Computational Complexity

Lecture 4
in which Diagonalization takes on itself,
and we enter Space Complexity
(But first Ladner's Theorem)

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  - No!

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- If $P \neq NP$, then are all non-P NP languages equally hard? (Are all NP-complete?)
  - No!
  - Can show an NP language which is neither in P, nor NP complete (unless P = NP)

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- To define H s.t. H(n) bounded by const. iff SAT_H in P

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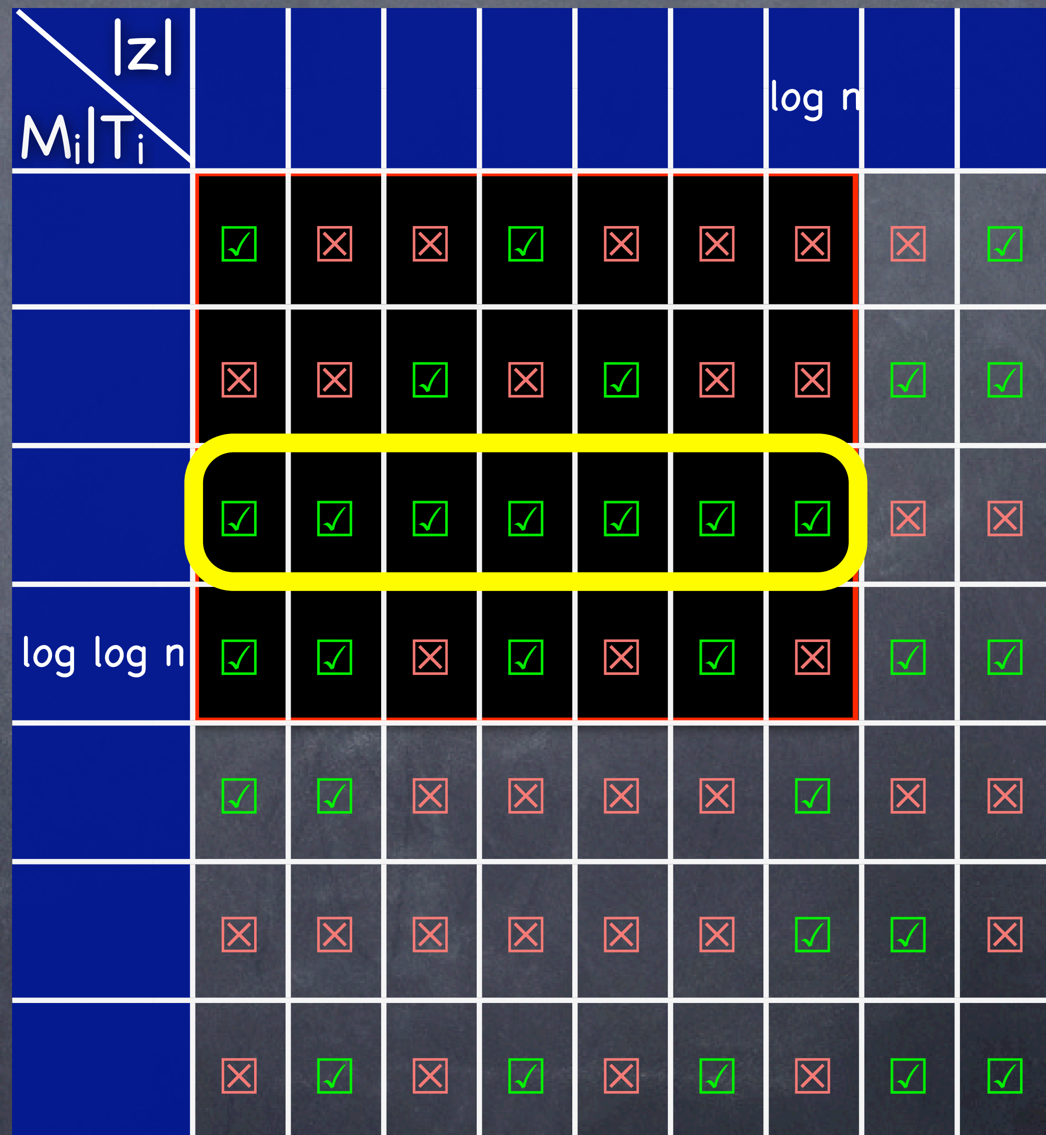

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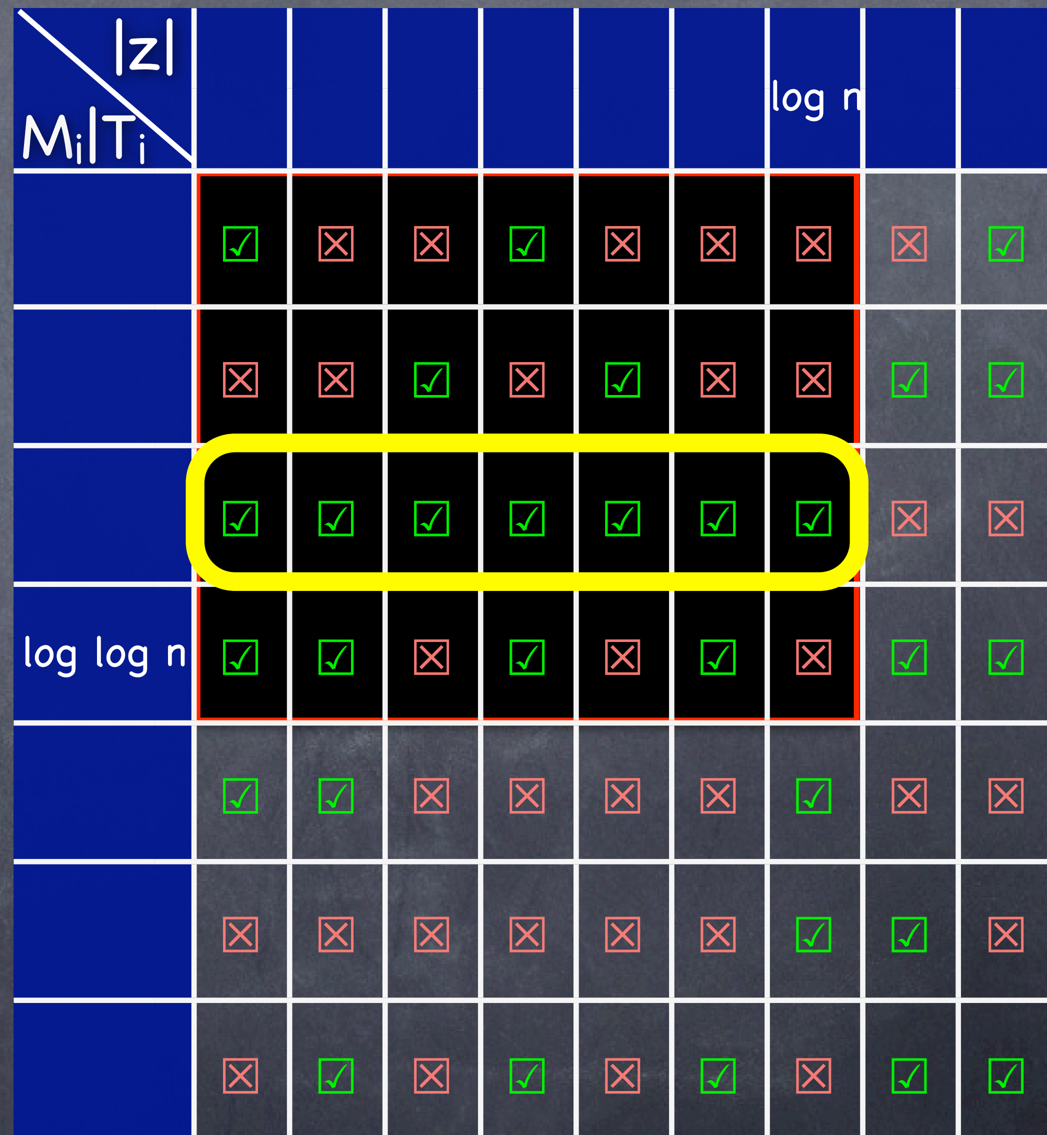

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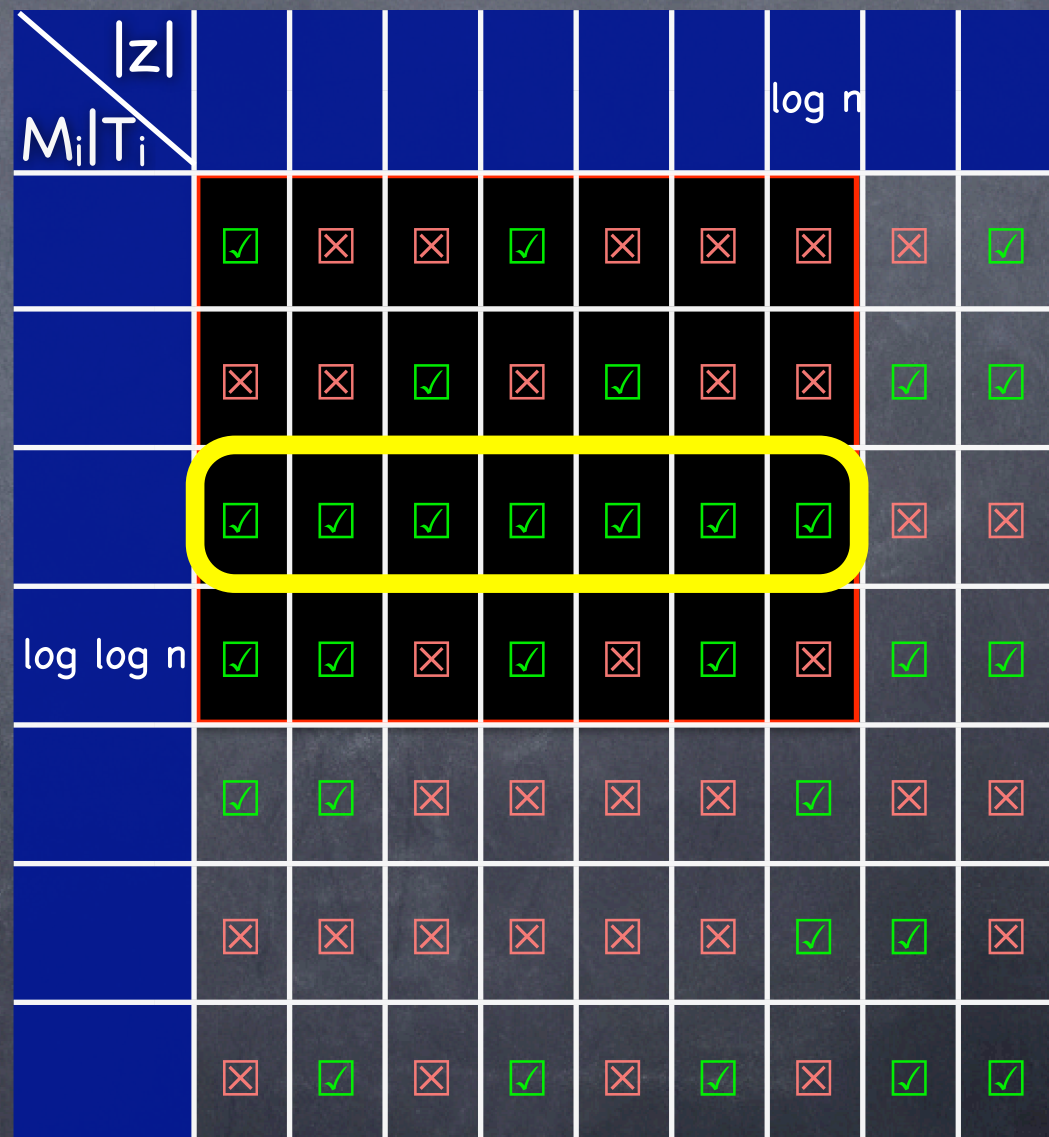

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Meta-Questions

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"Real" Questions

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"Real" Questions ⟷ "Meta" Questions

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"Real" Questions $\longleftrightarrow$ "Meta" Questions

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Under-the-hood stuff

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Equivalence carries over!

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    - There exist languages A, B such that $P^A = NP^A$, but $P^B \neq NP^B$!

