Computational Complexity

Lecture 3

in which we come across

Diagonalization and Time-hierarchies

(But first some more of NP-completeness)

NP-Complete Languages

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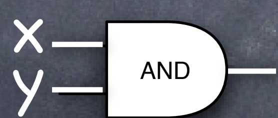
    - e.g. 

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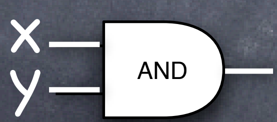
    - e.g. 

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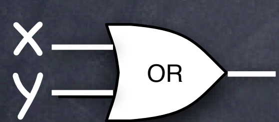
    - and 

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- Reduction not parsimonious (can you make it? [Exercise])

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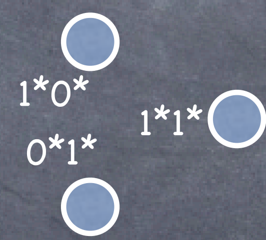

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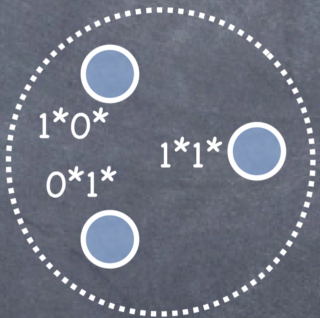

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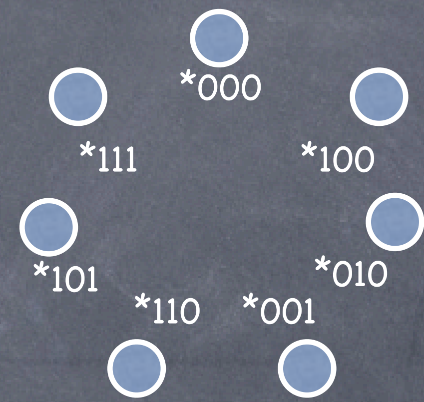

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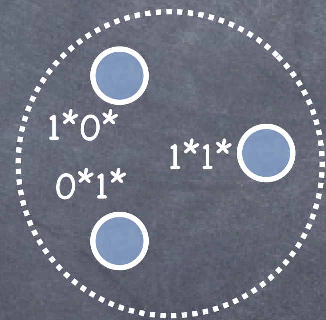

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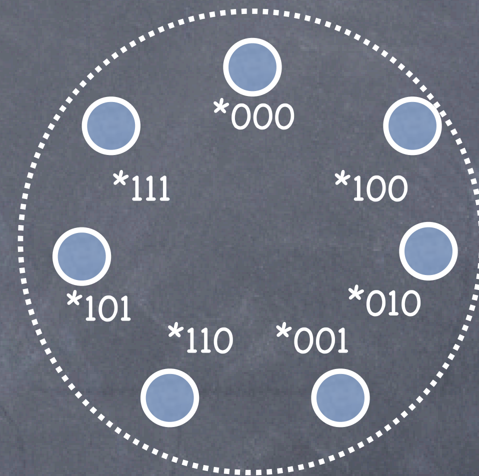

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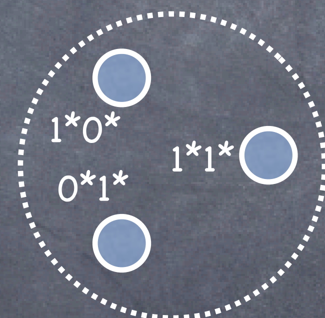

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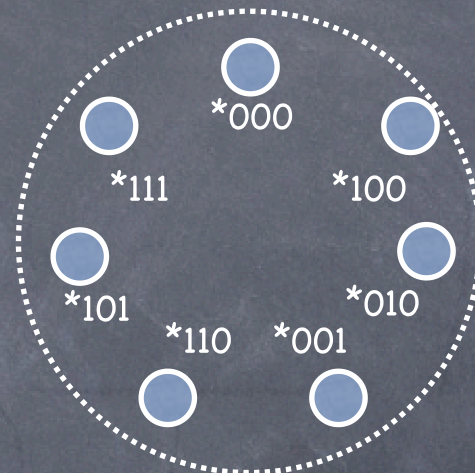

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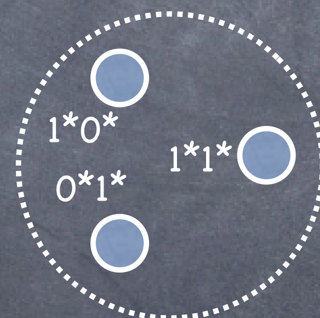

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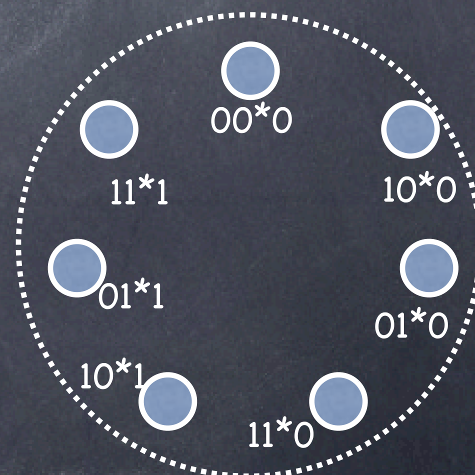

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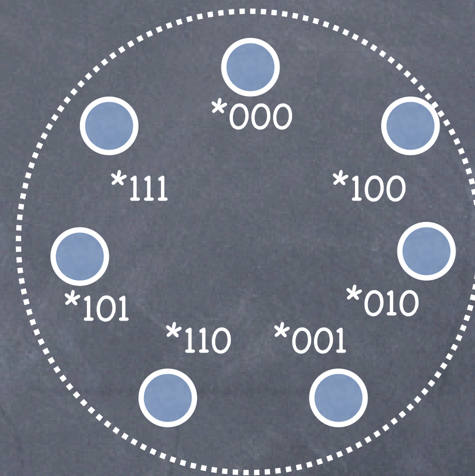

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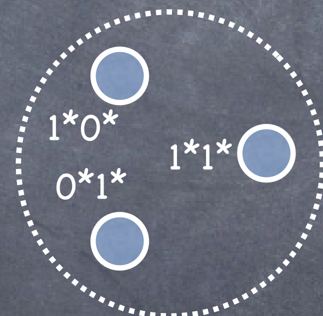

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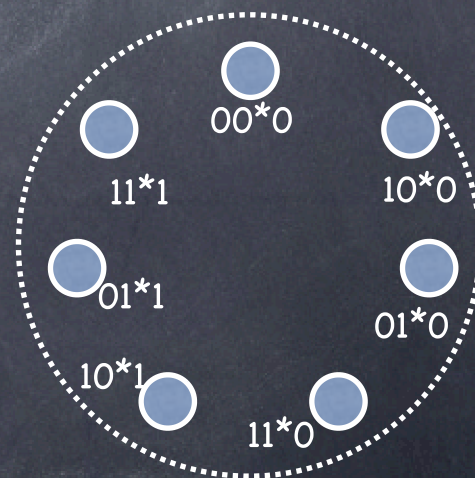

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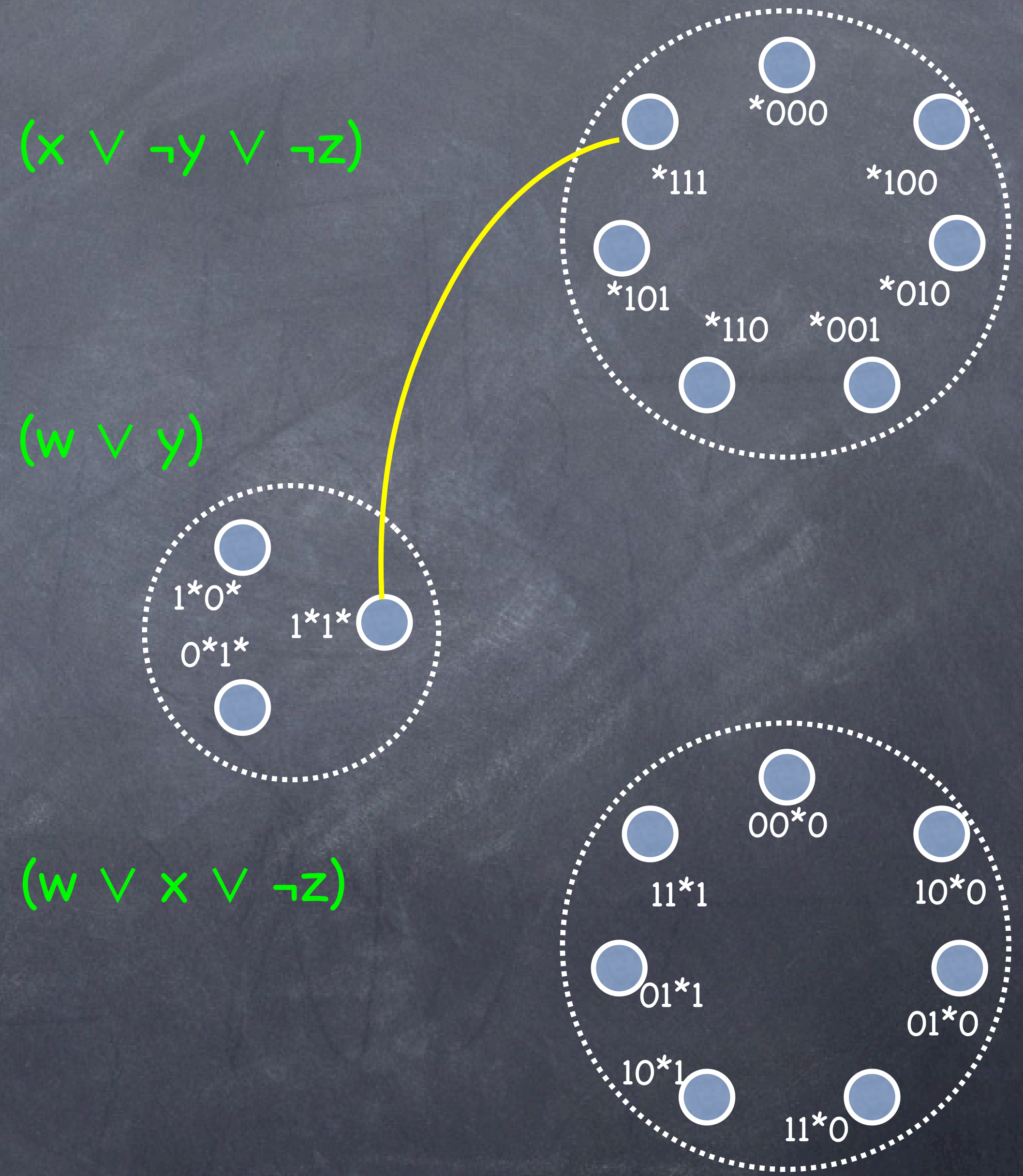

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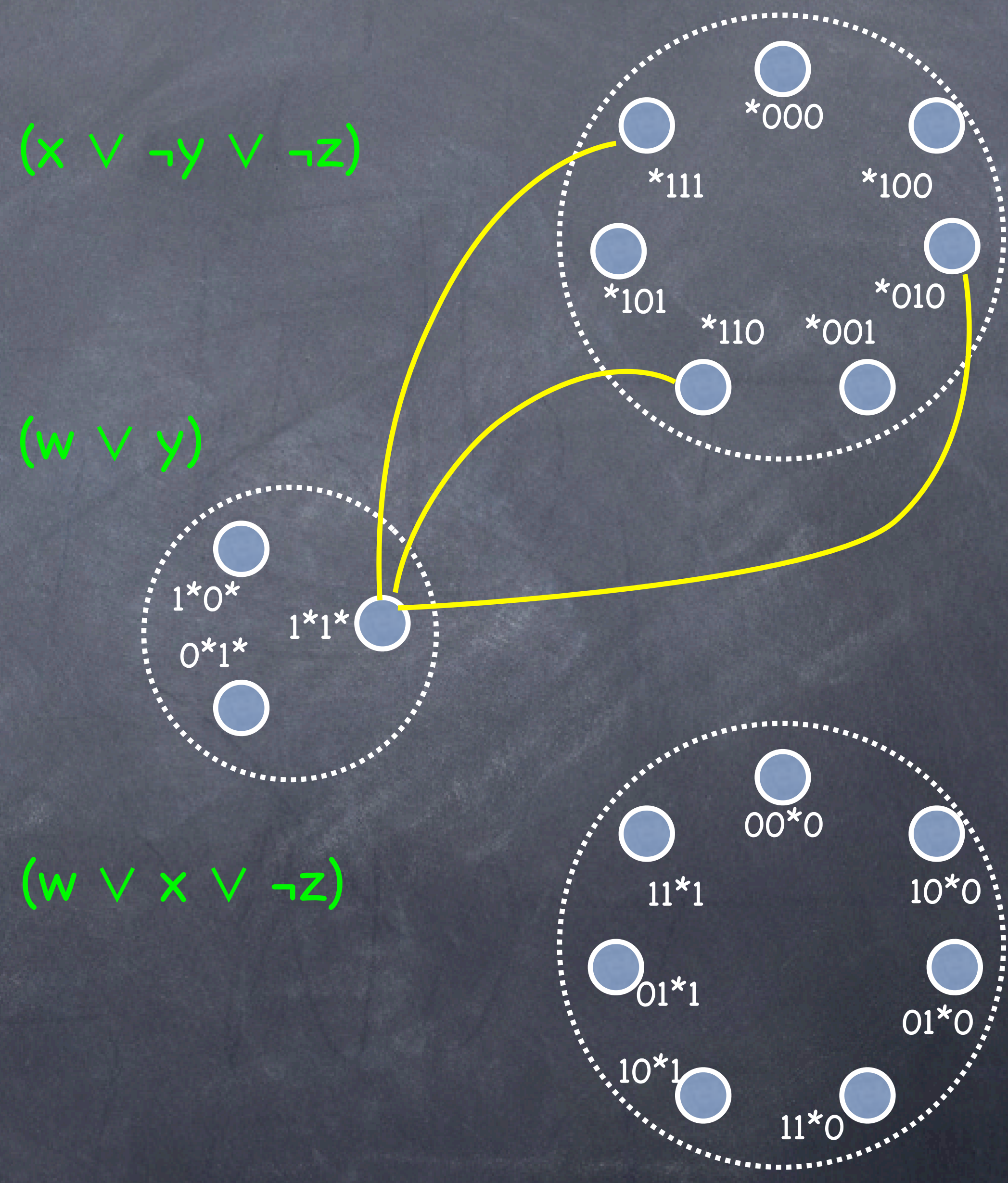

