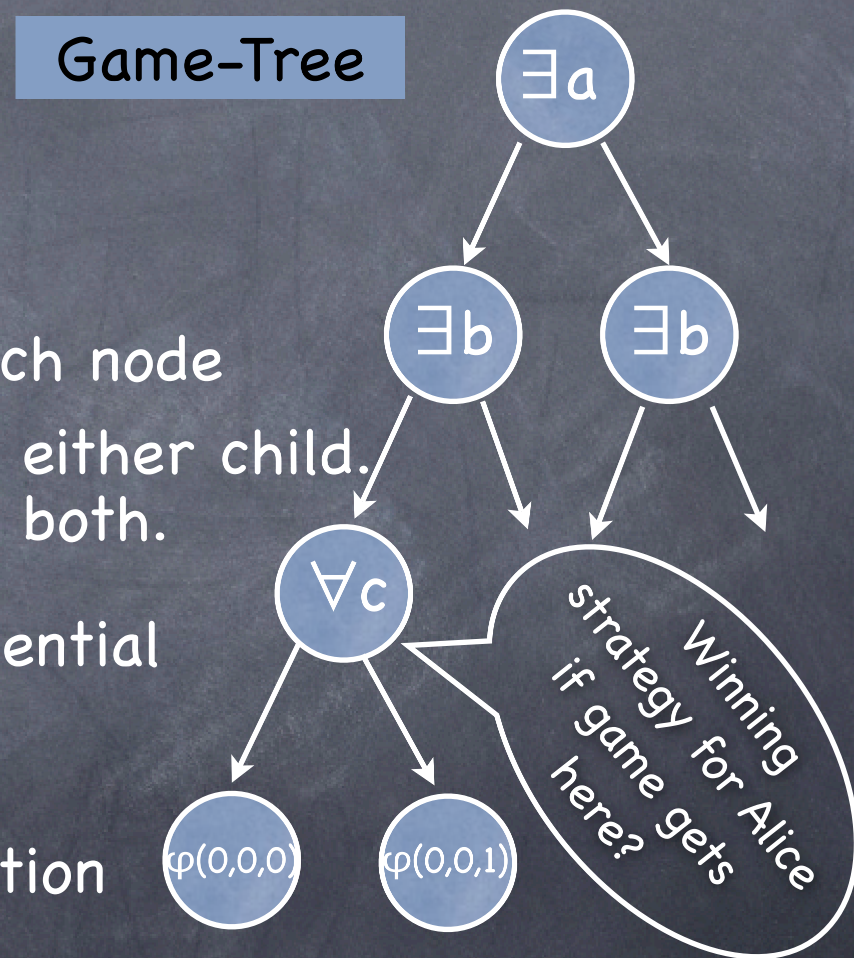

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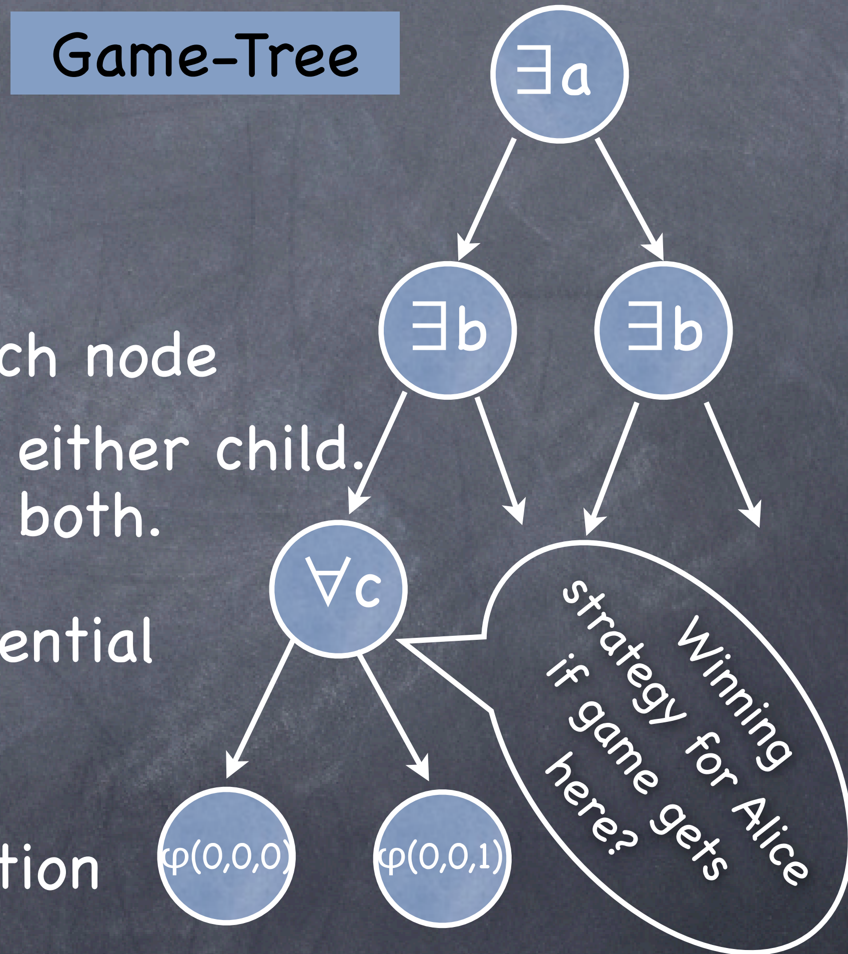
      - Space needed = O(depth) + for evaluation = poly(|QBF|)



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    - Use power of quantification to write it succinctly

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      - In fact, same as naive formula!

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  - Coming soon!

Today

Today

- Zoo (more later)

Today

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- TQBF

Today

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Today

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