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  - EXPTM = { $(M,x,1^n)$ | TM represented by M accepts x within time 2^n }

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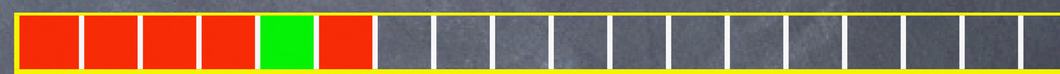

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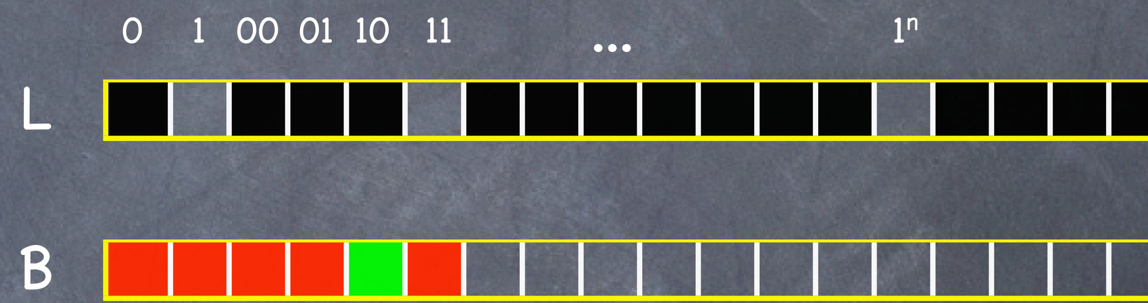

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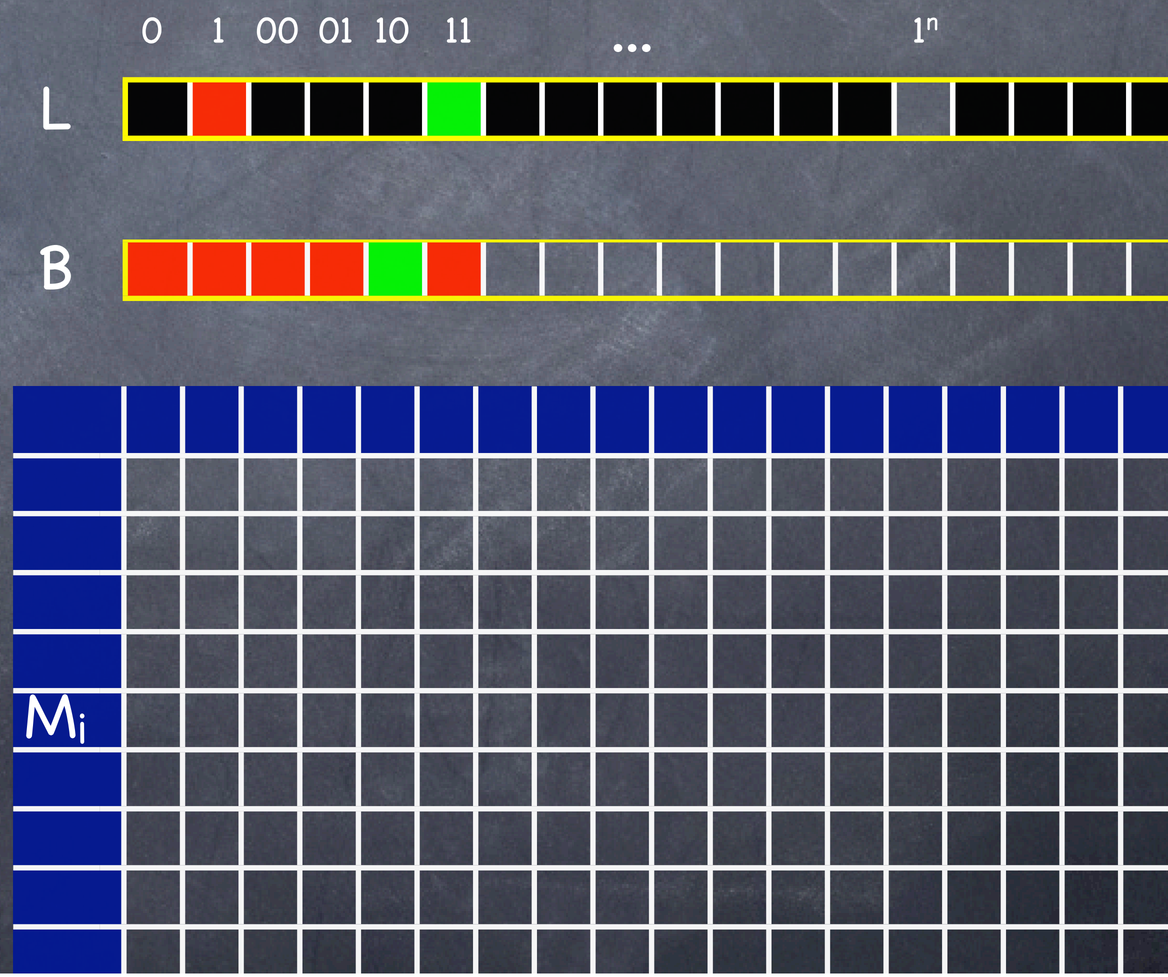

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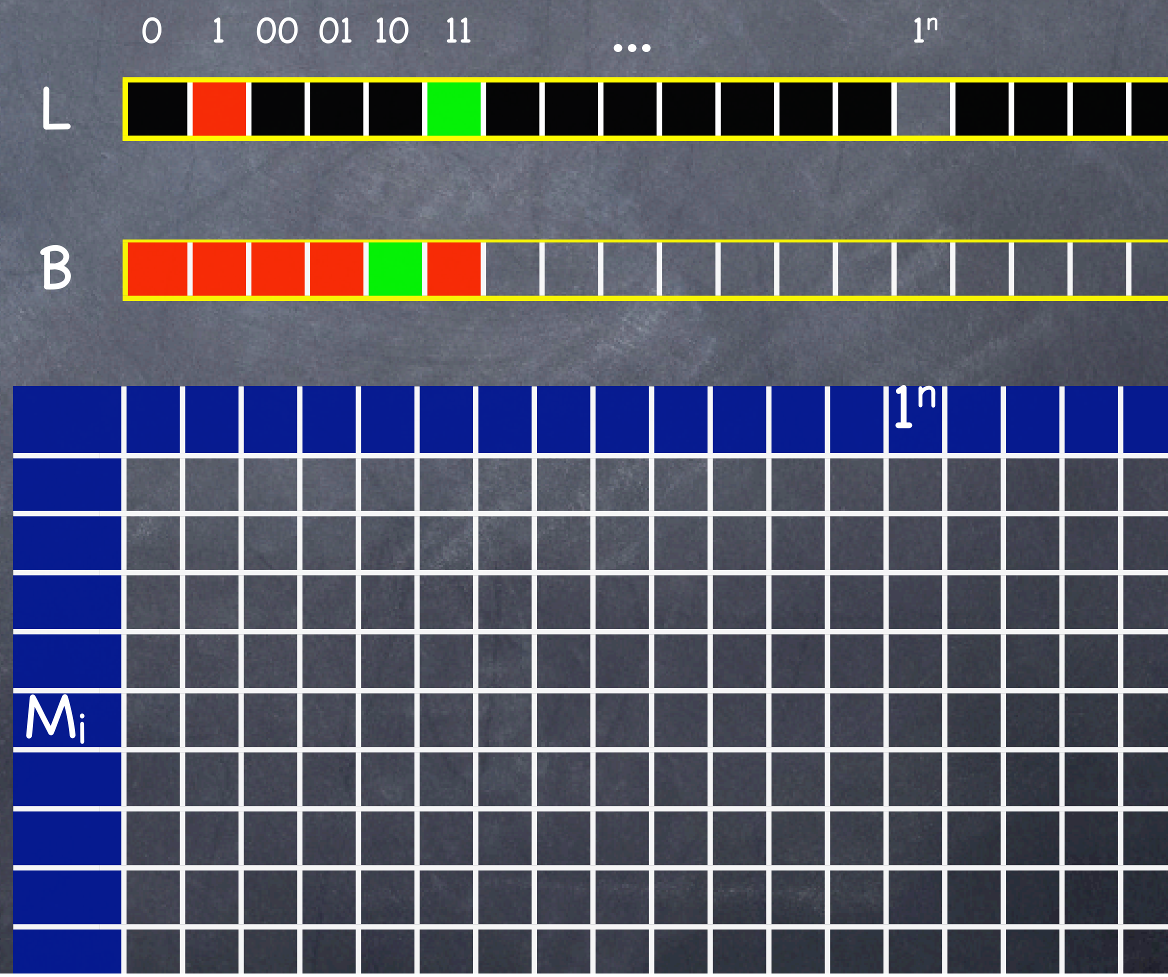

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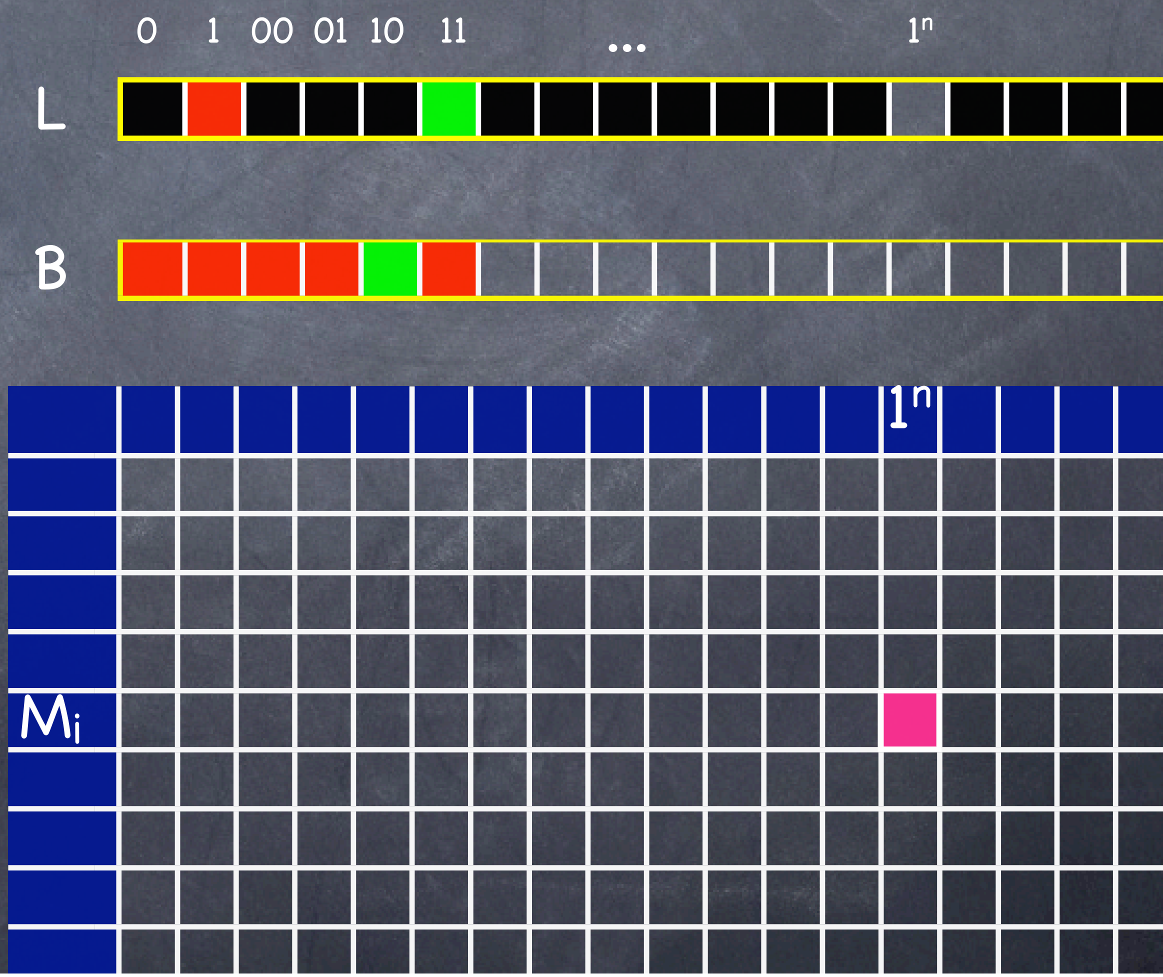

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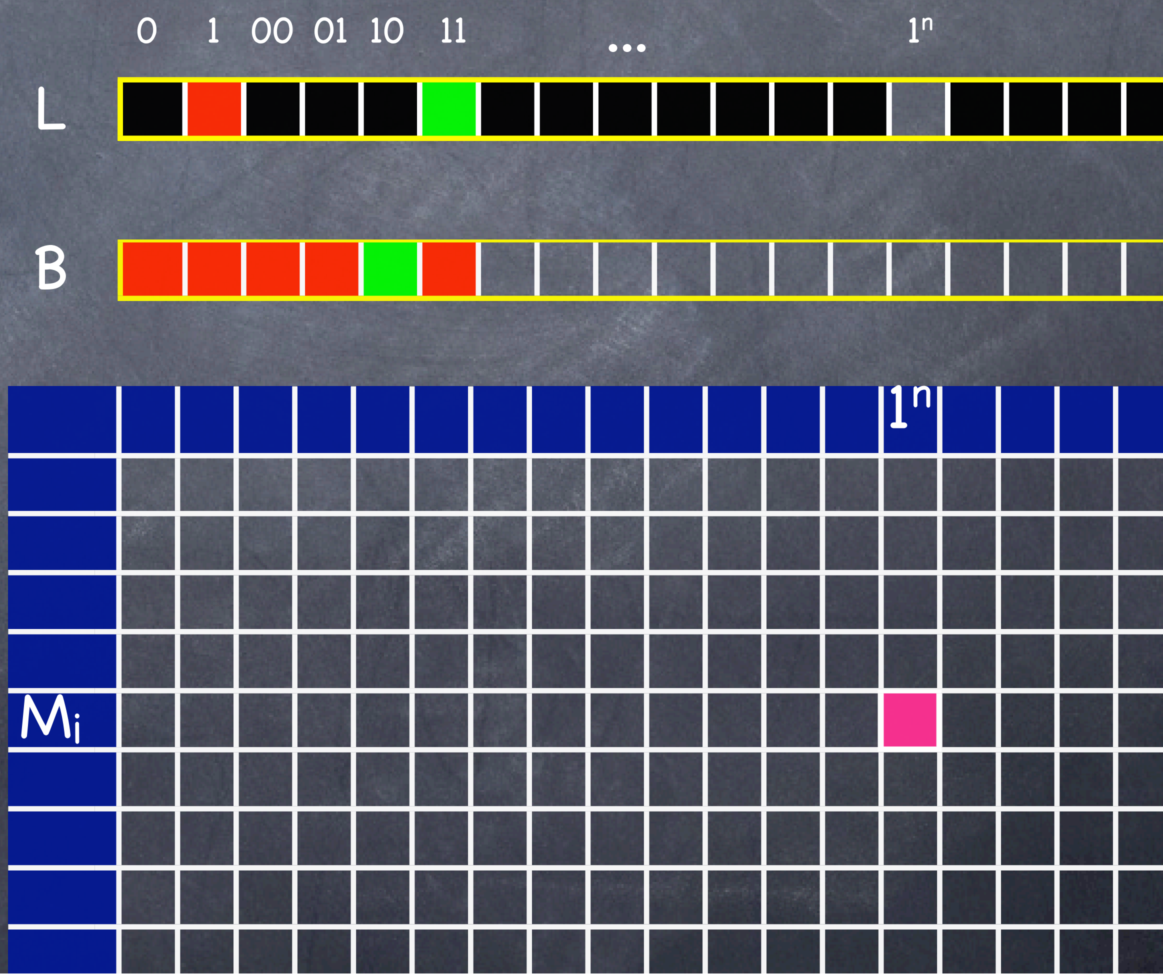