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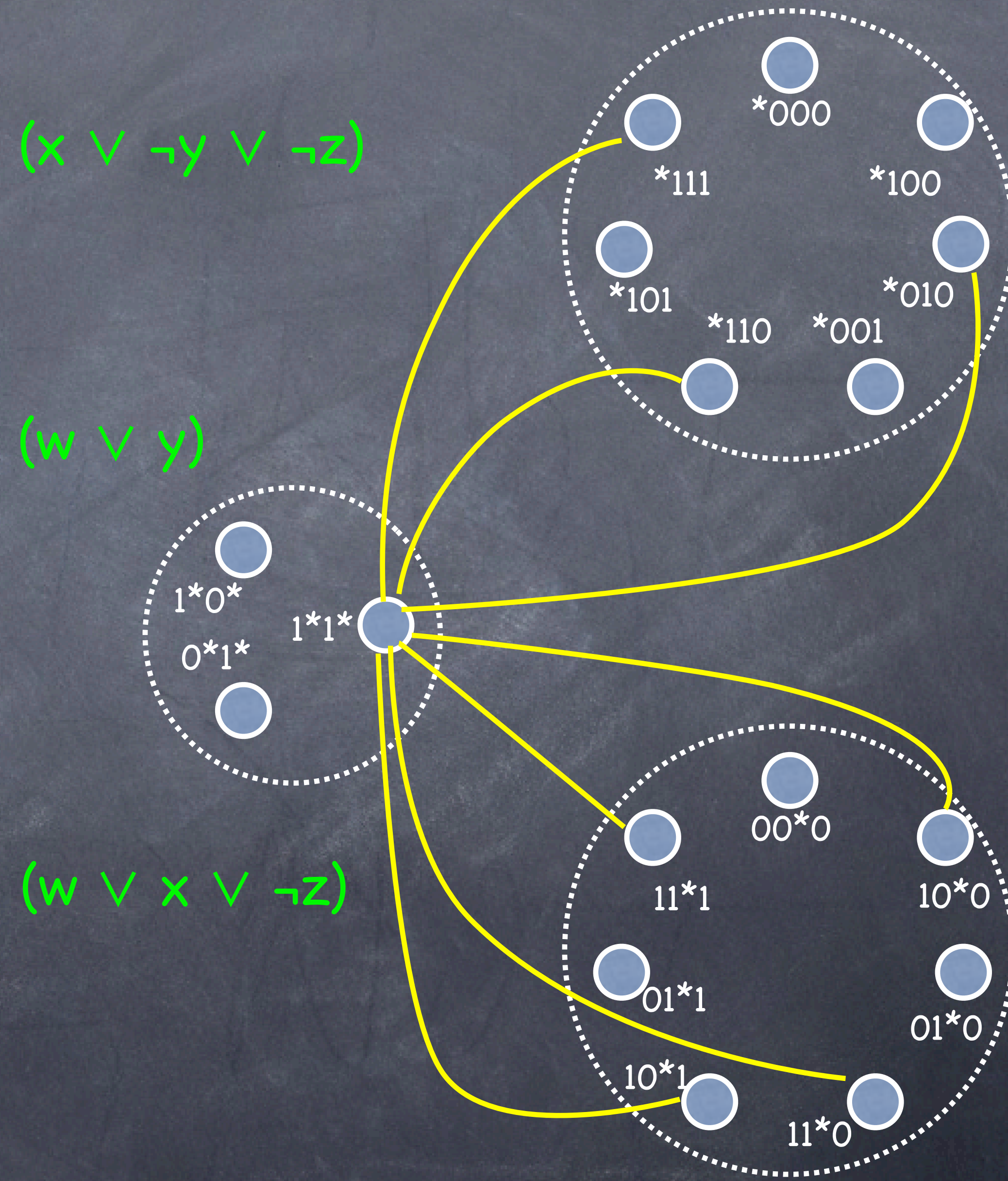

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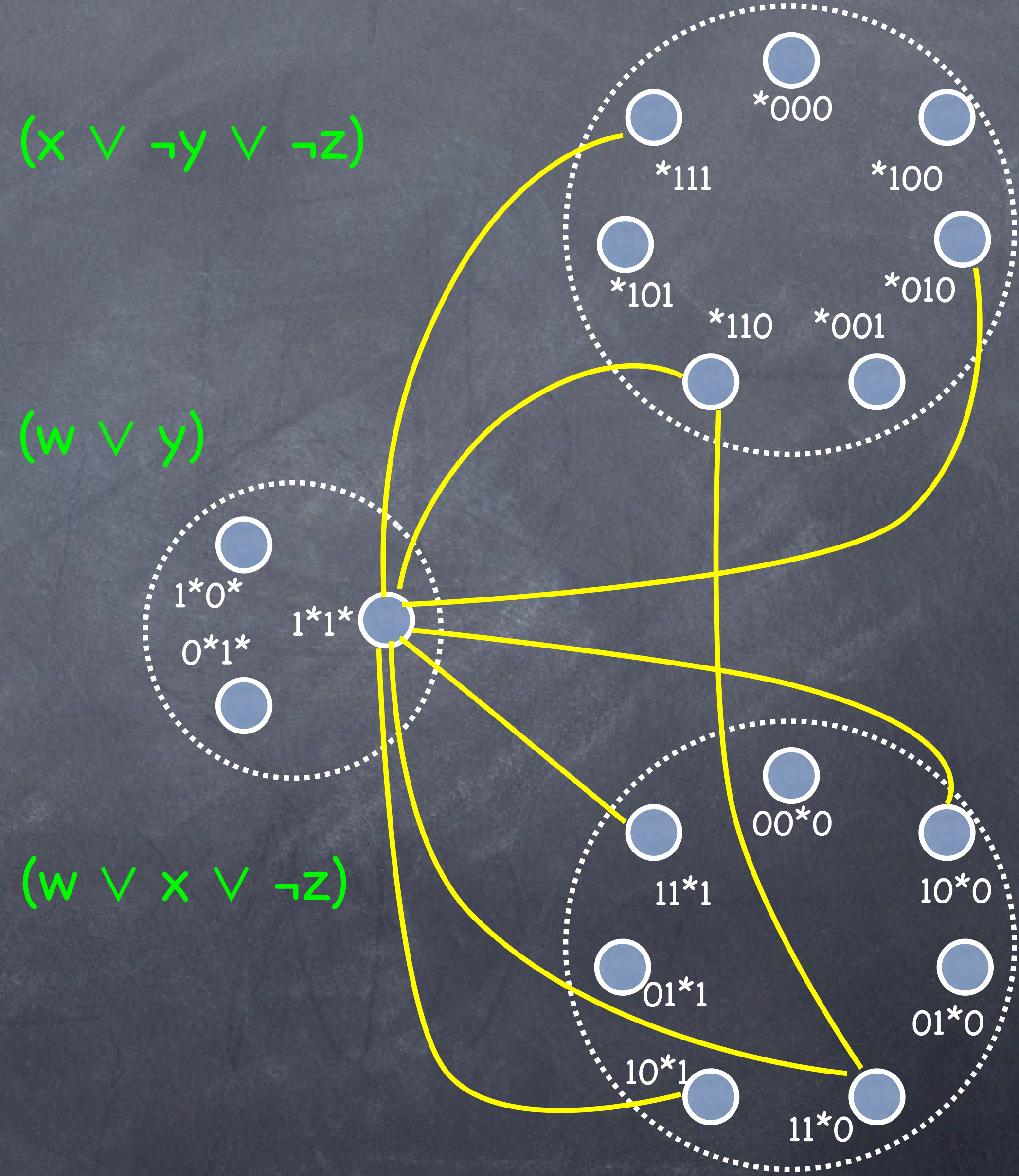

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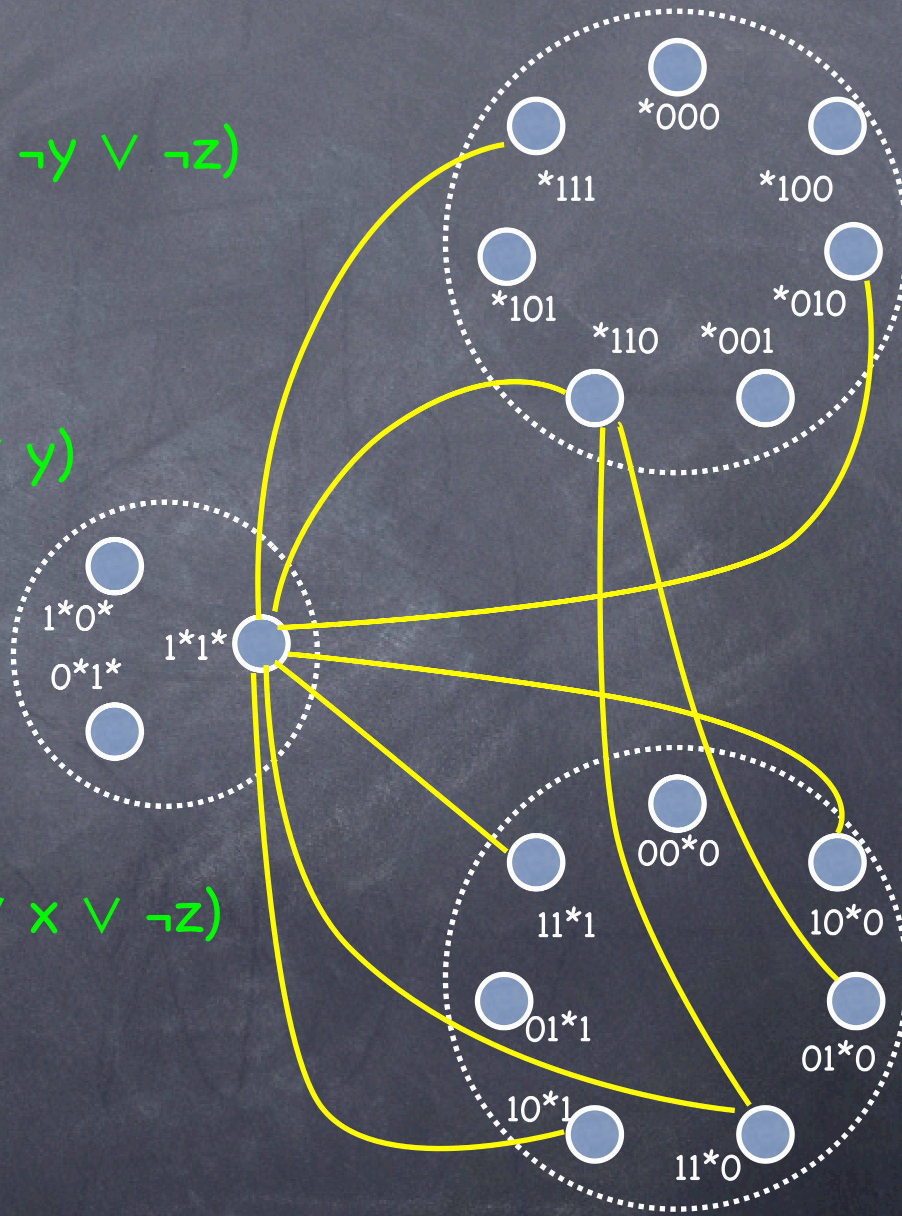