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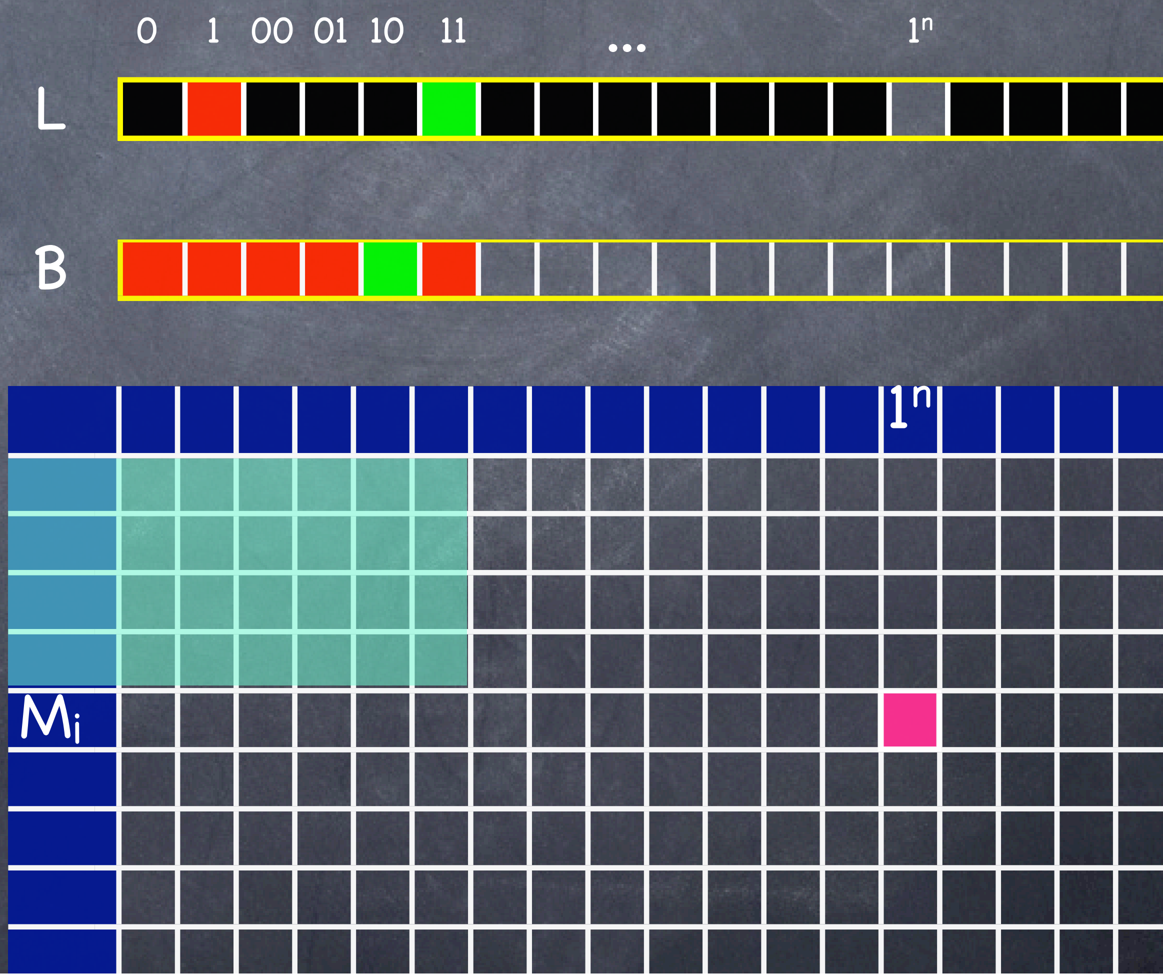

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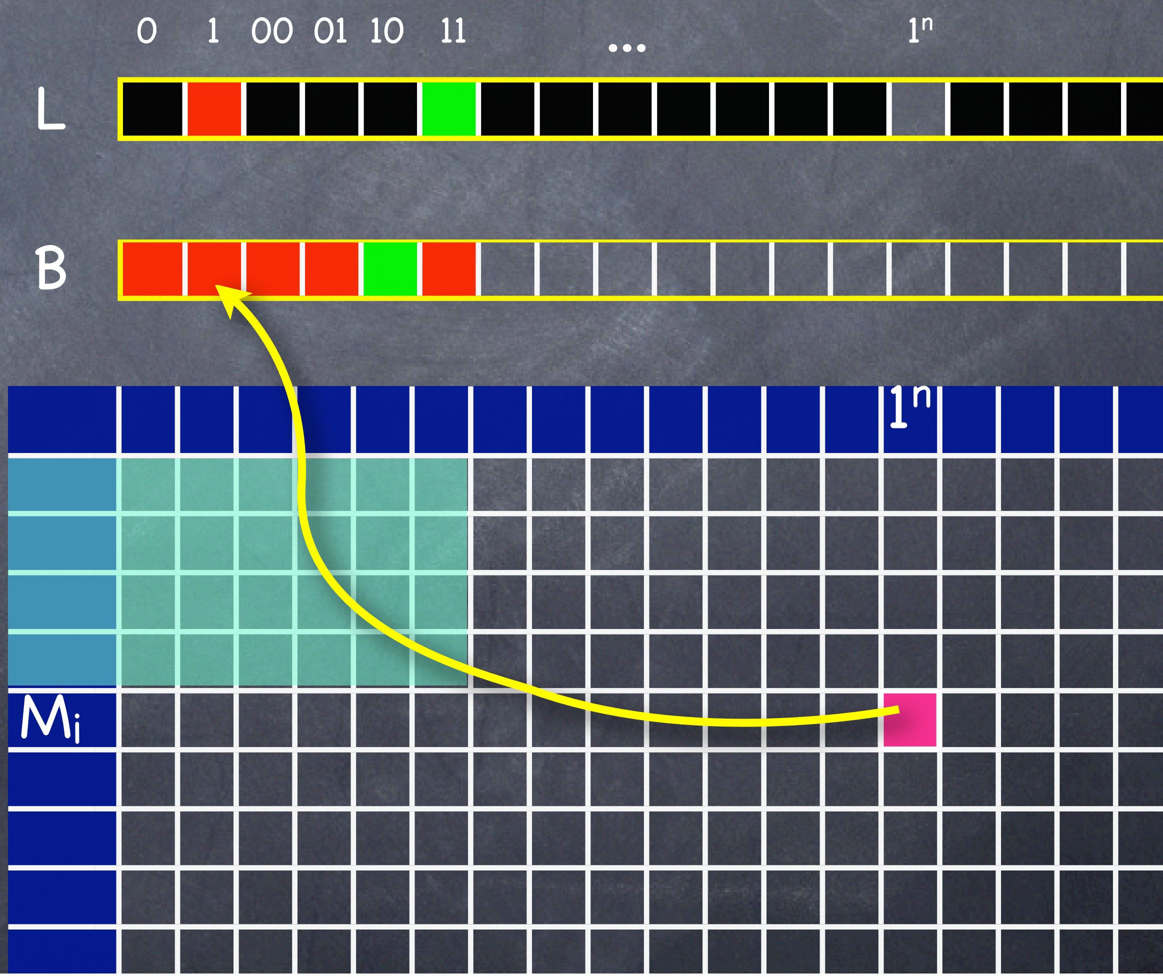

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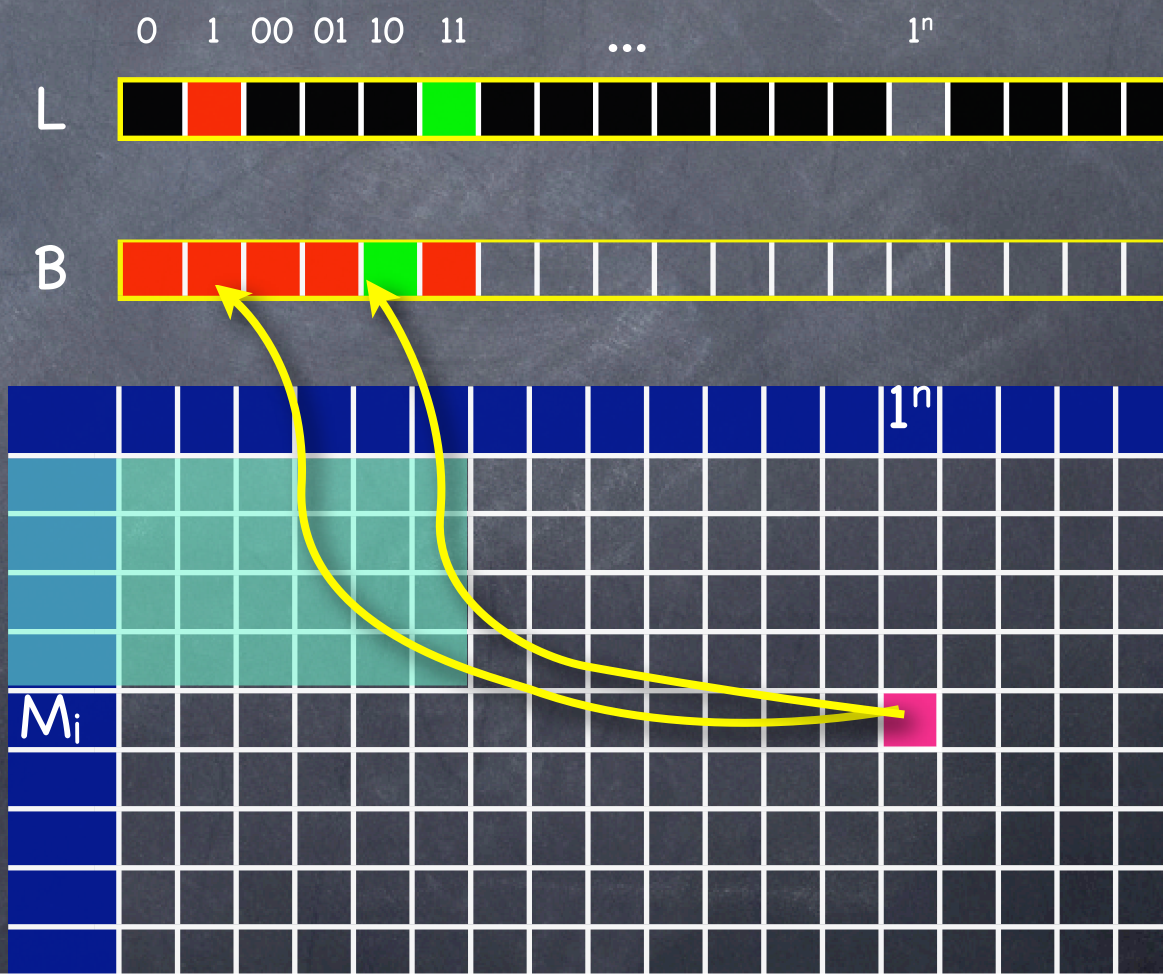

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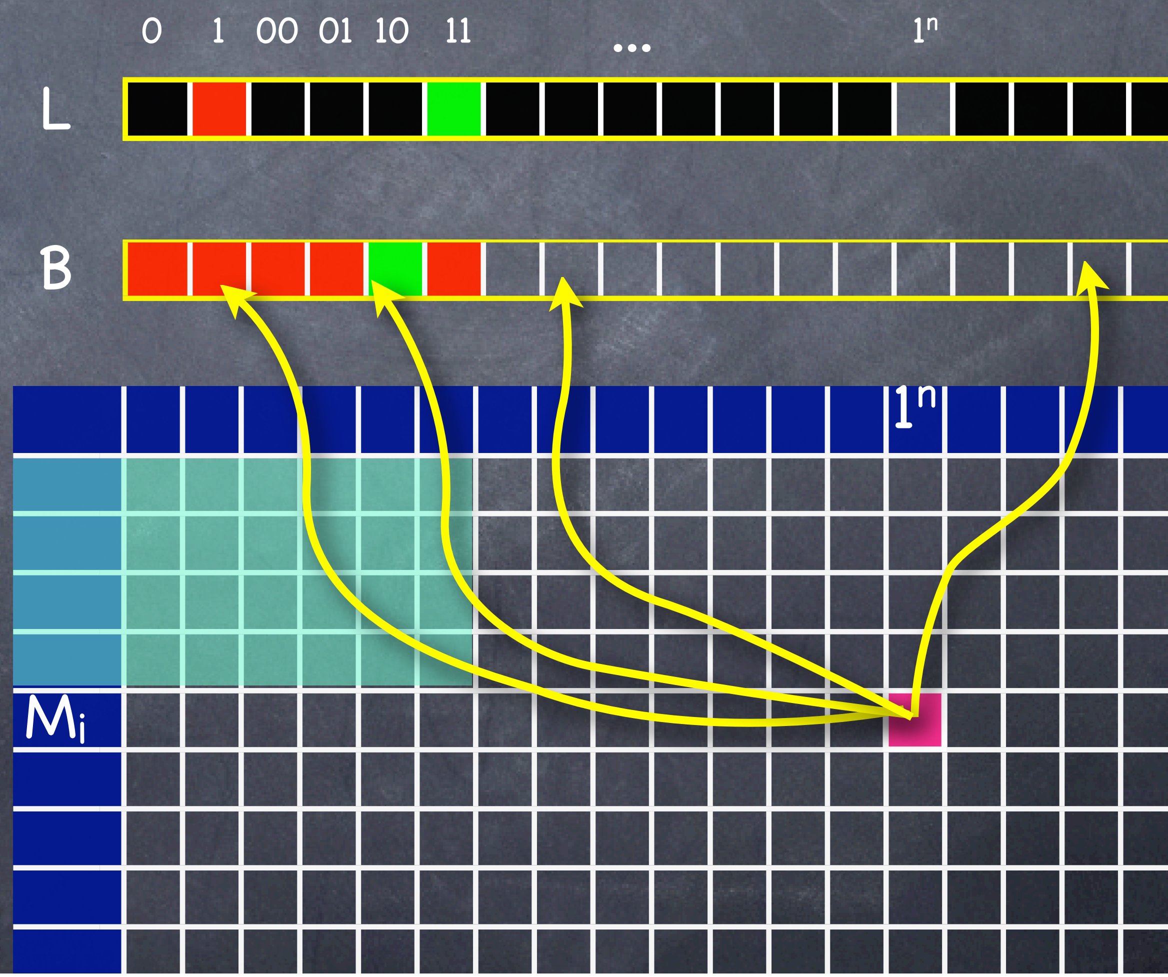

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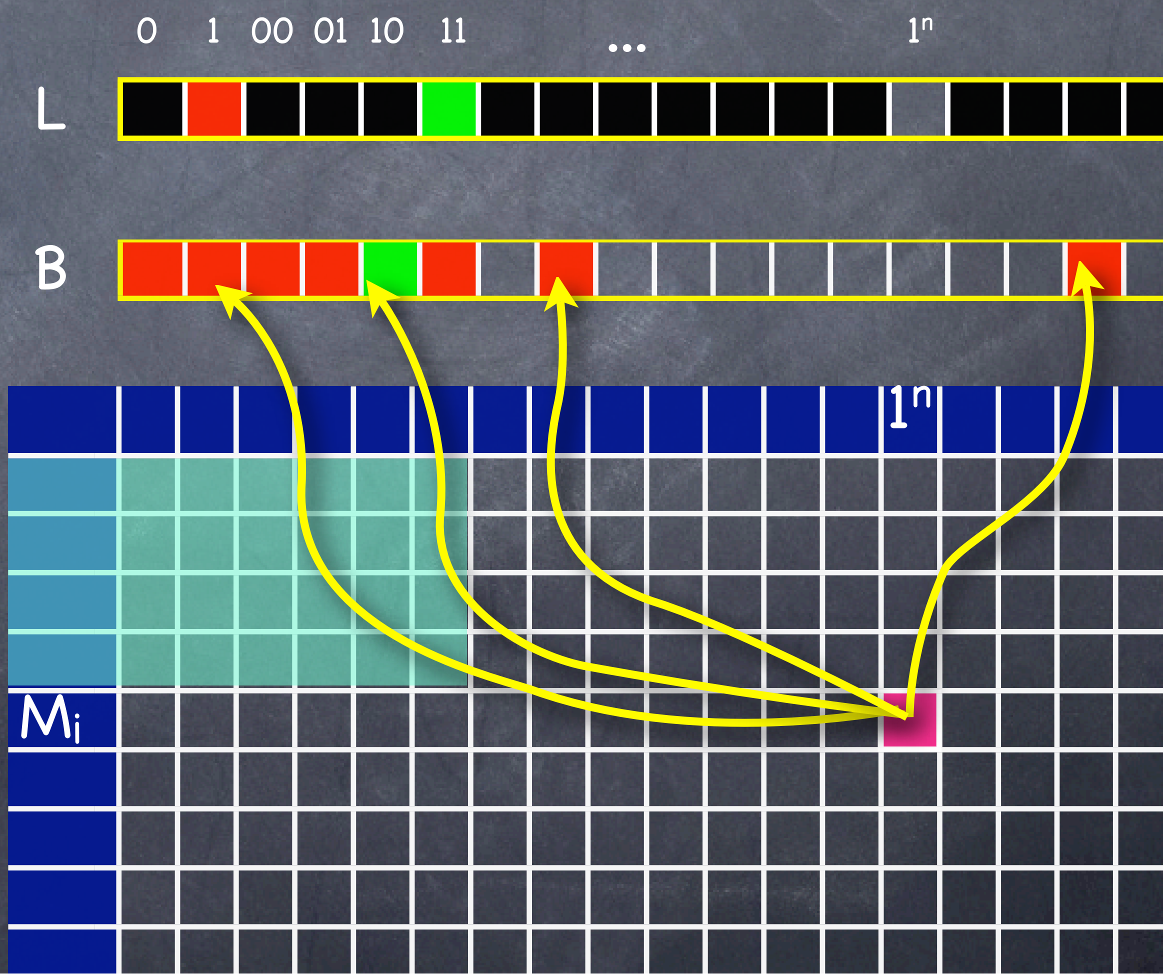

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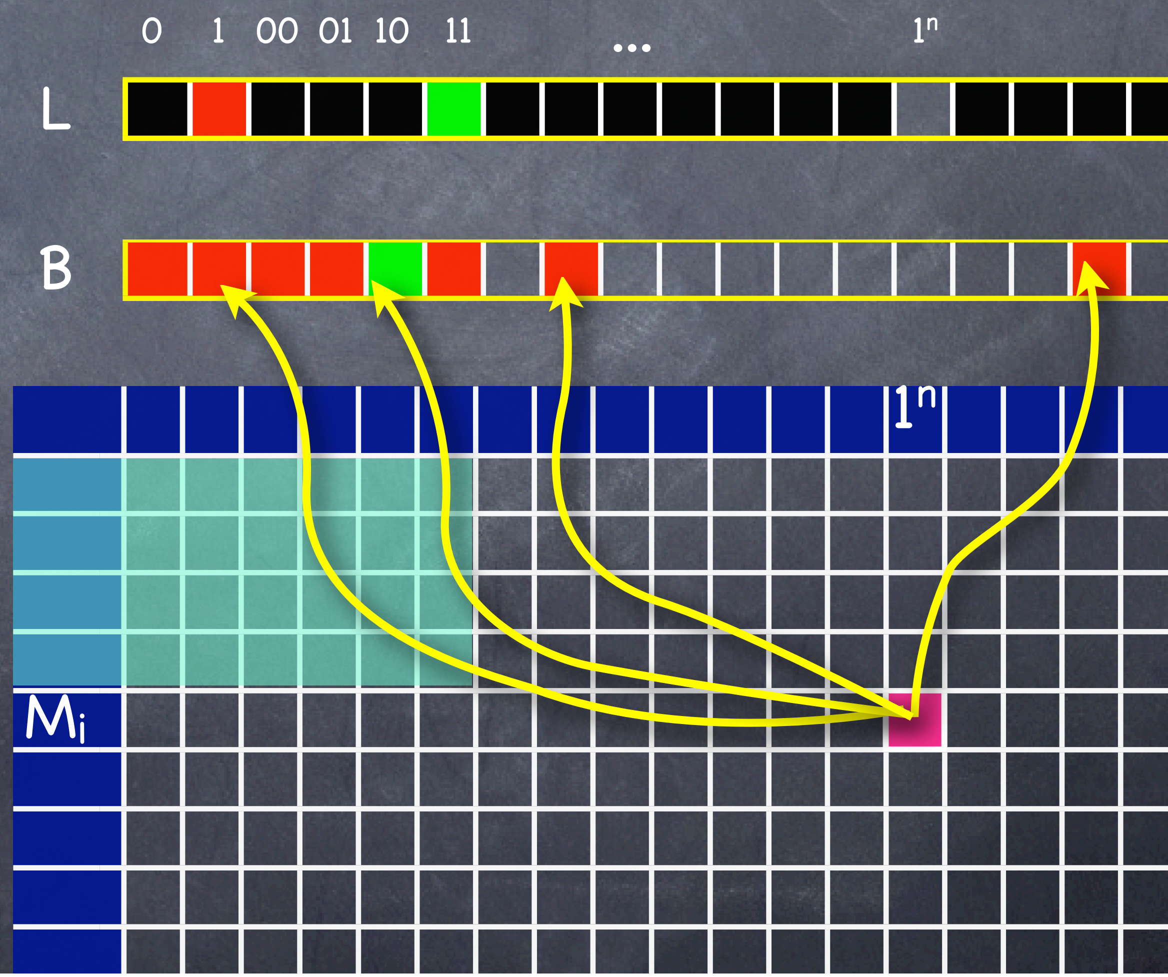

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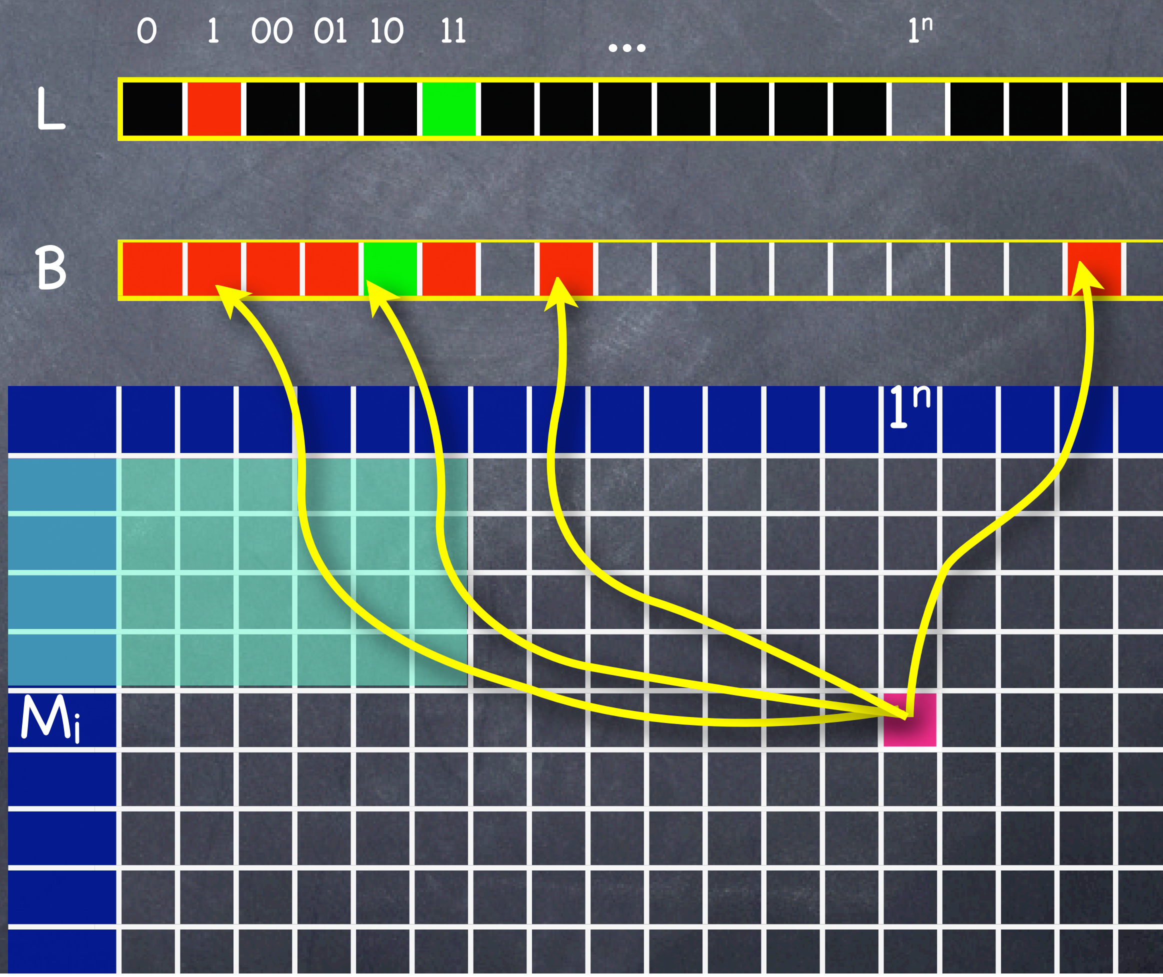

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  - e.g. of non-relativizing proof: that of Cook-Levin theorem

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    - Turns out, often a little memory can go a long way (if we can spare the time)

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- "L and NL are to space what P and NP are to time"

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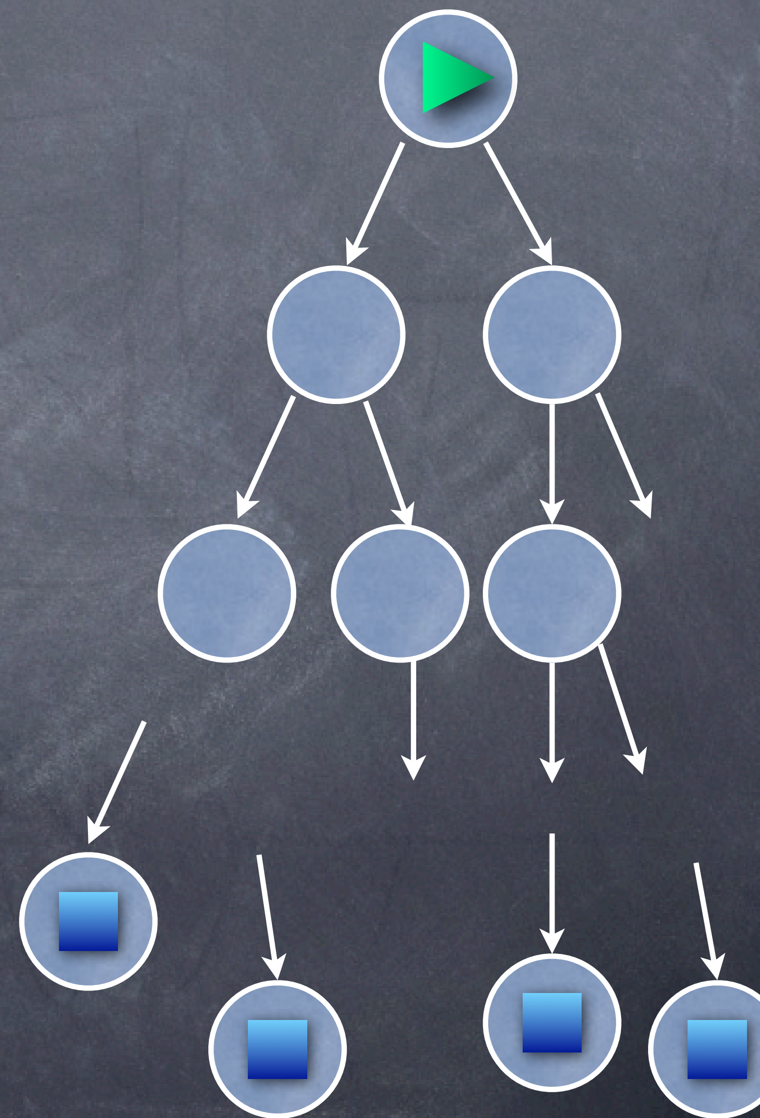