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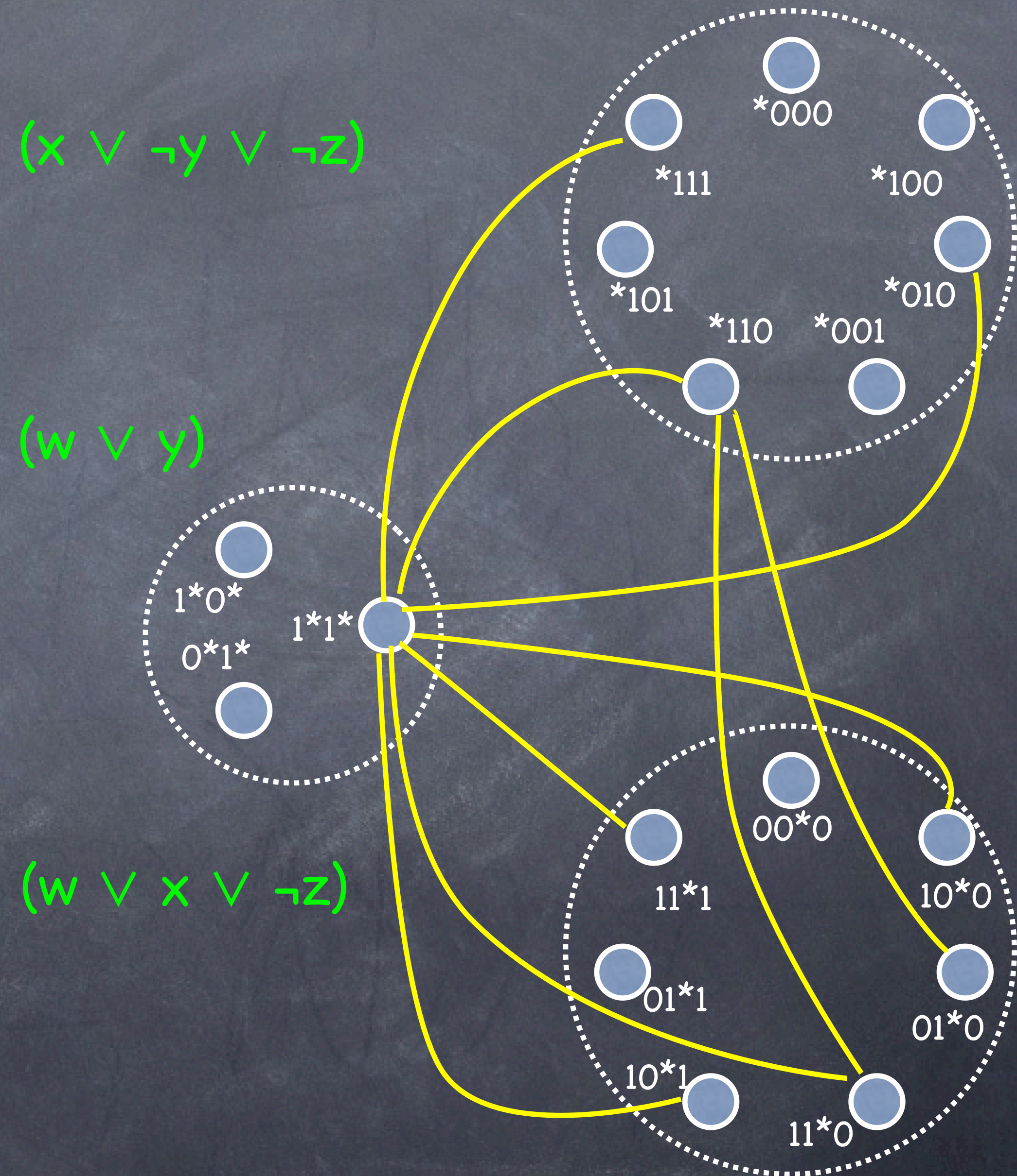

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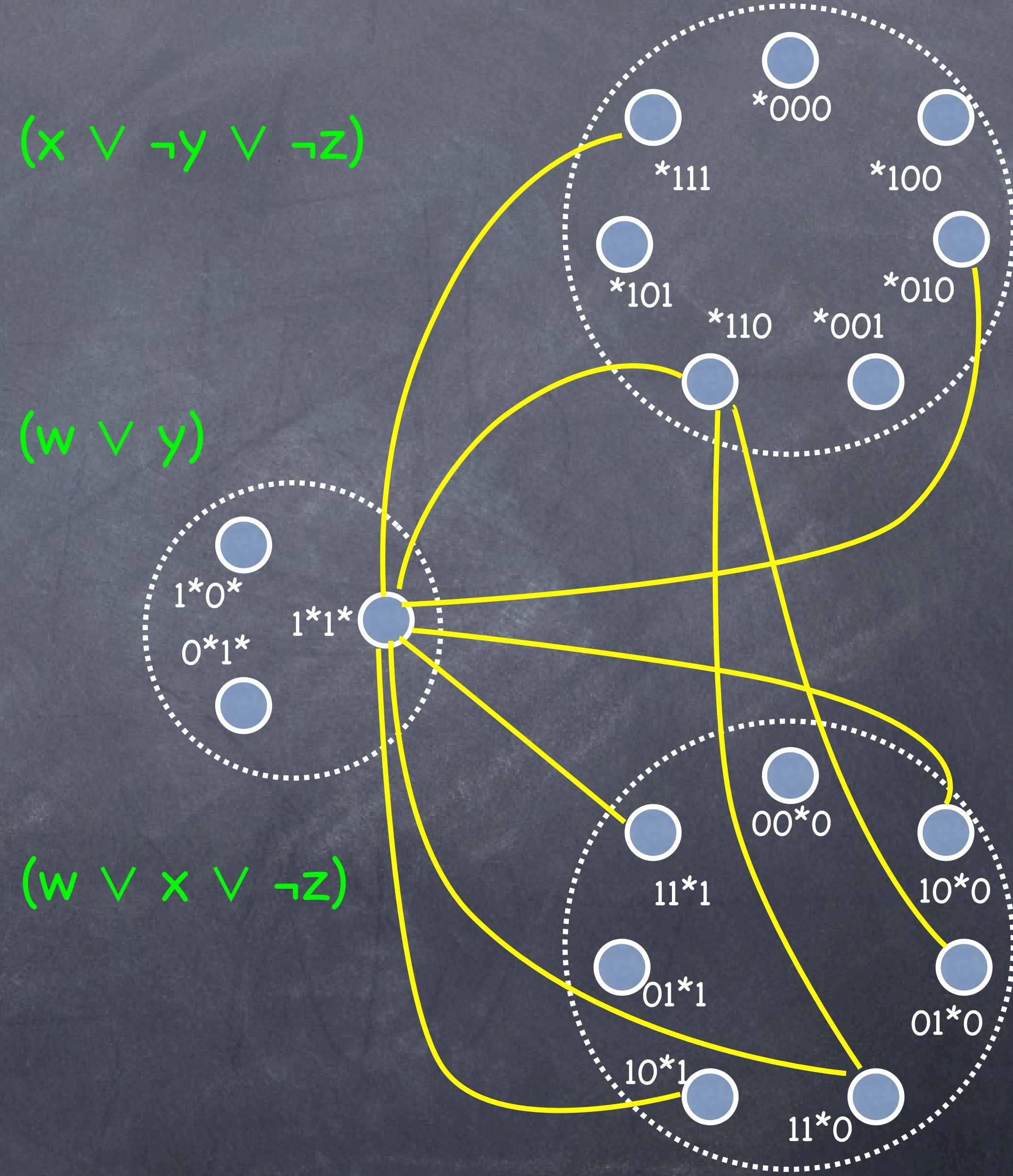

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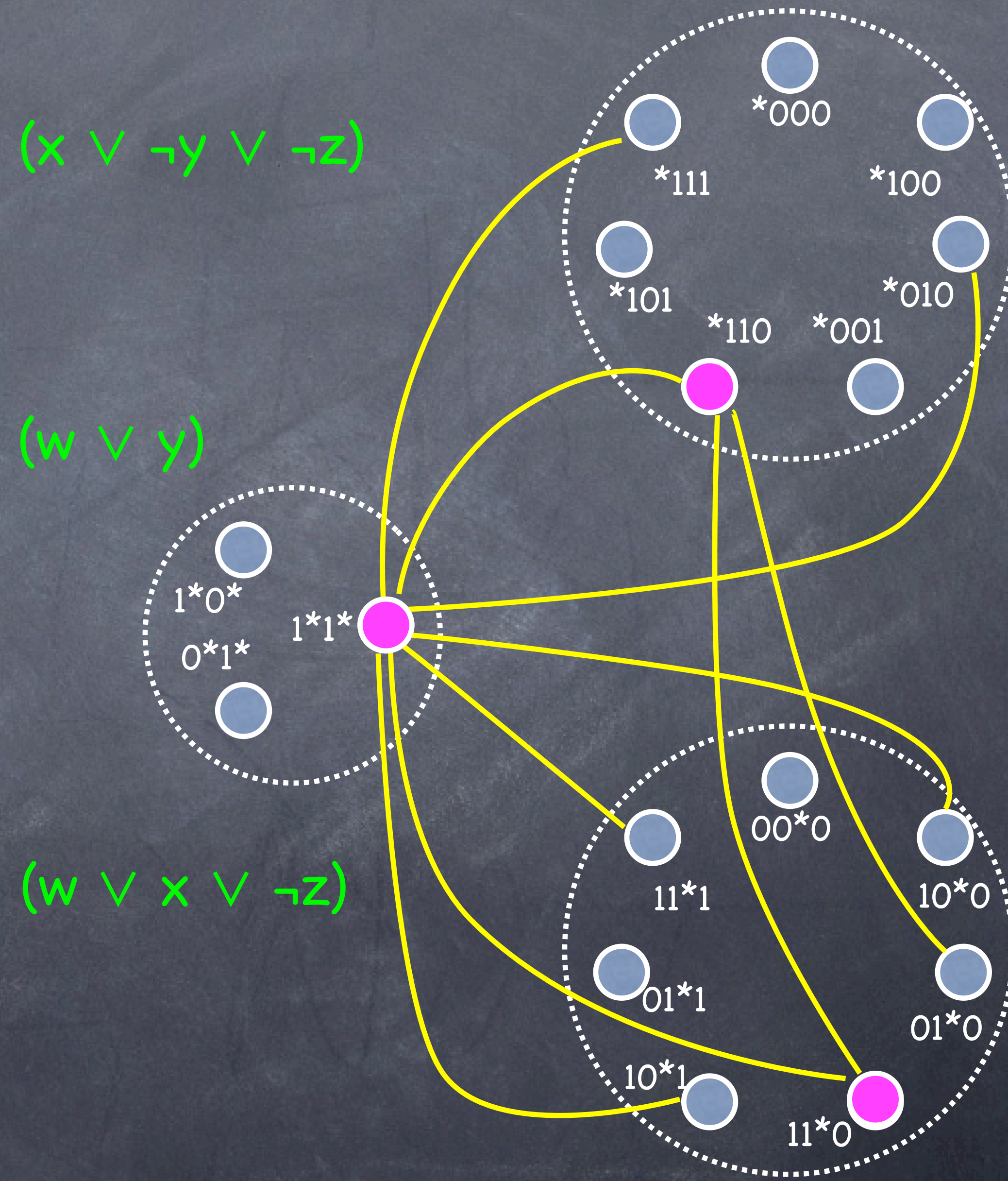