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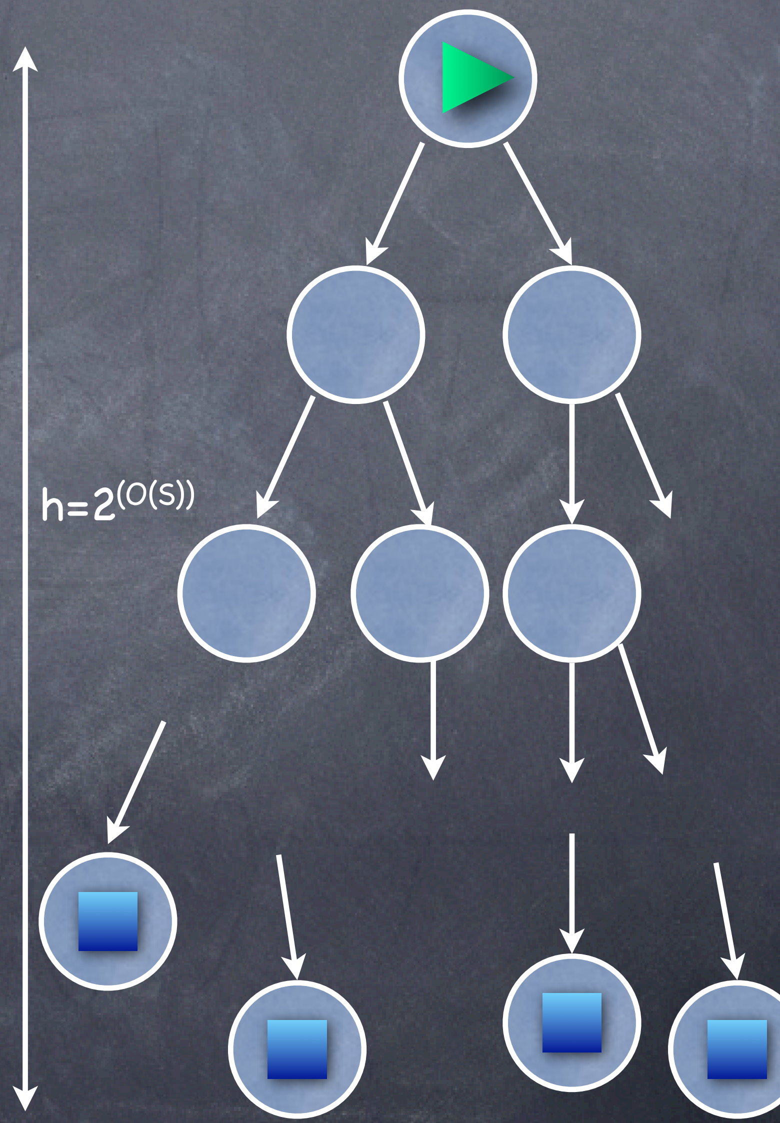

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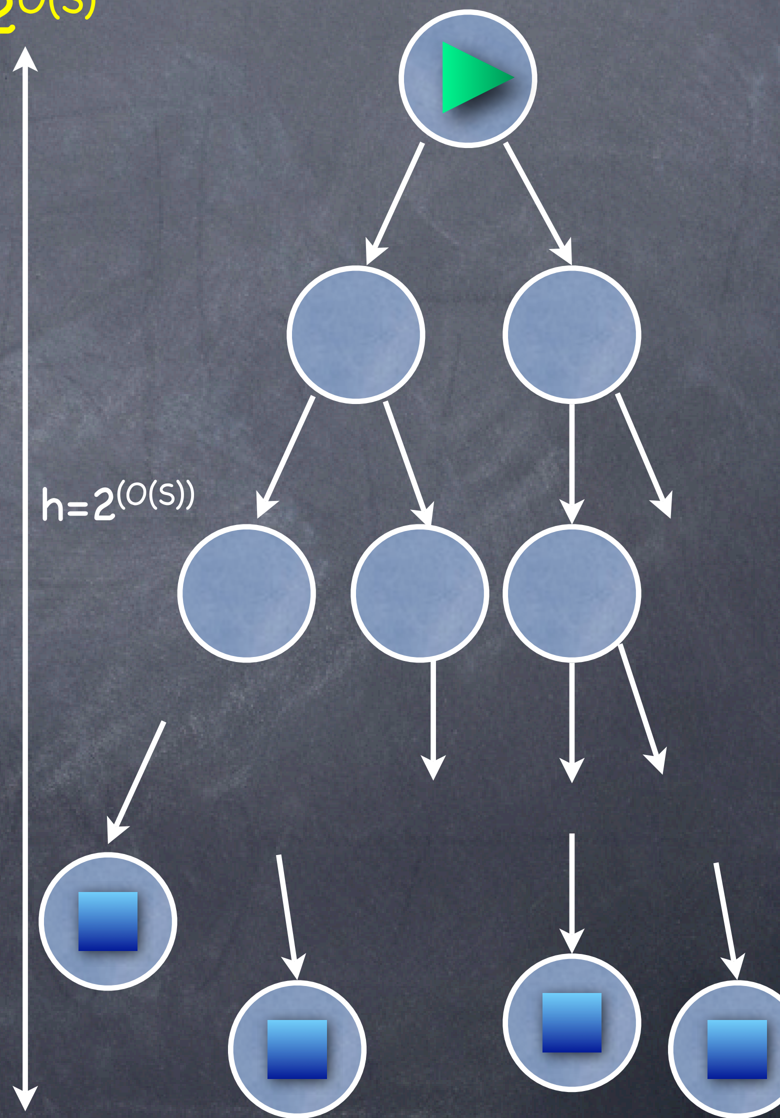

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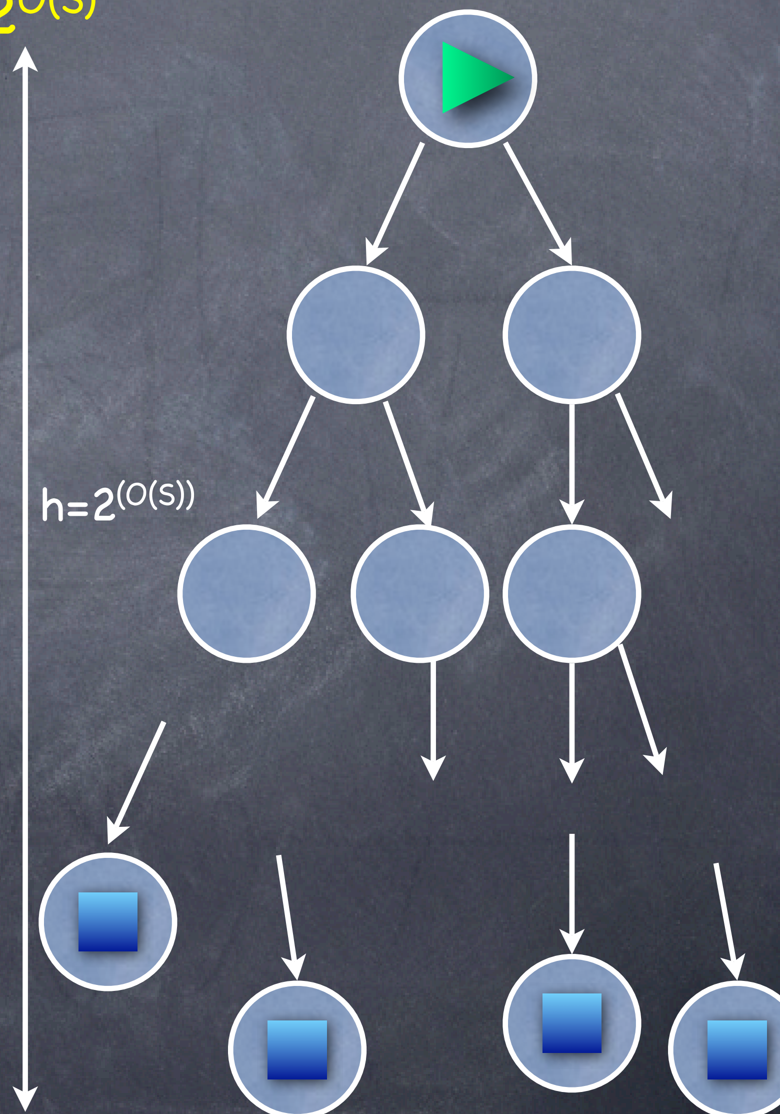

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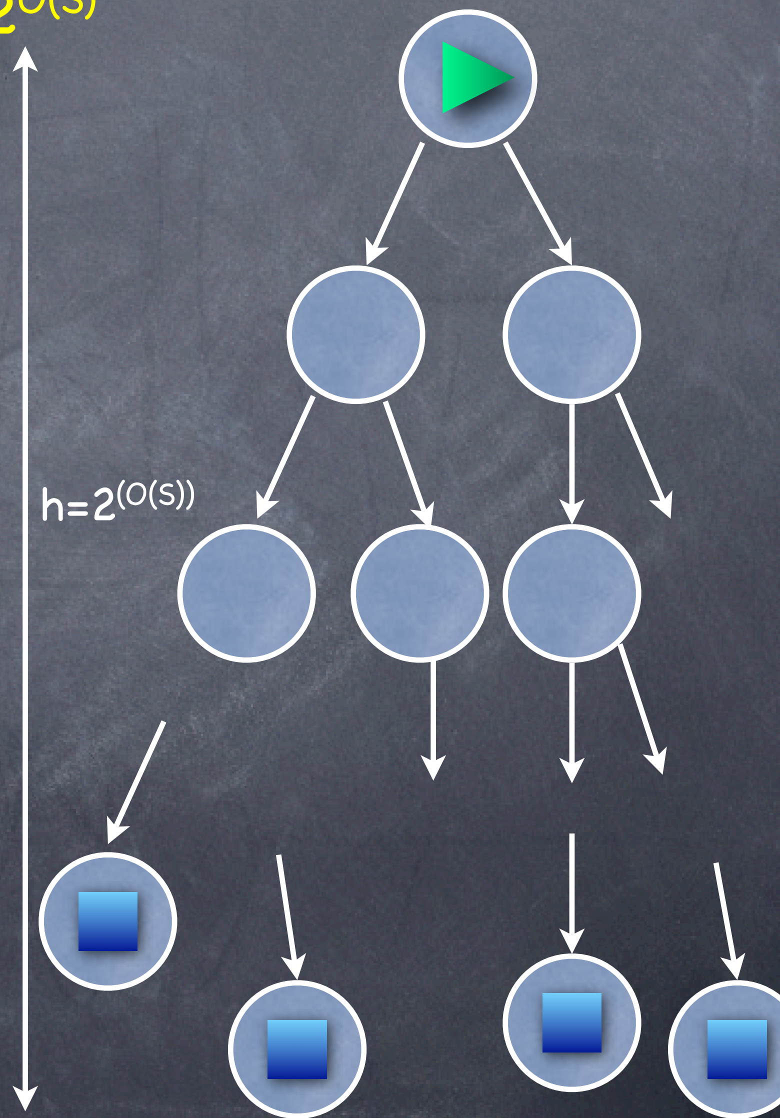

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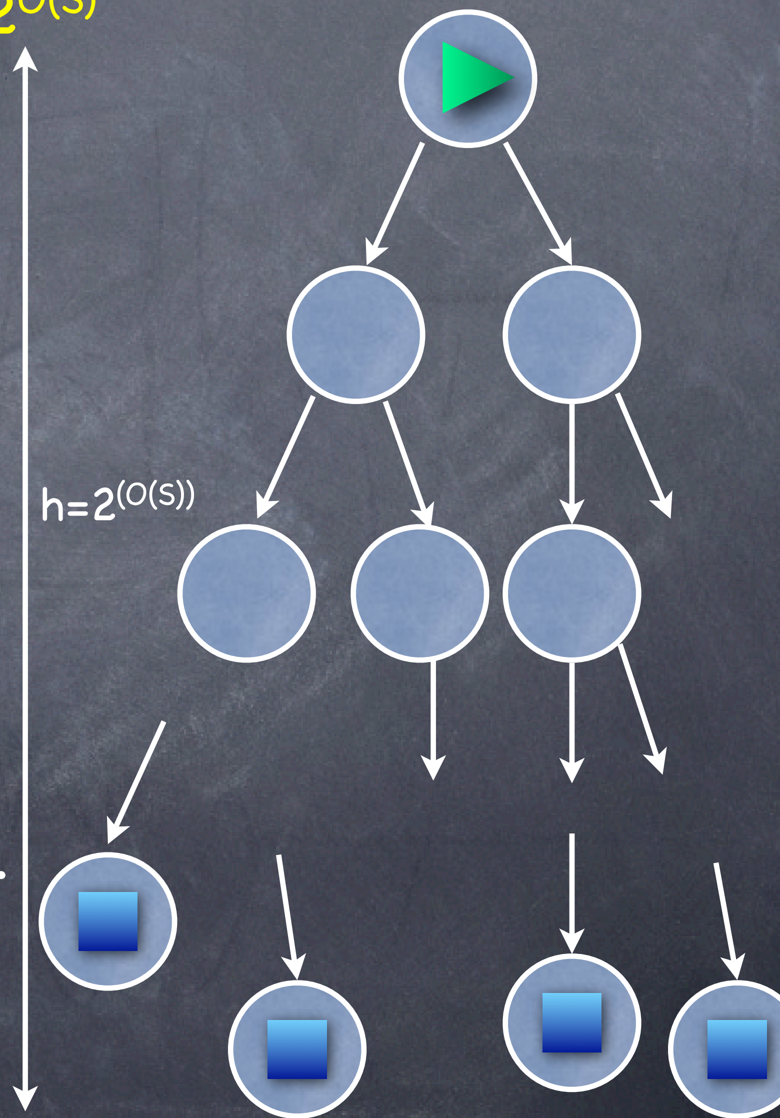

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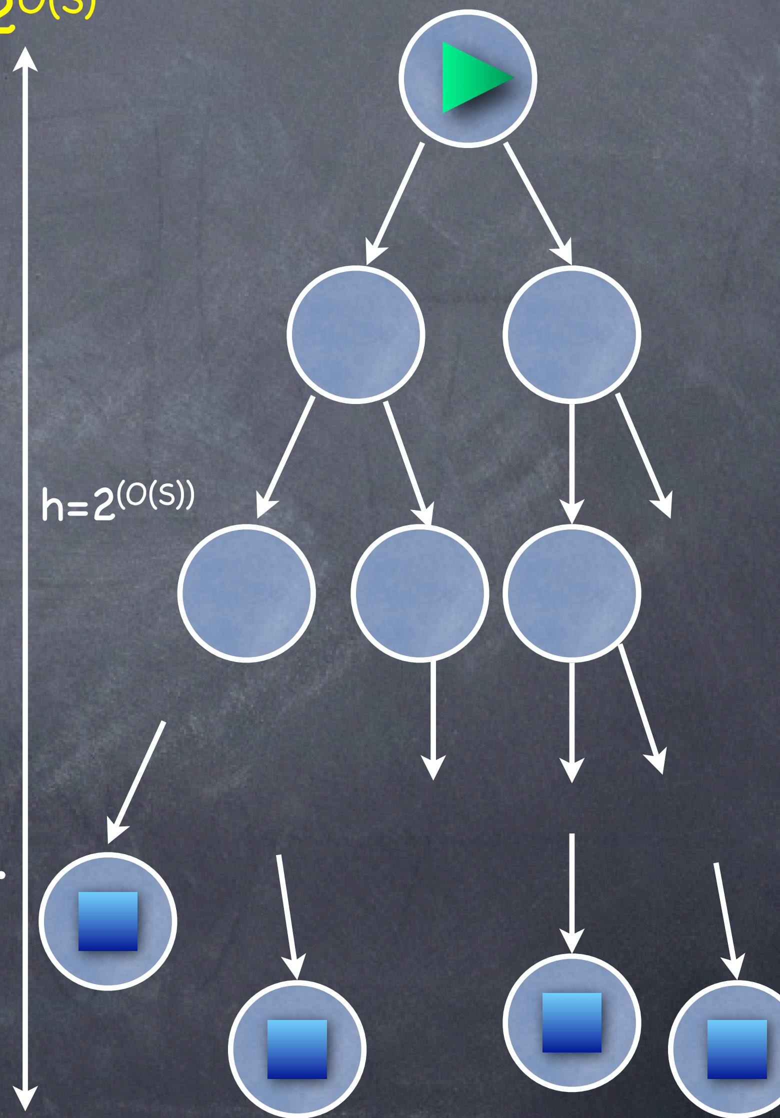

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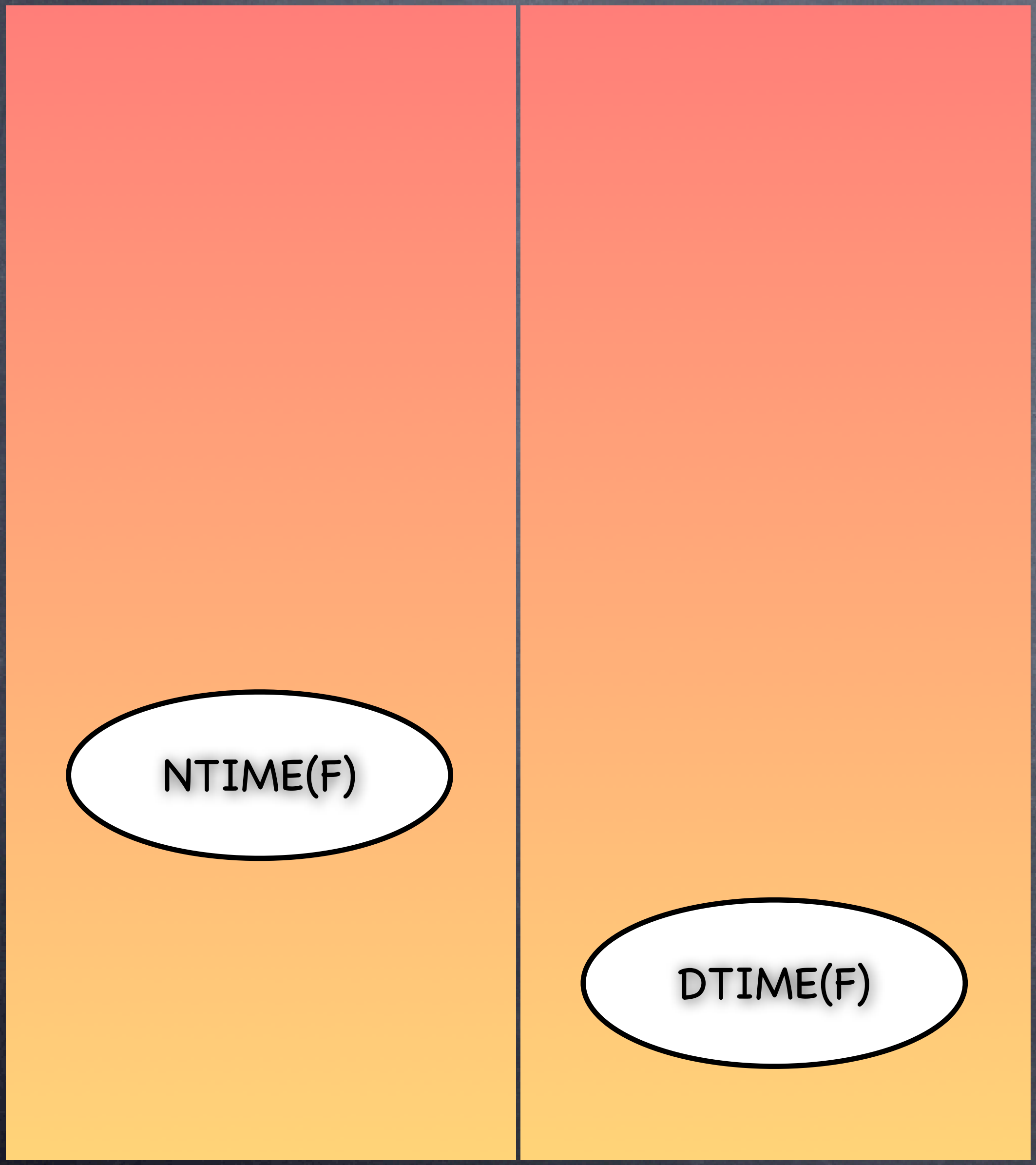

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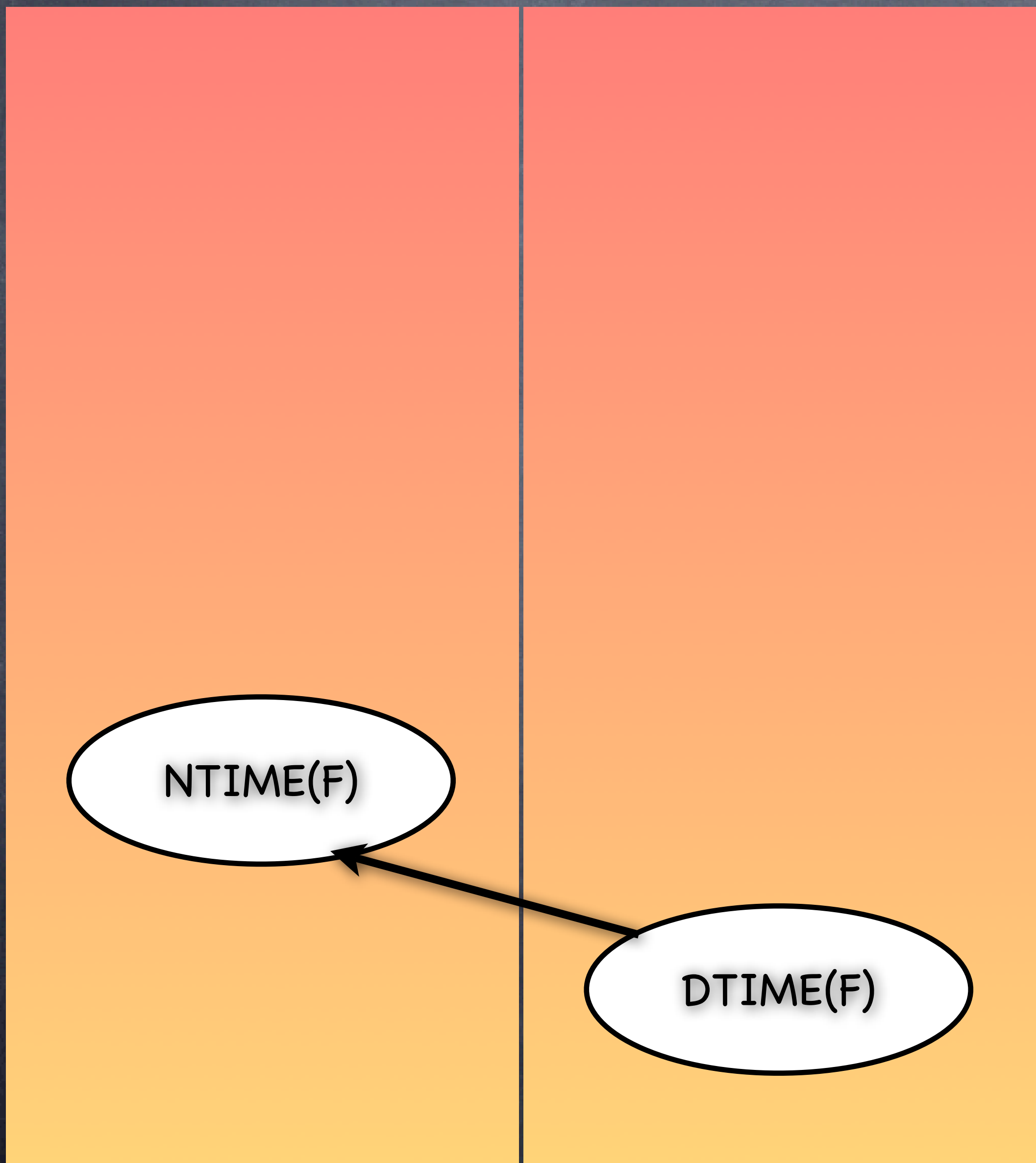

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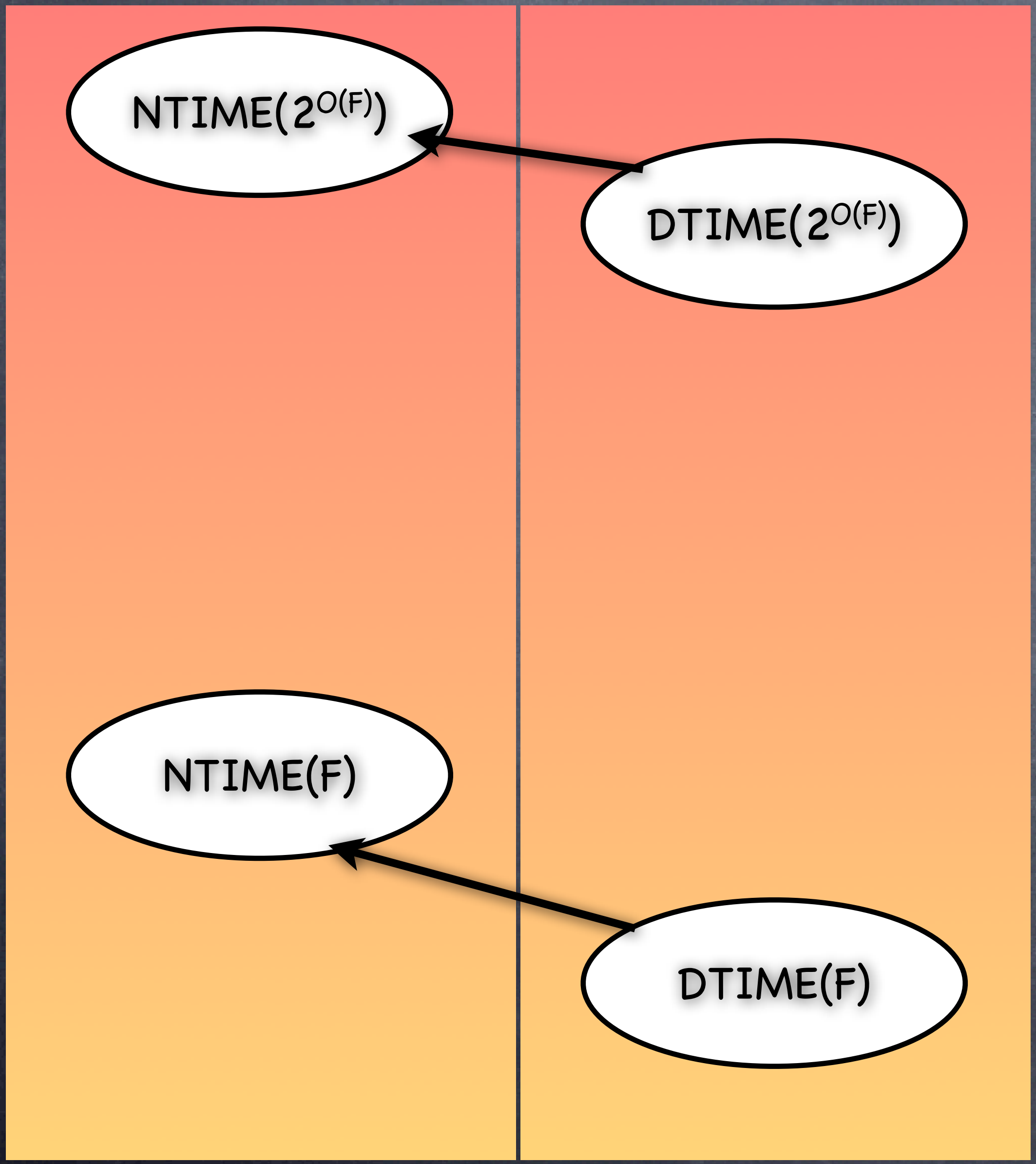