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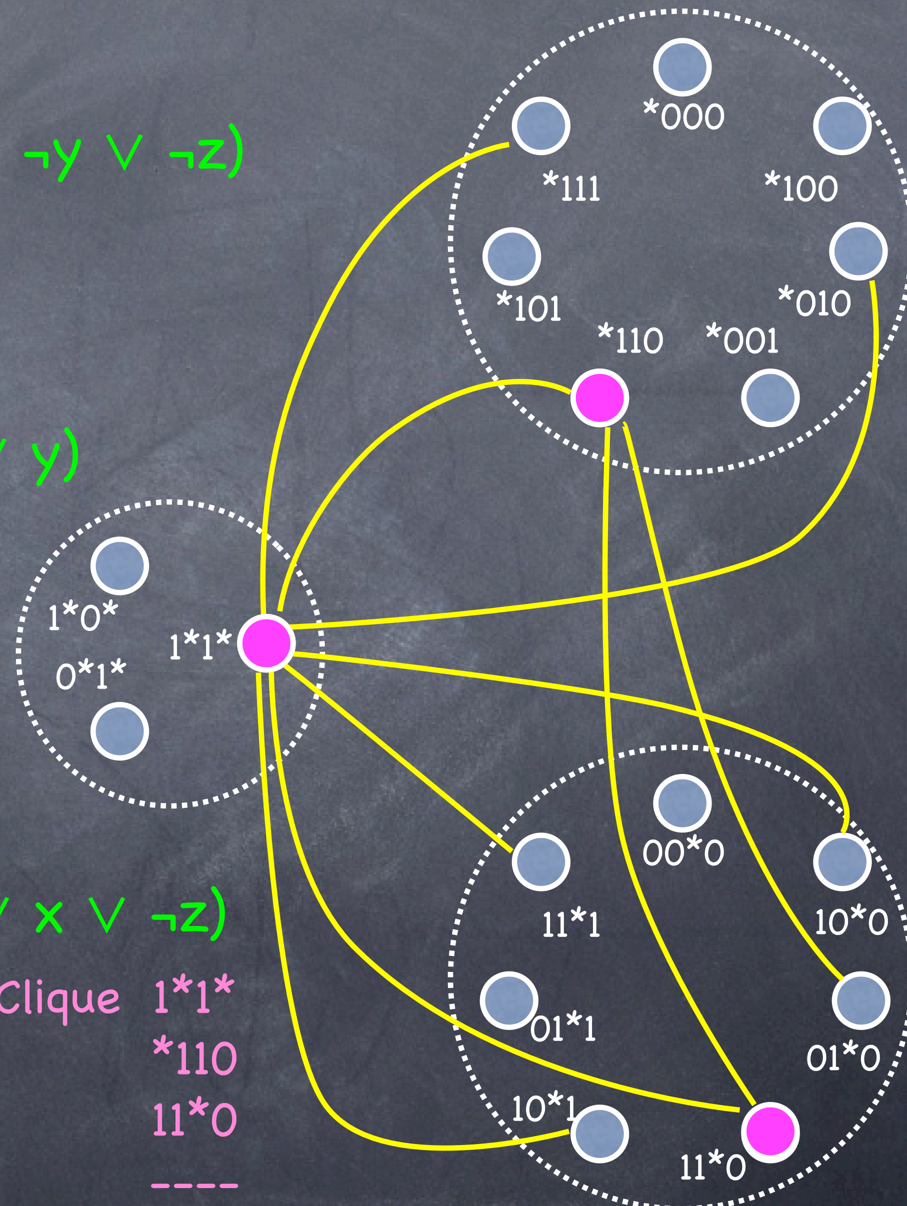

3-Clique 1*1*
*110
11*0
----
sat assignment 1110

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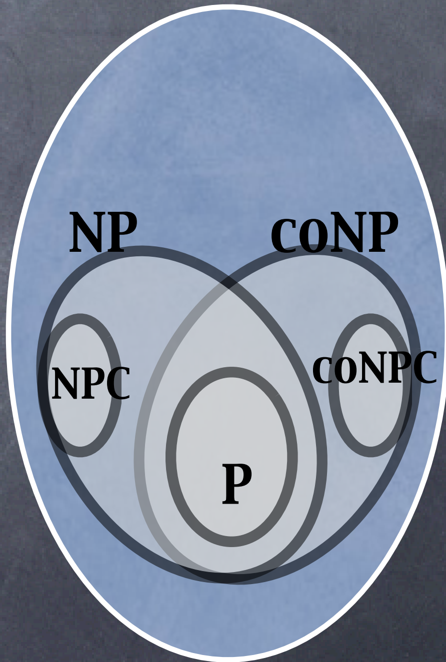
NP, P, co-NP and NPC



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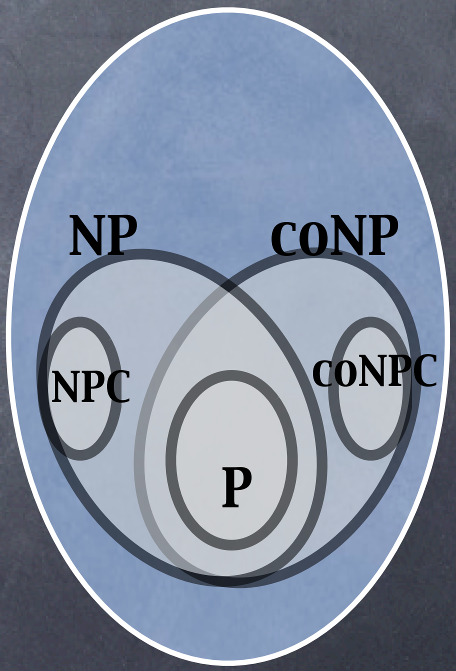

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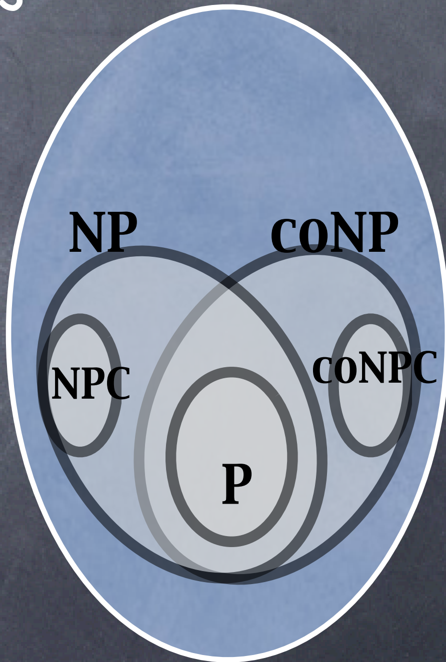

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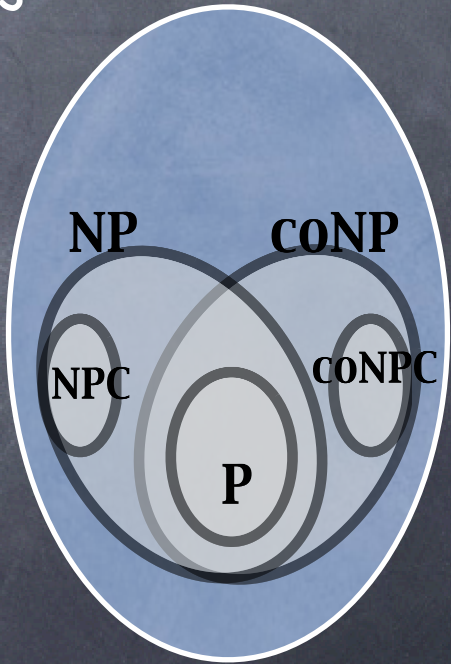

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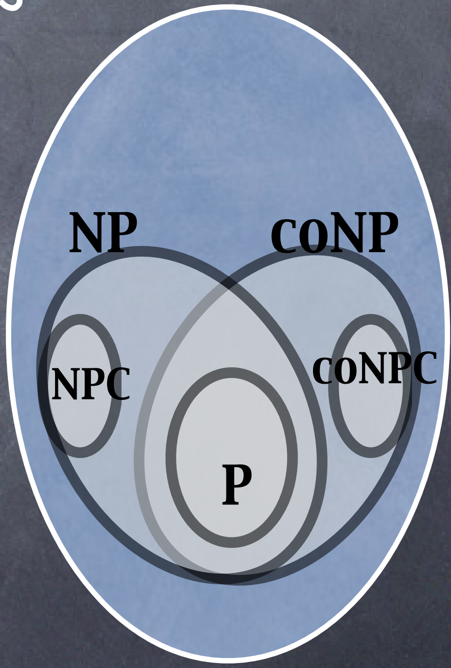

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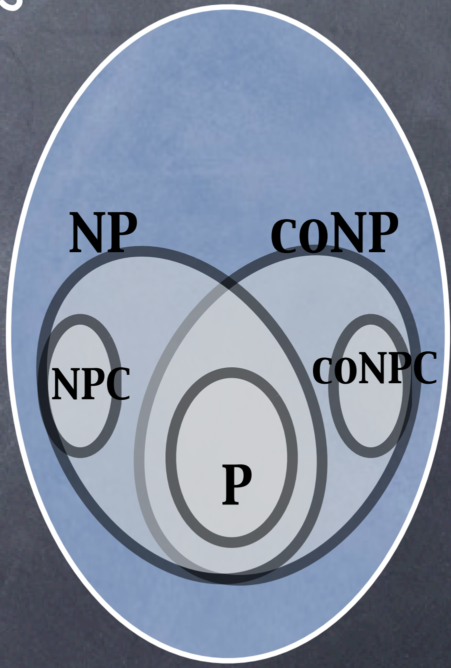

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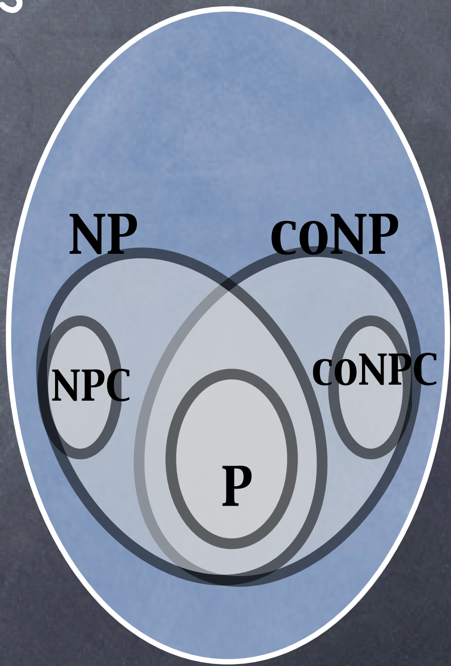

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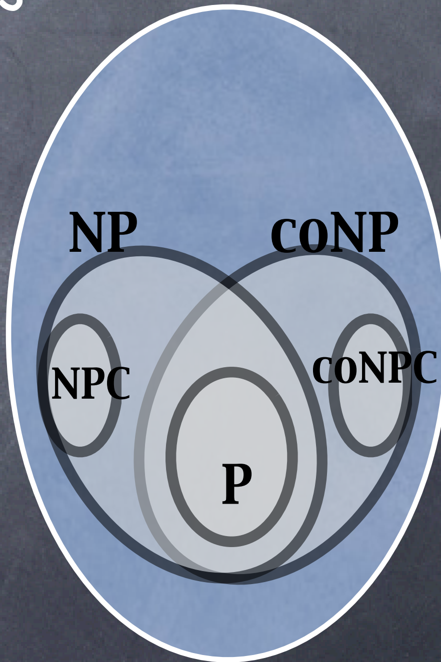

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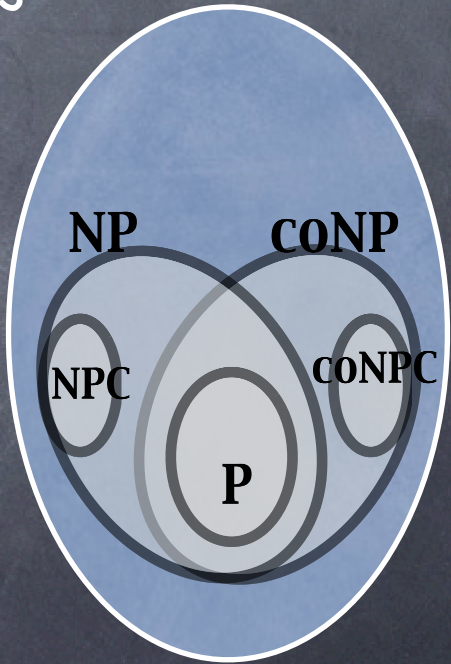

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  - Comparing infinite sets: diagonalization!