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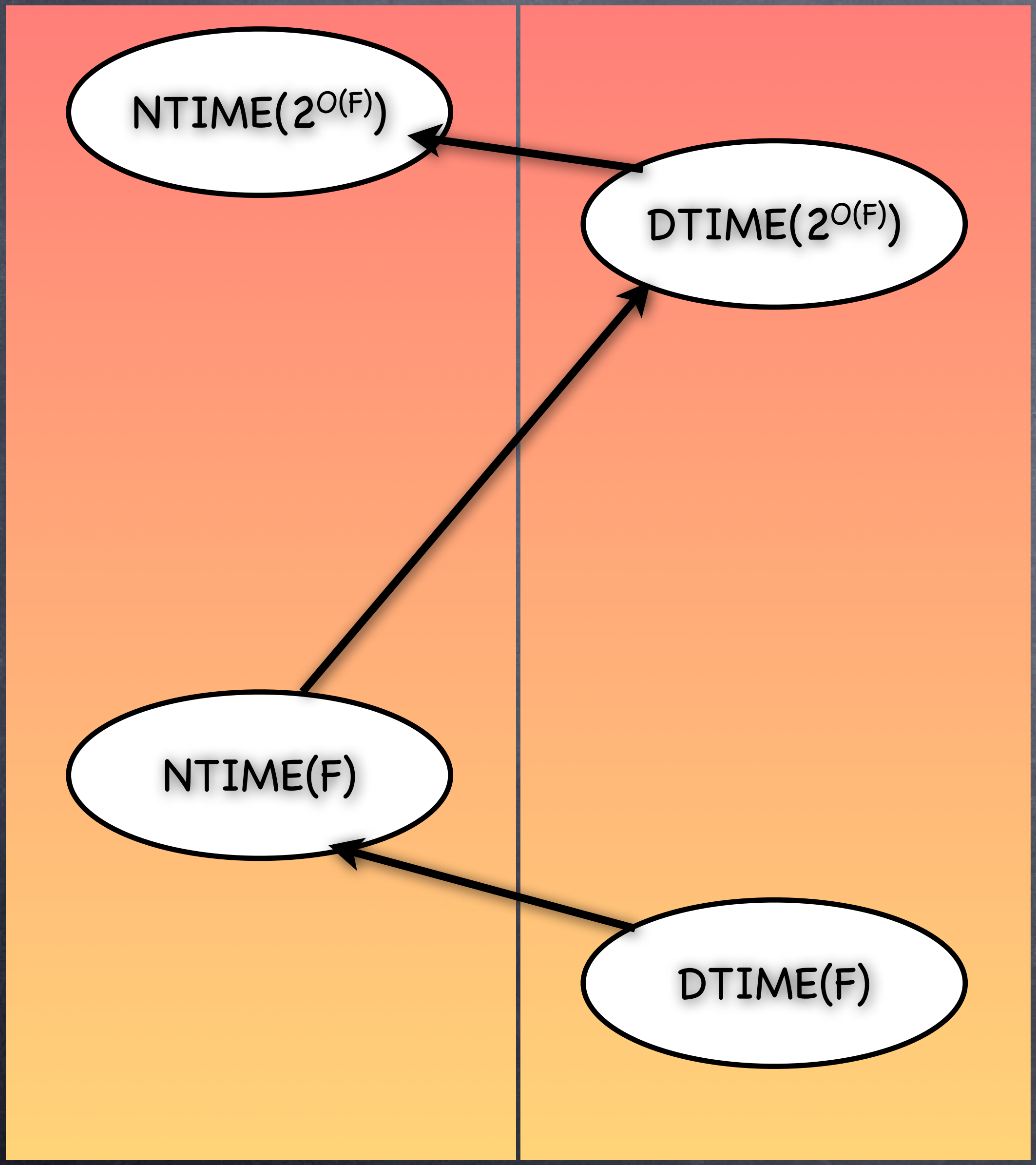

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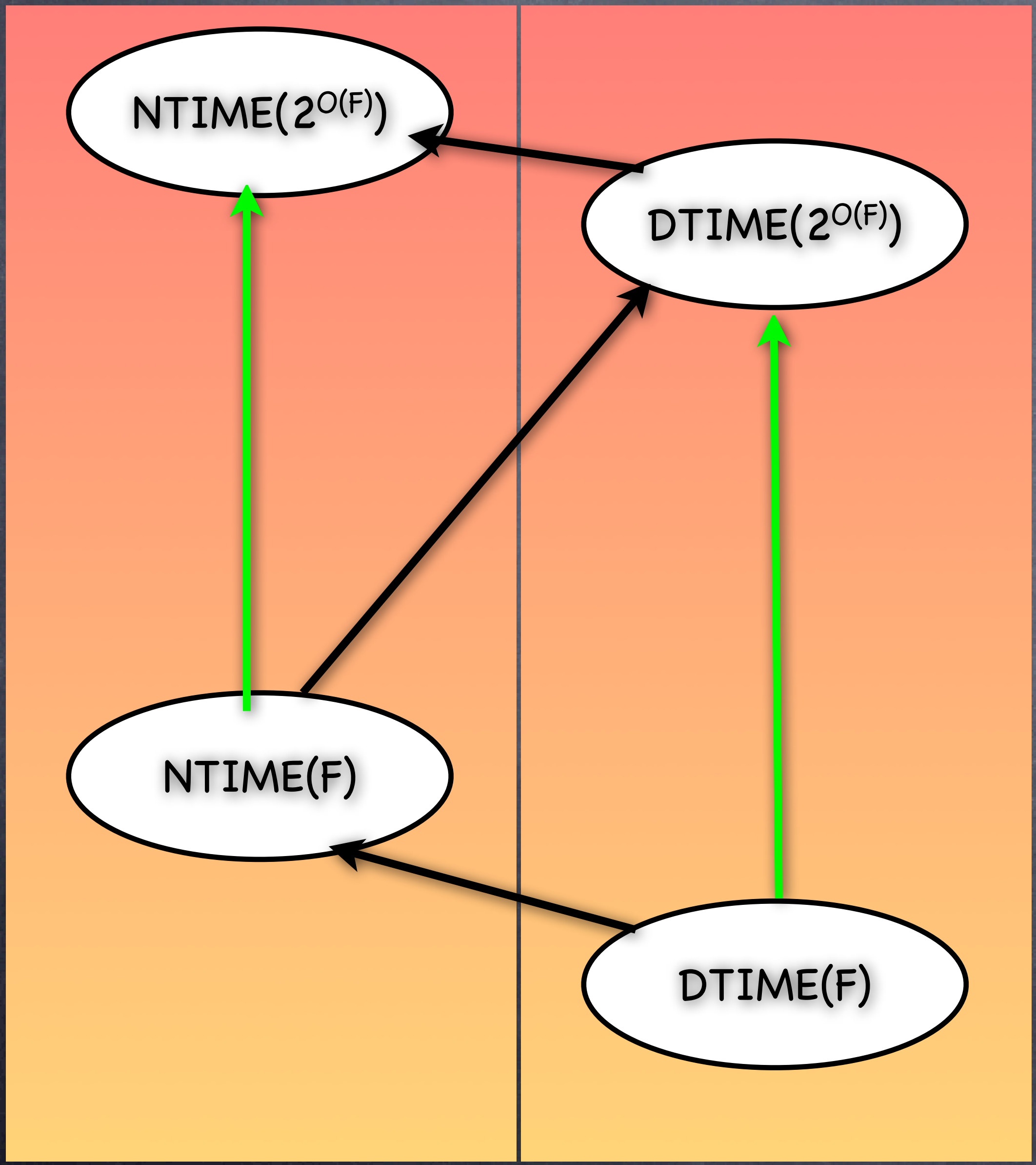

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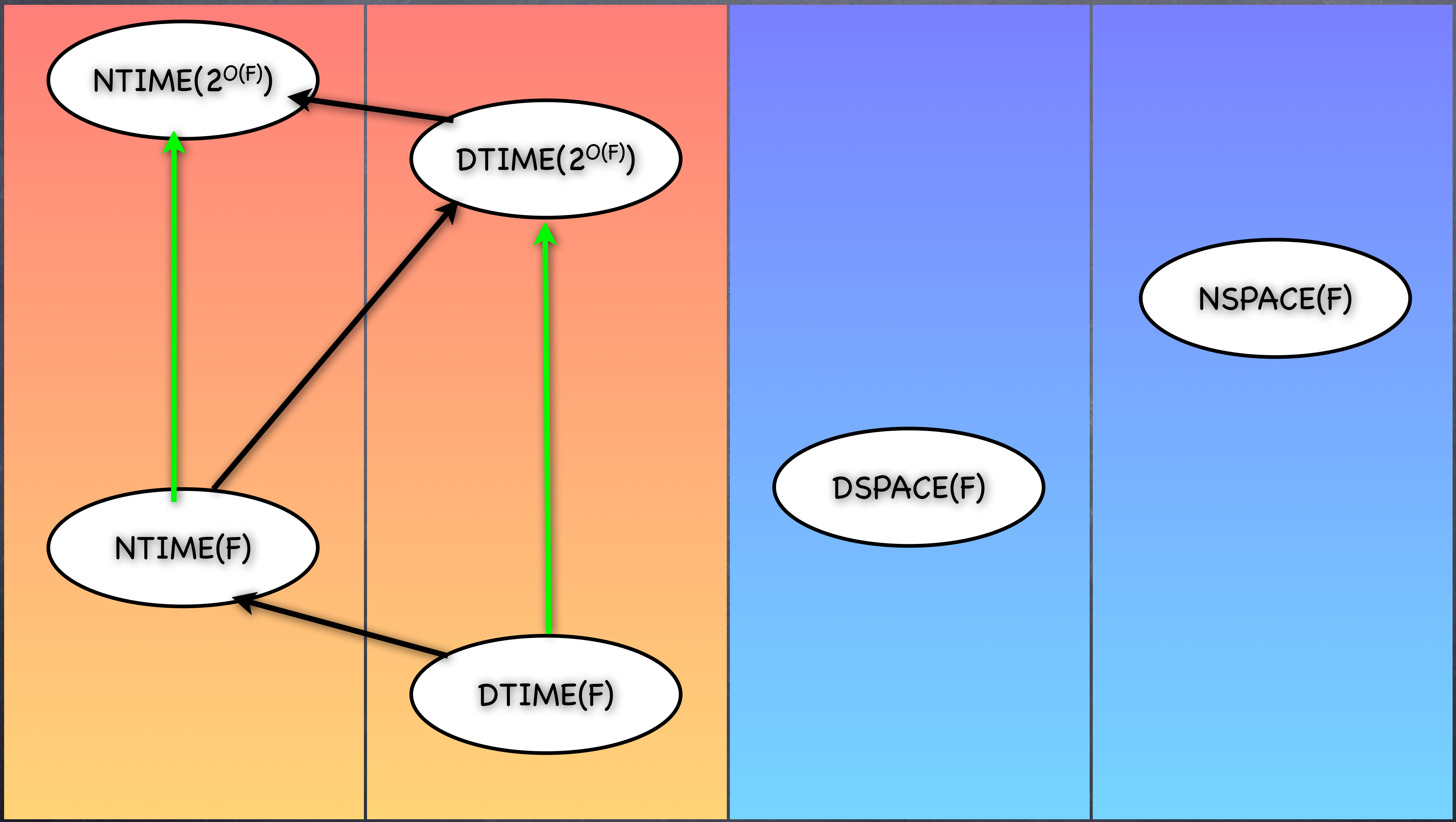