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- Are real numbers (say in the range [0,1)) countable?
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Cantor's Diagonal Slash

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Diagonalization to Separate Classes

- Diagonalization can separate the class of decidable languages (from the class of all languages)
  - Plan: Use similar techniques to separate complexity classes

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Think DTIME(T) $\subseteq$ rows

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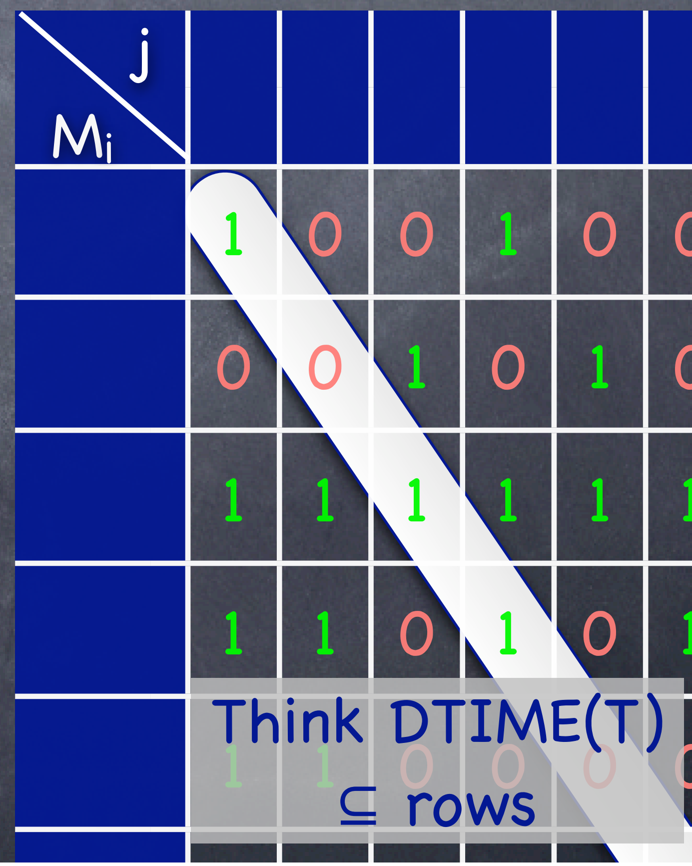

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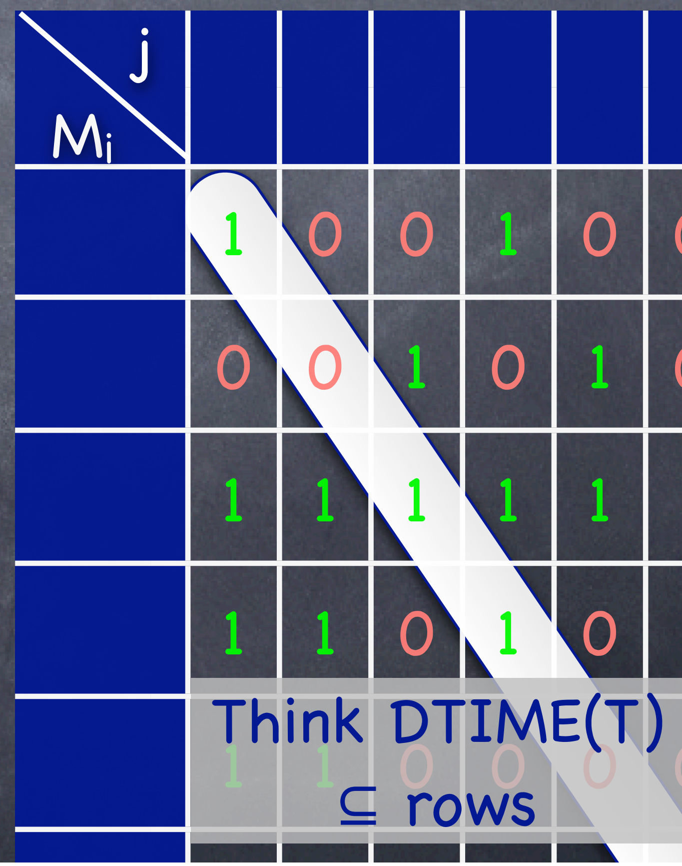

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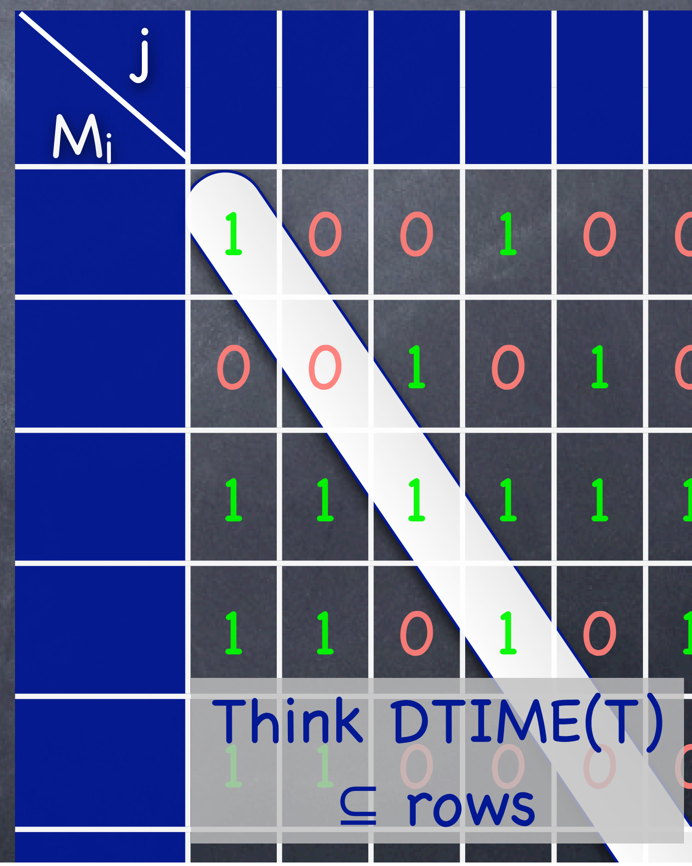

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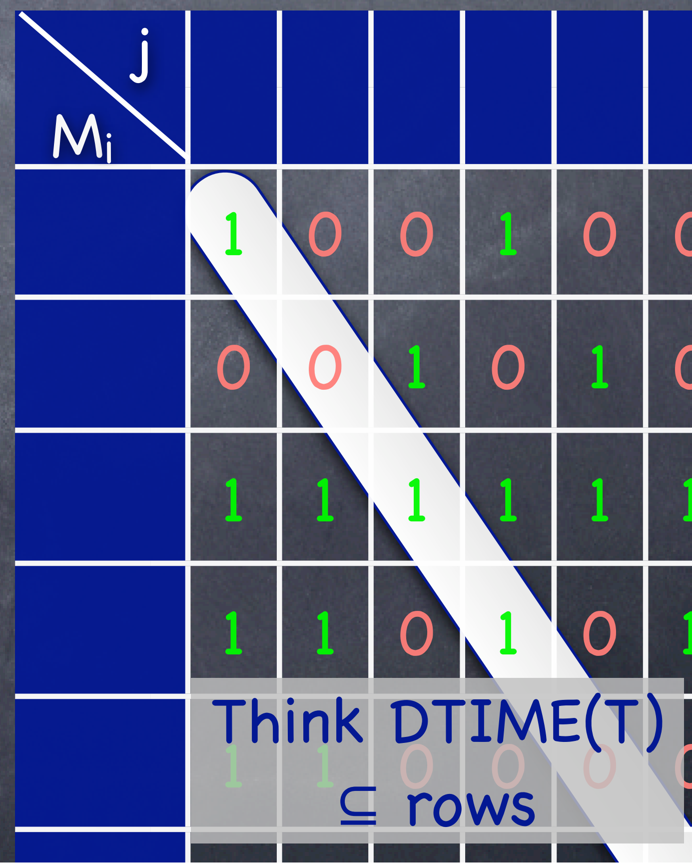

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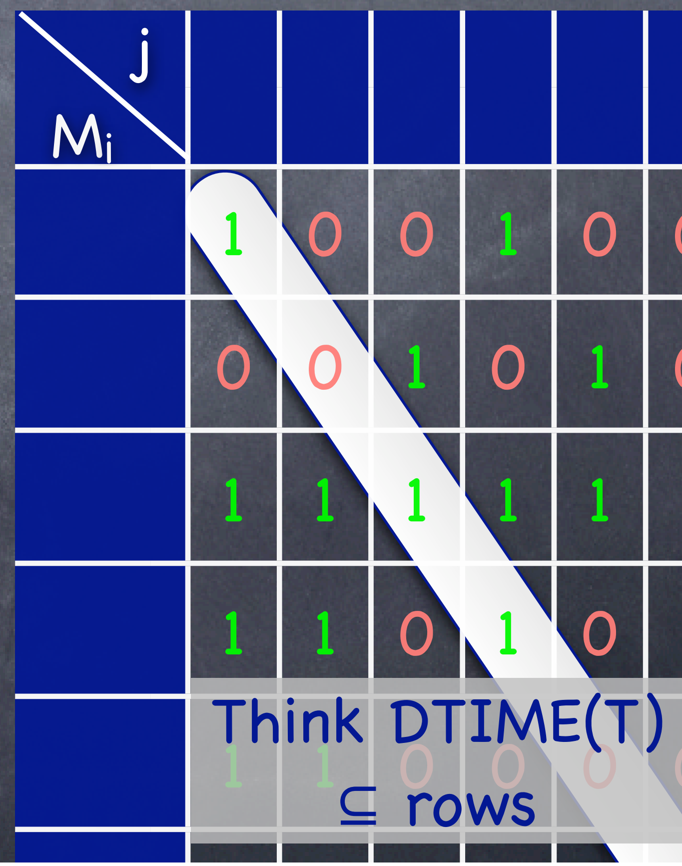

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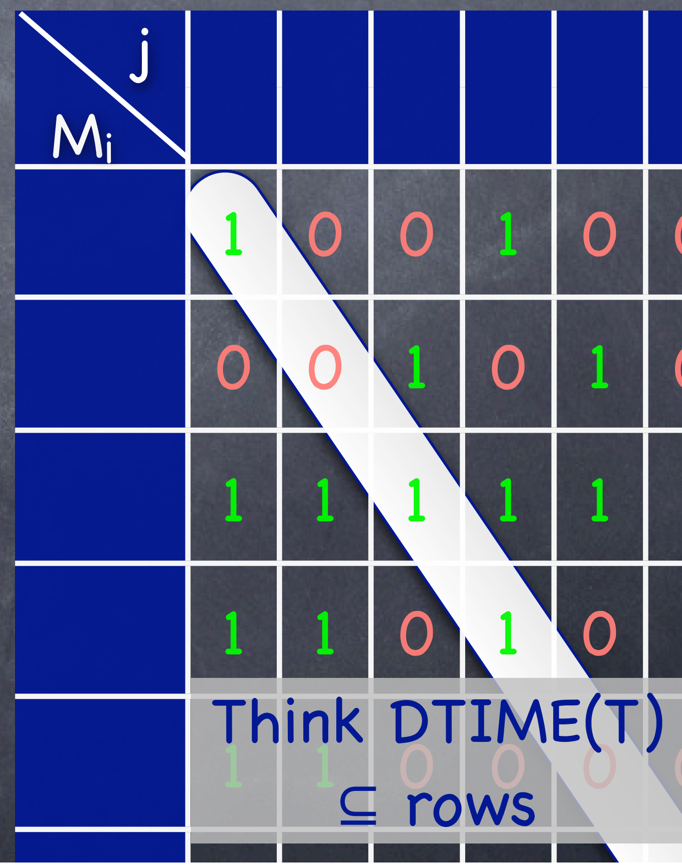
    - If M accepts L′ in time T, then for sufficiently large i s.t. M_i=M, UTM can finish simulating $M_i(i)$. Then table(i,i)=L′(i)!



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In fact,
$T(n+1) = o(T'(n))$

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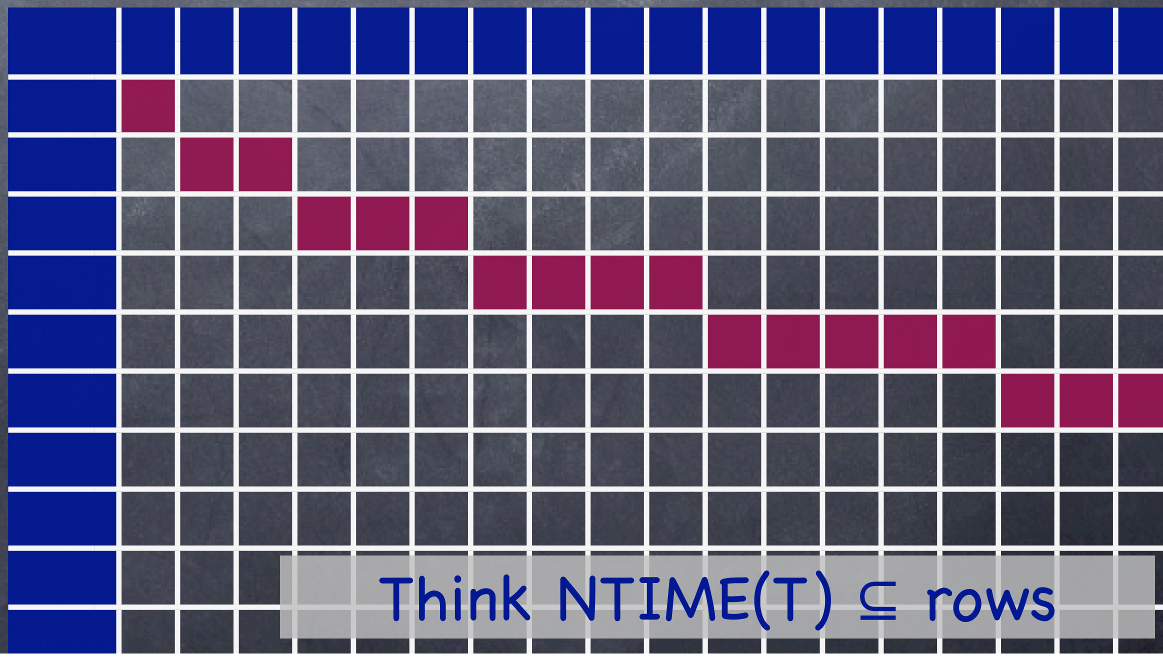