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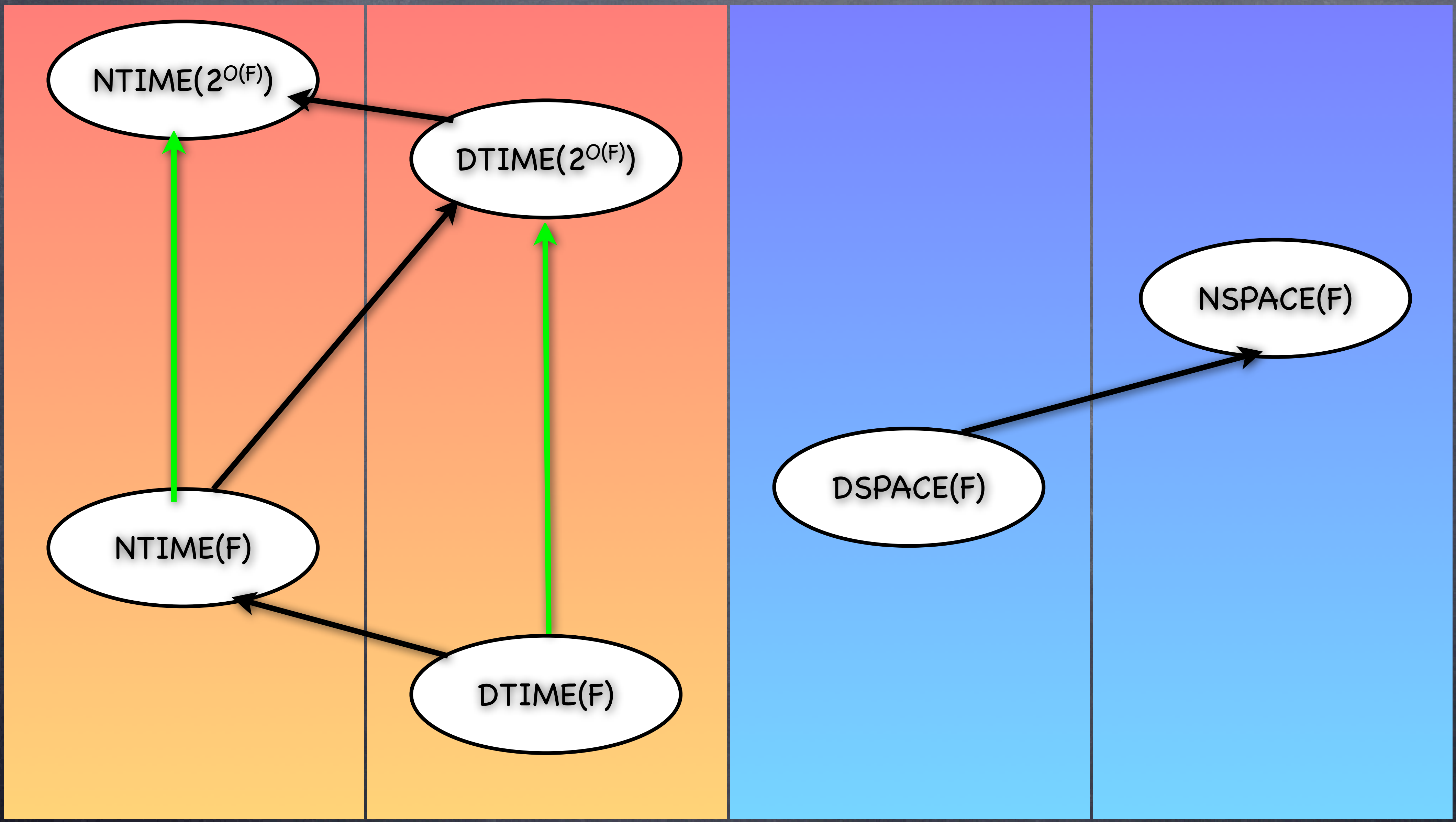

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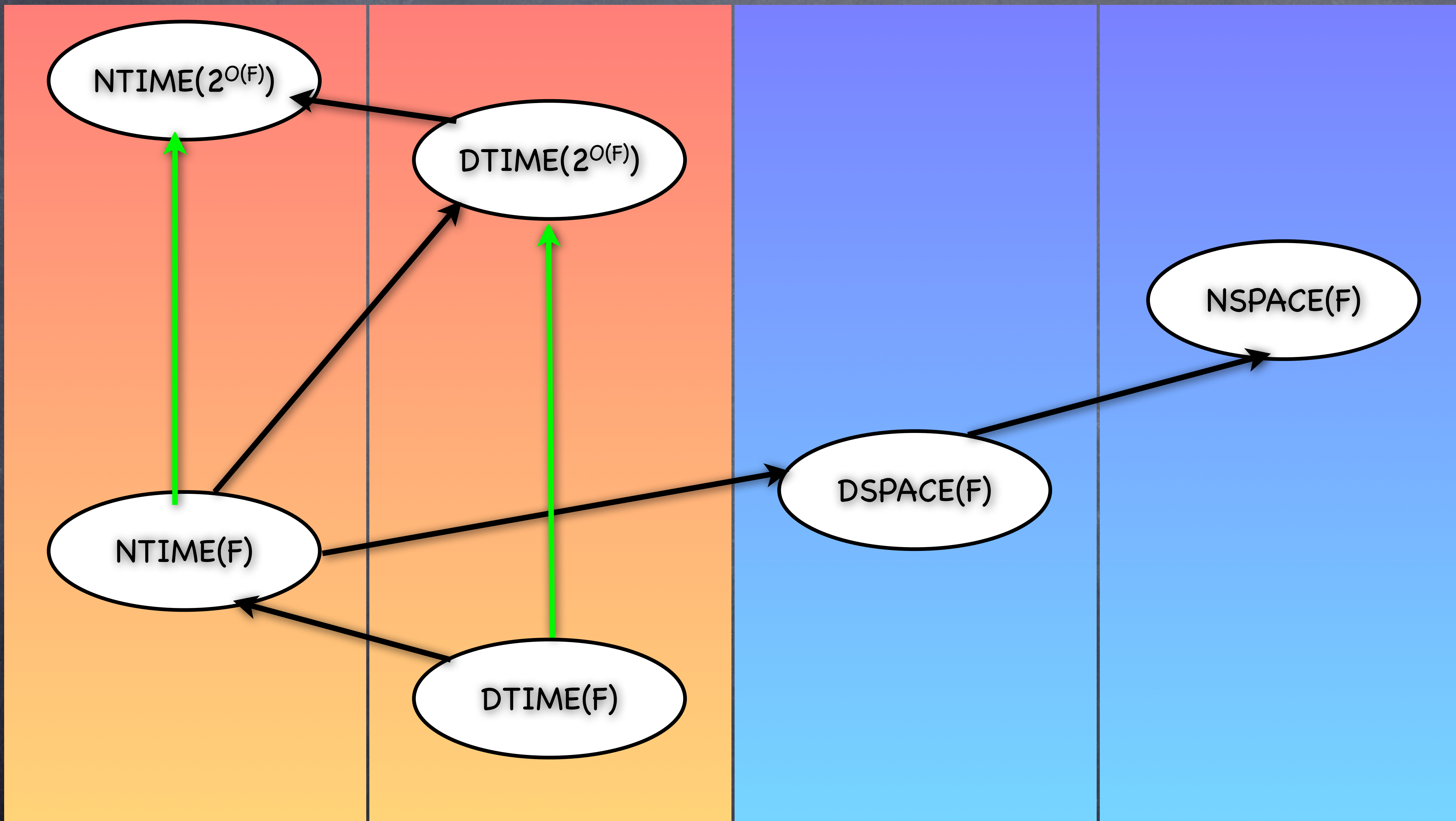

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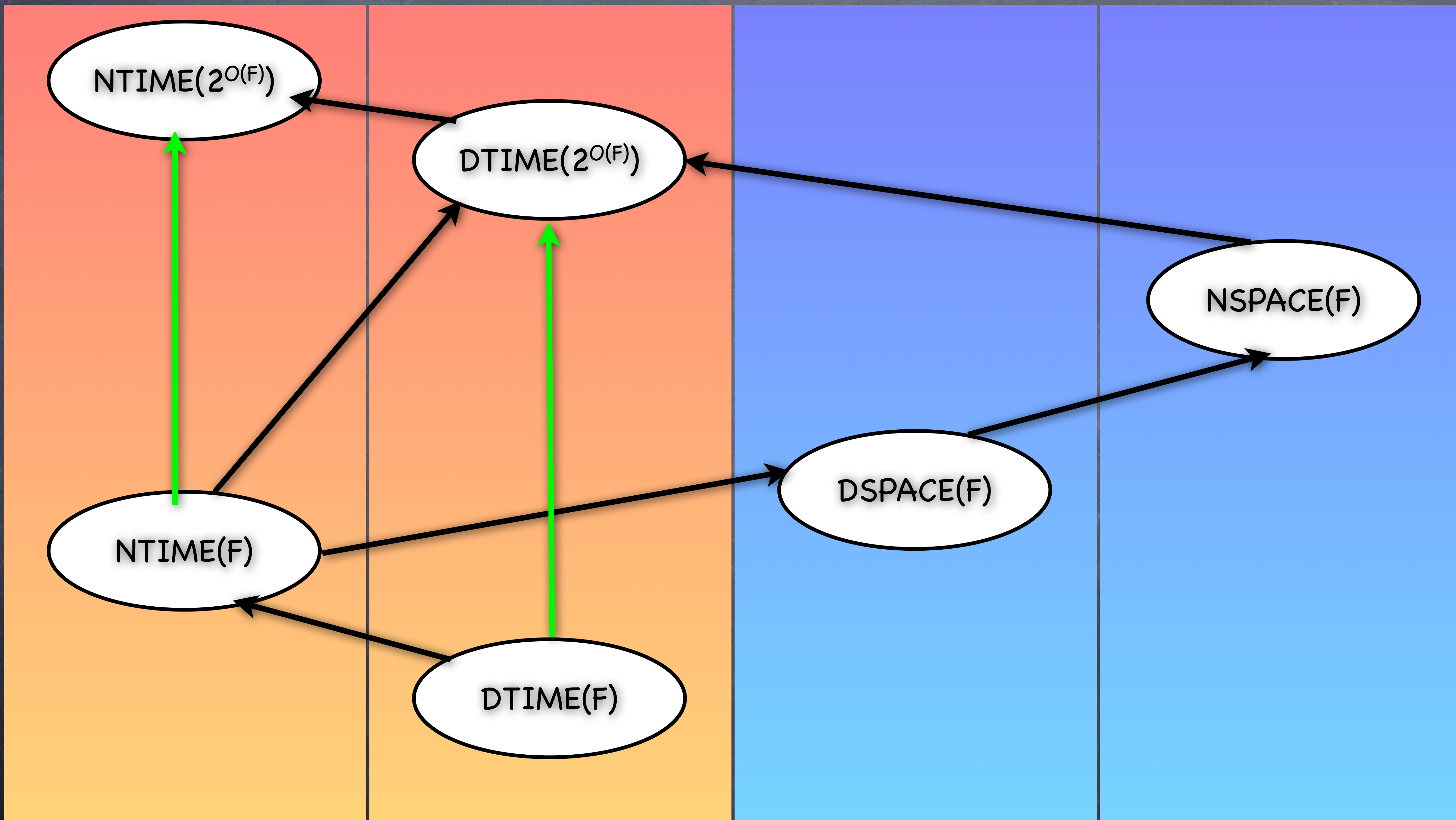

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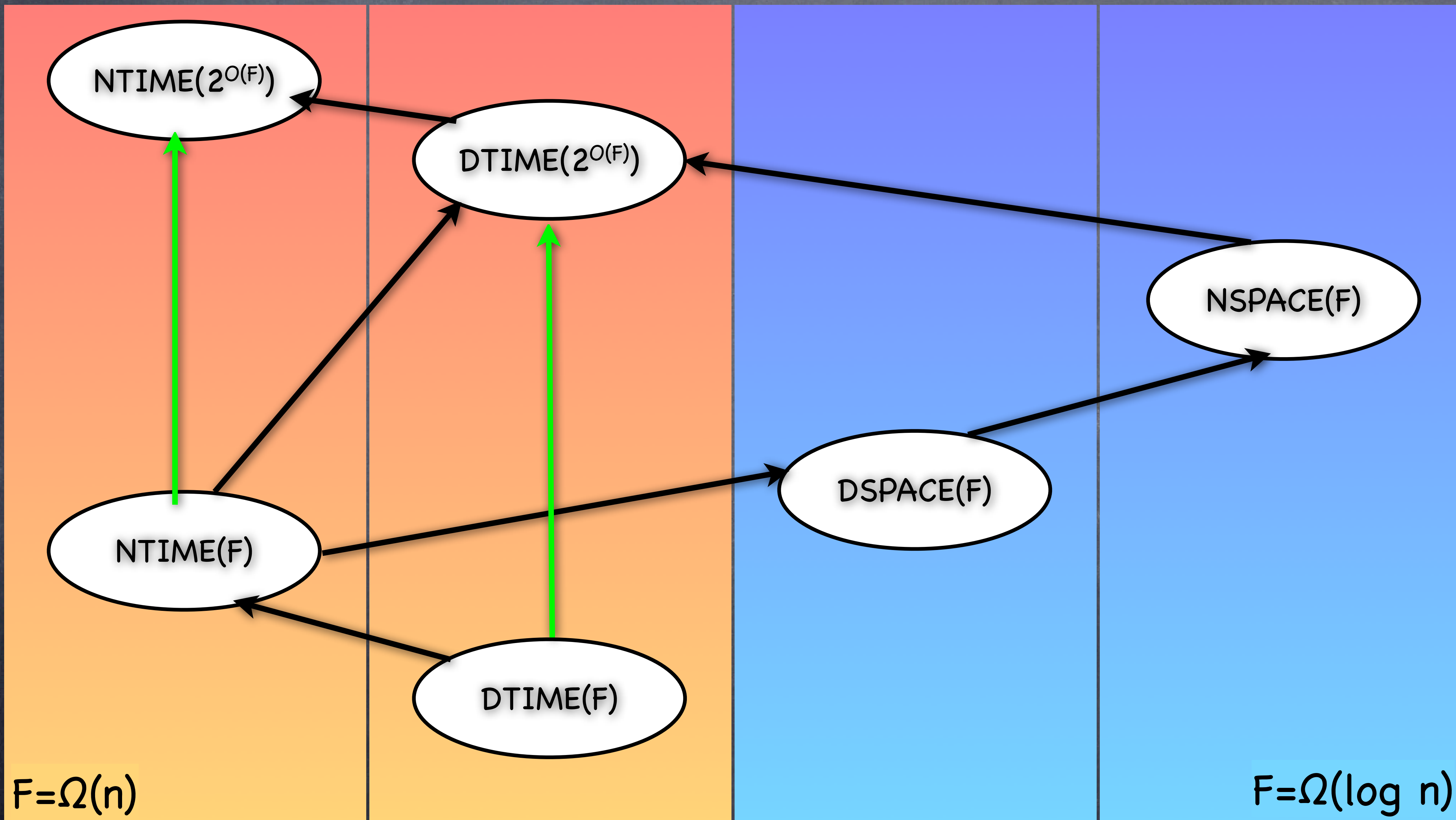

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