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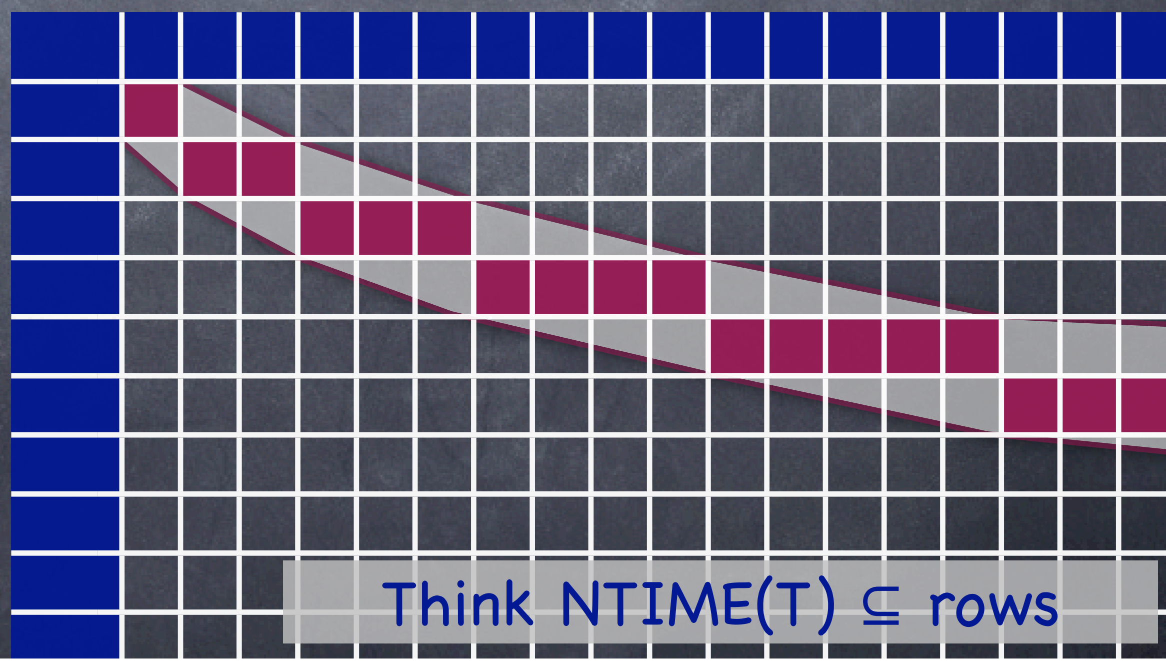

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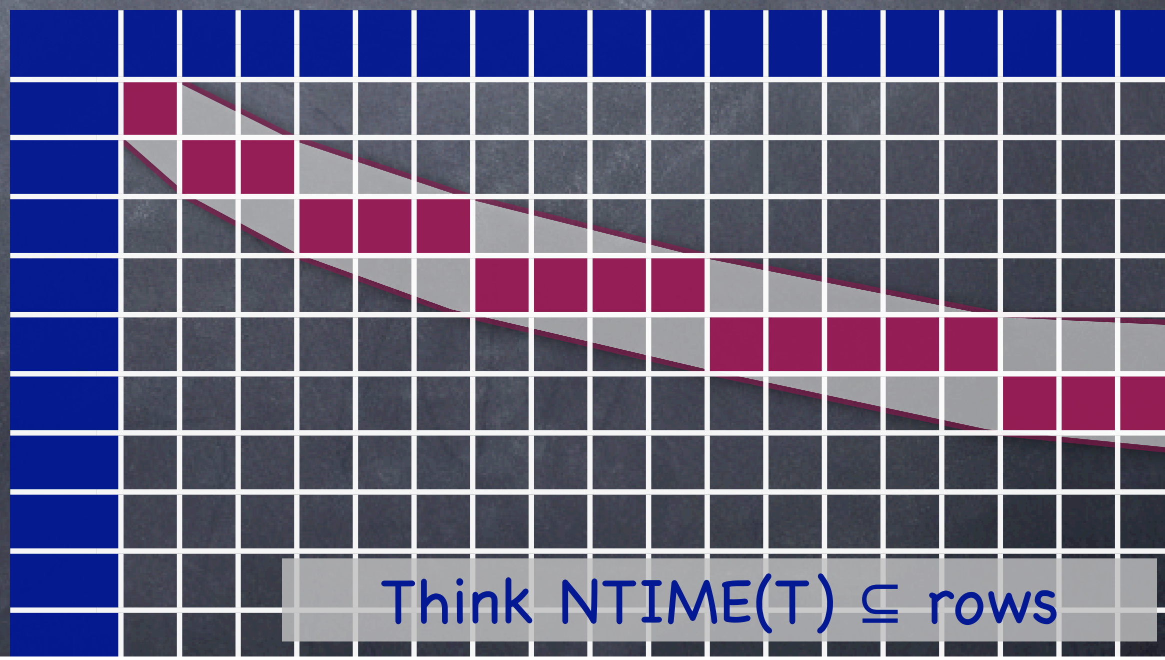

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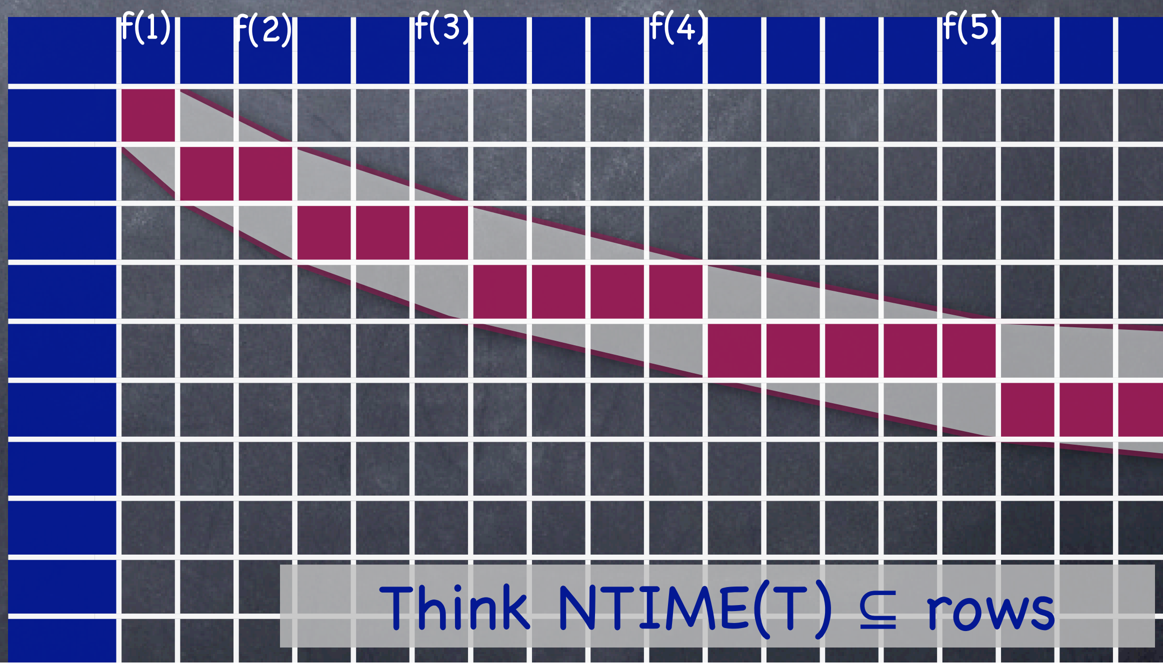

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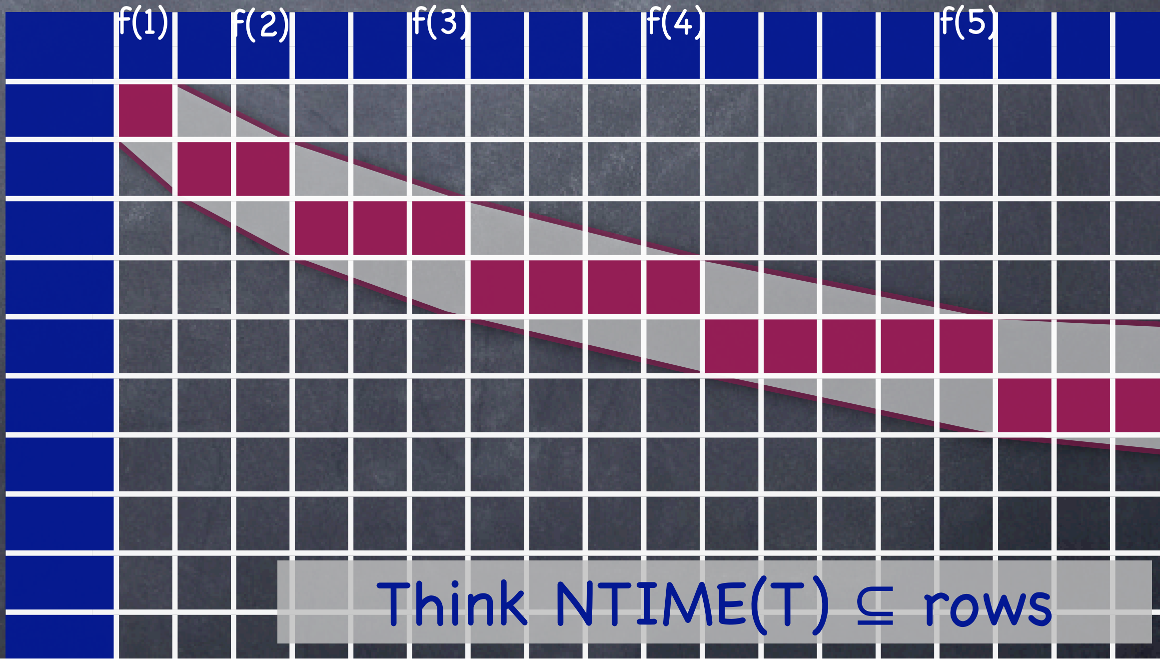

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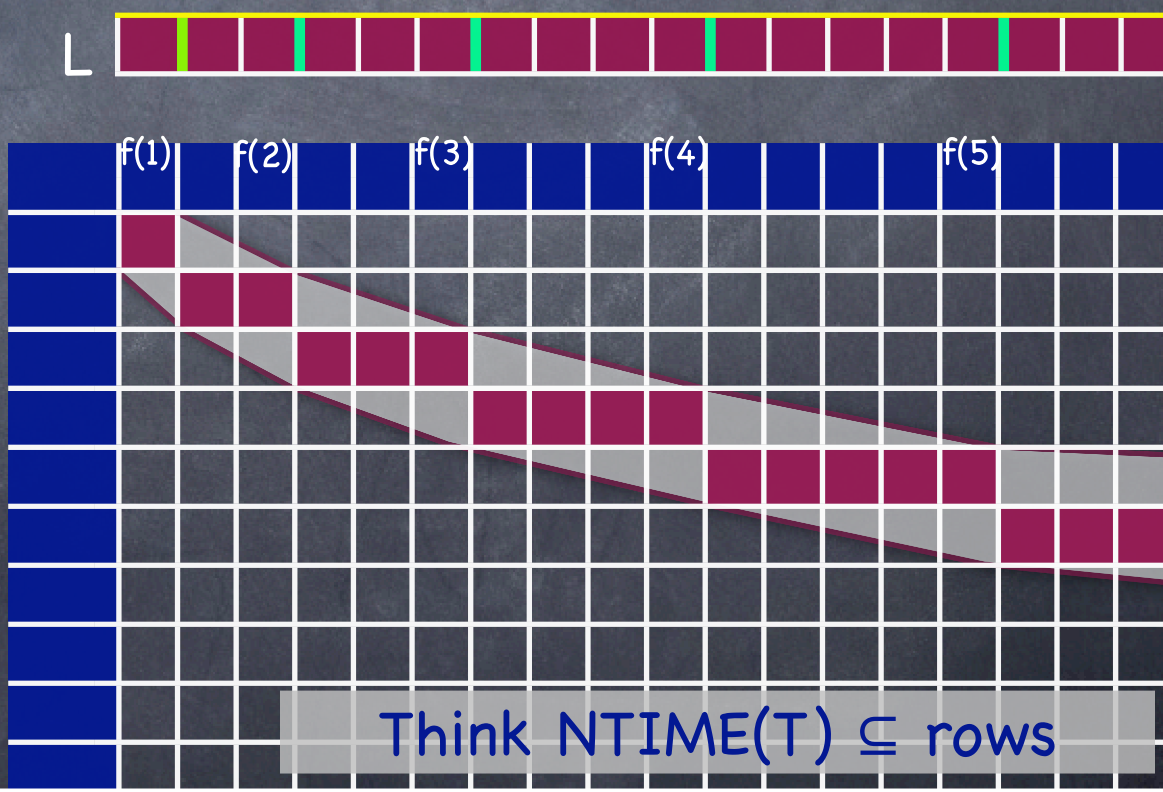

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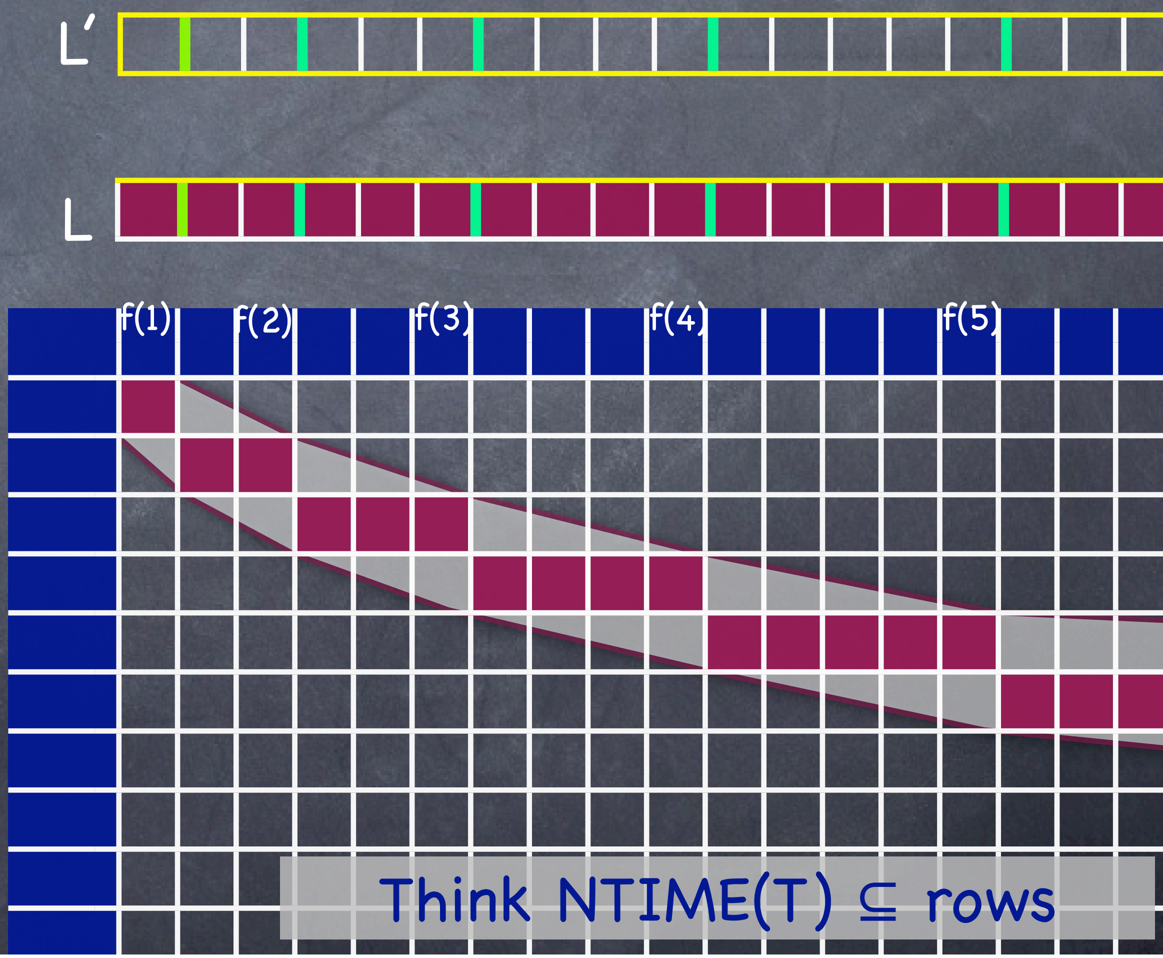

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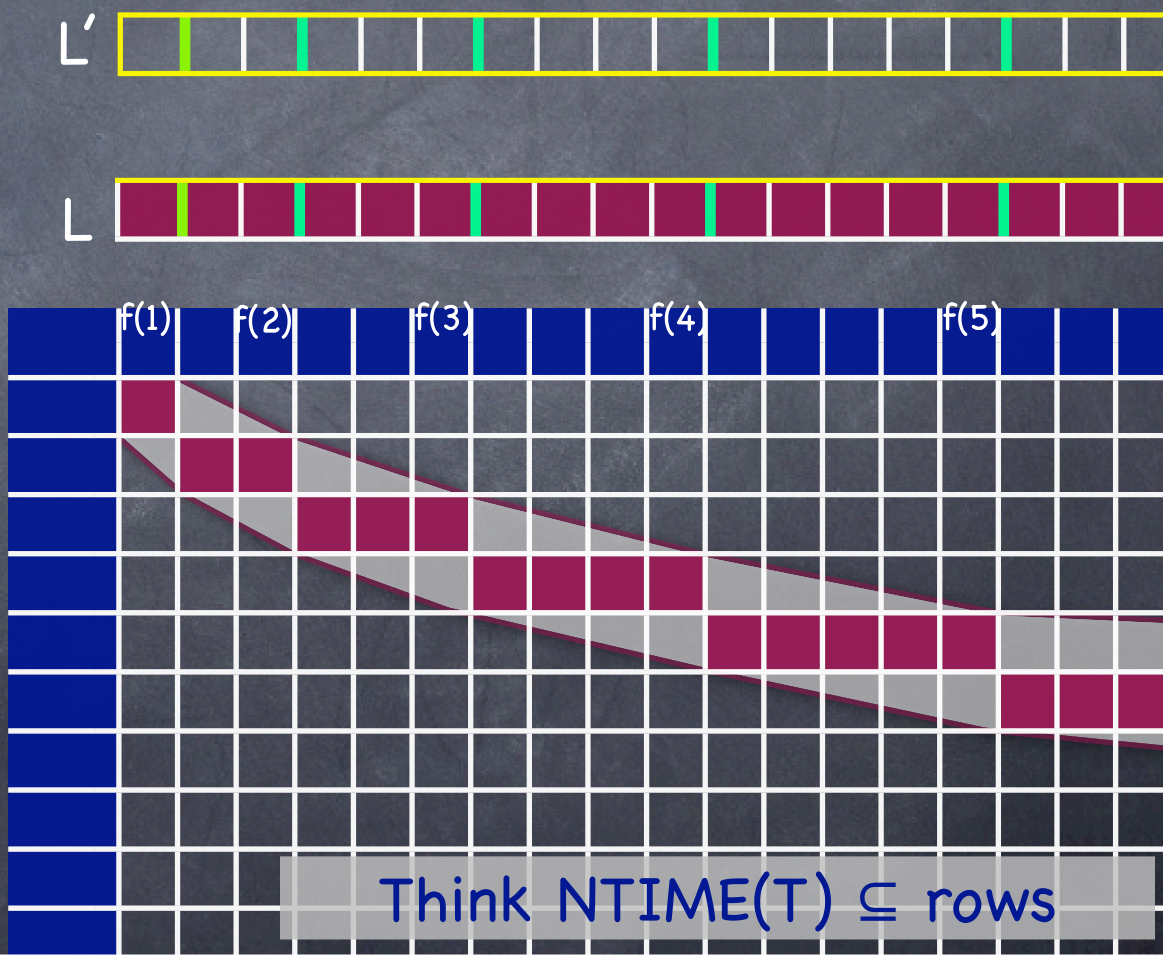

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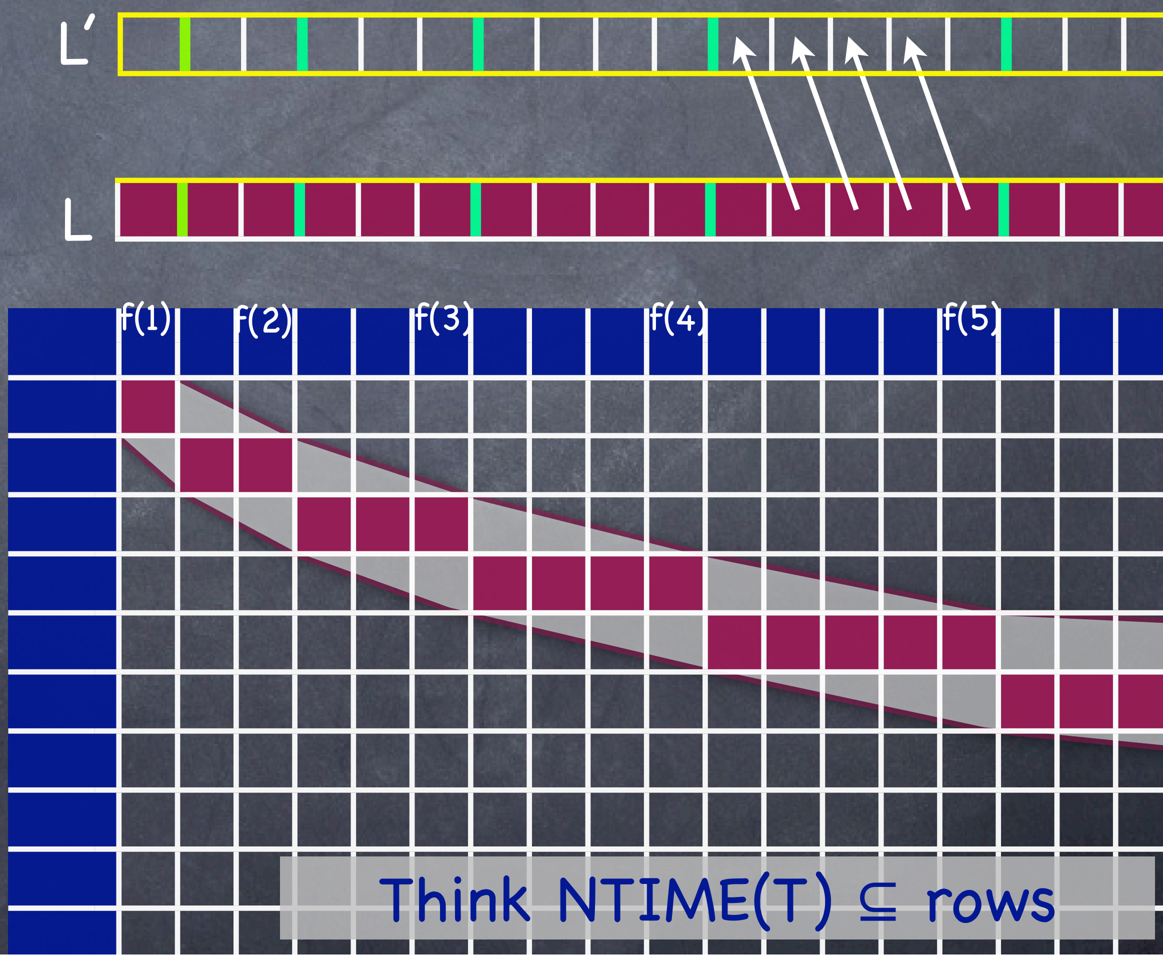

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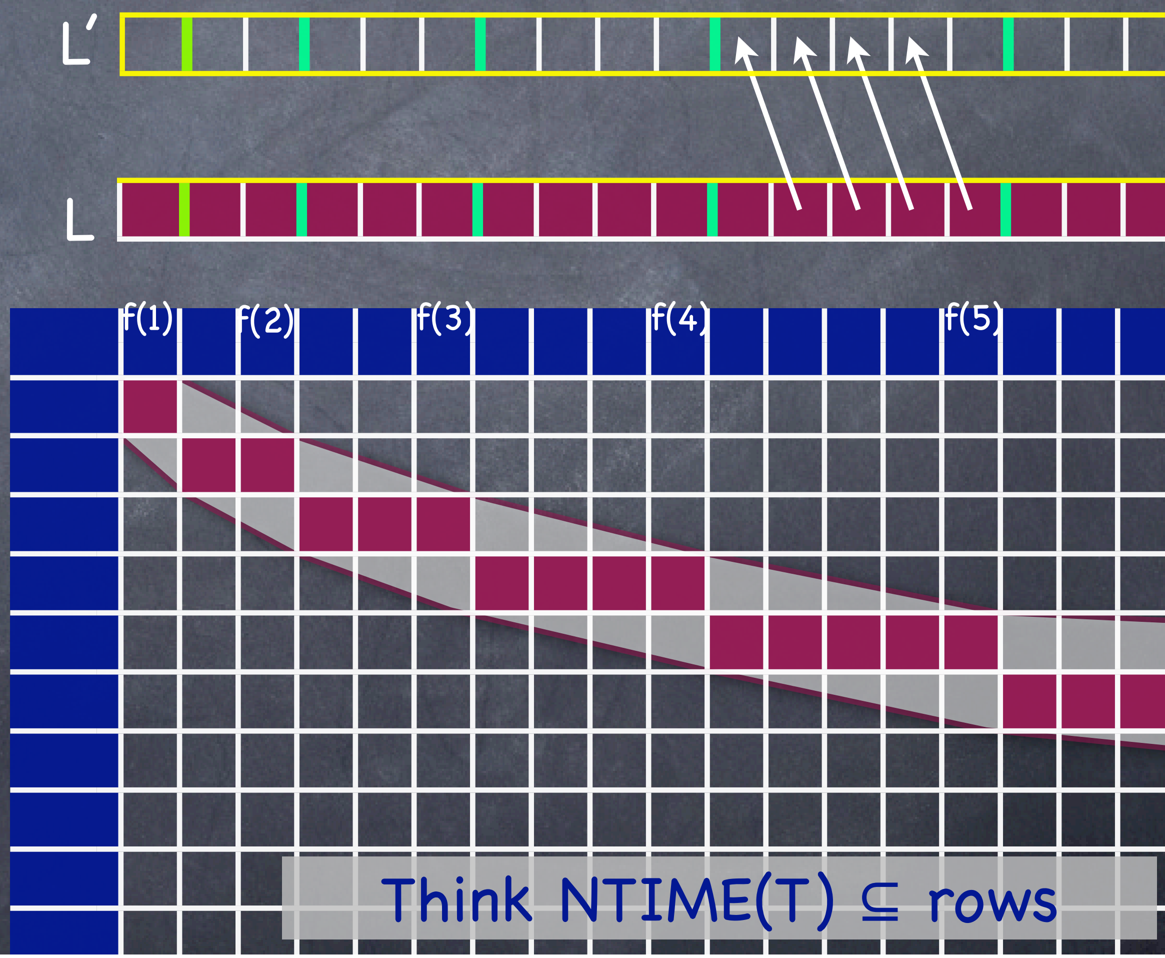

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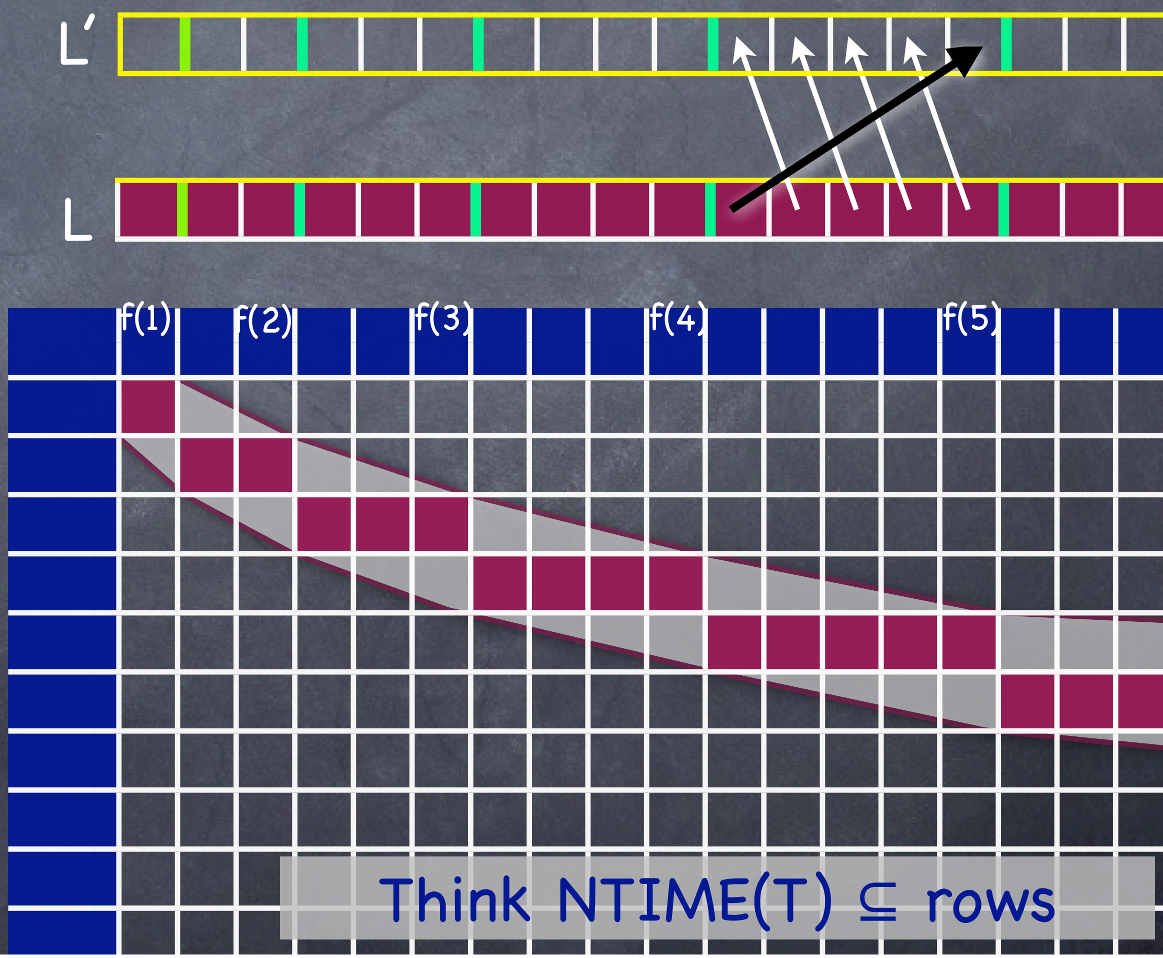

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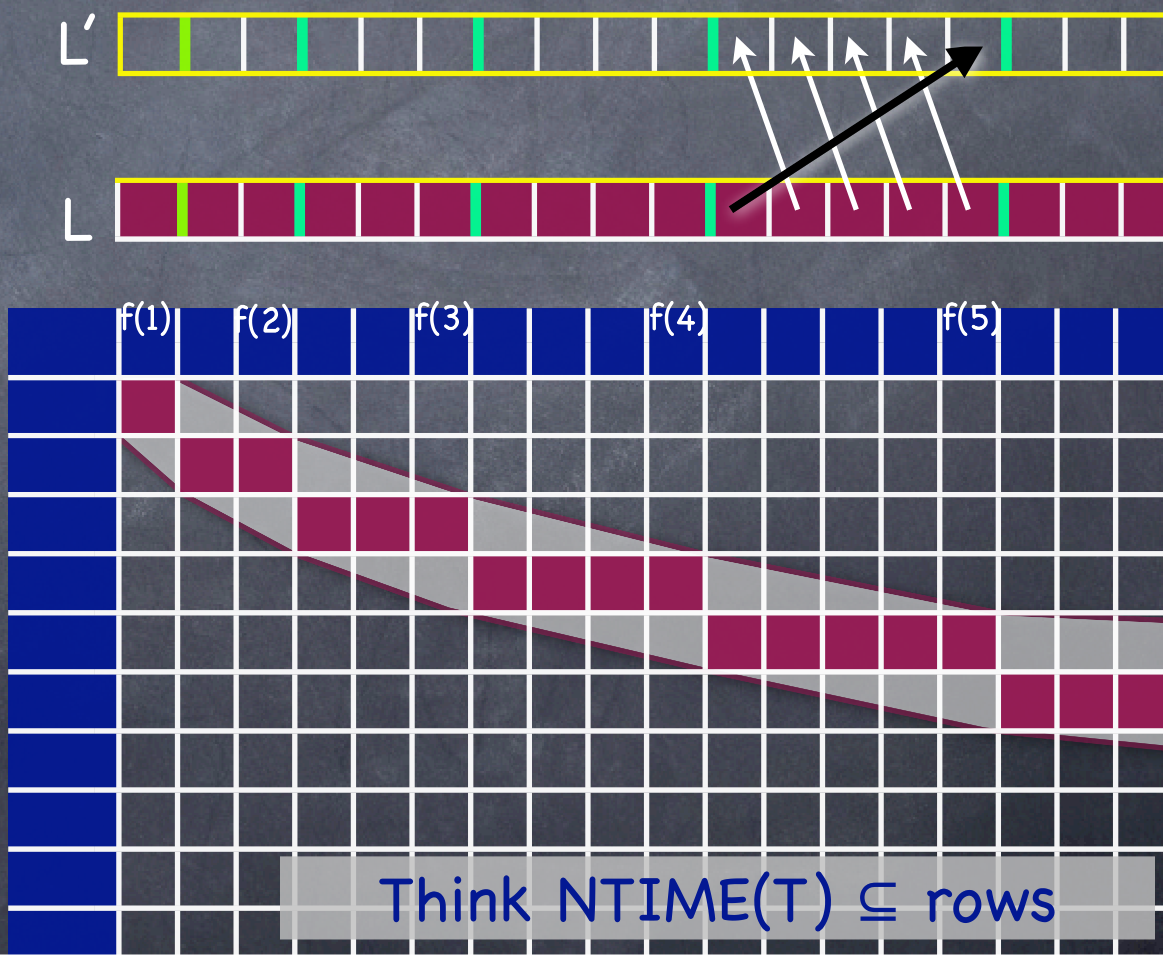

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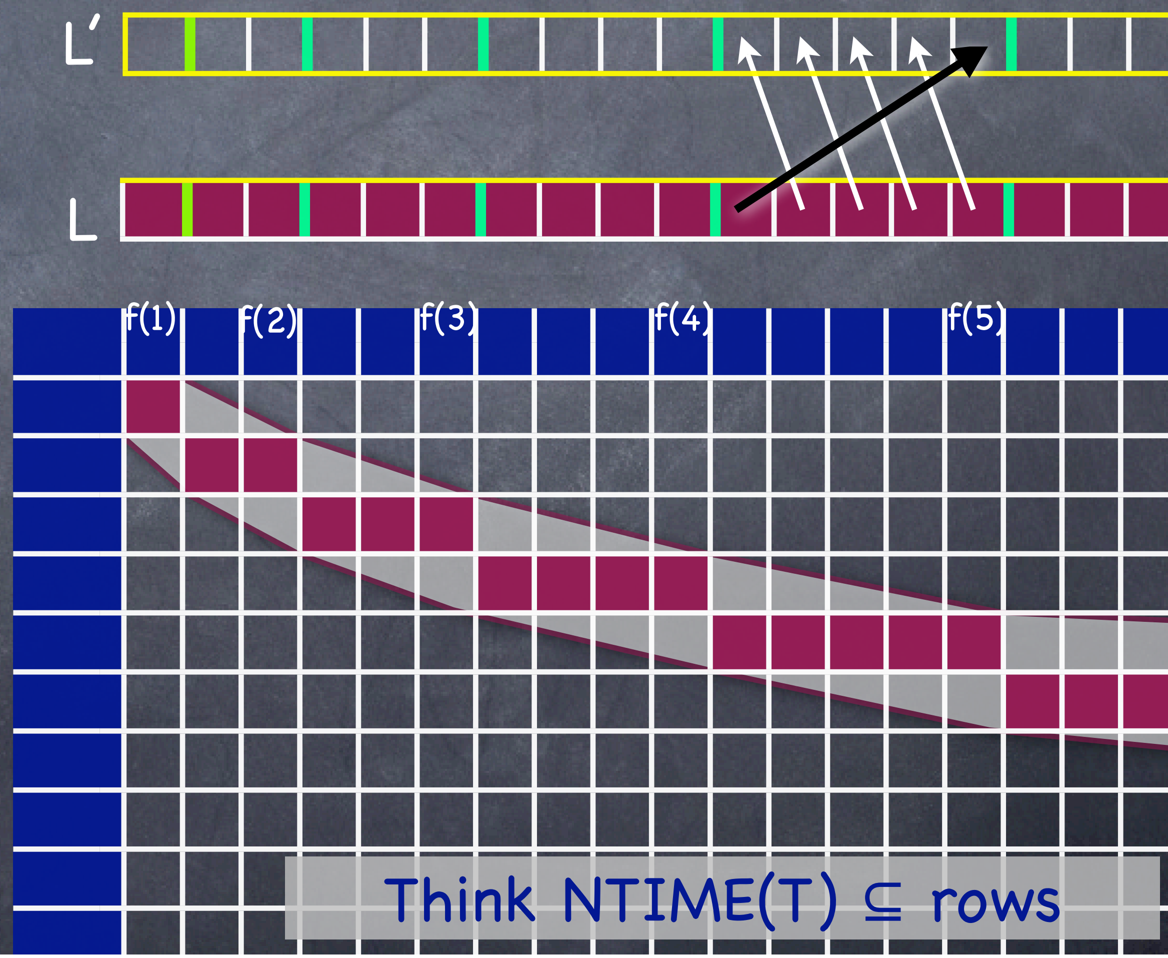

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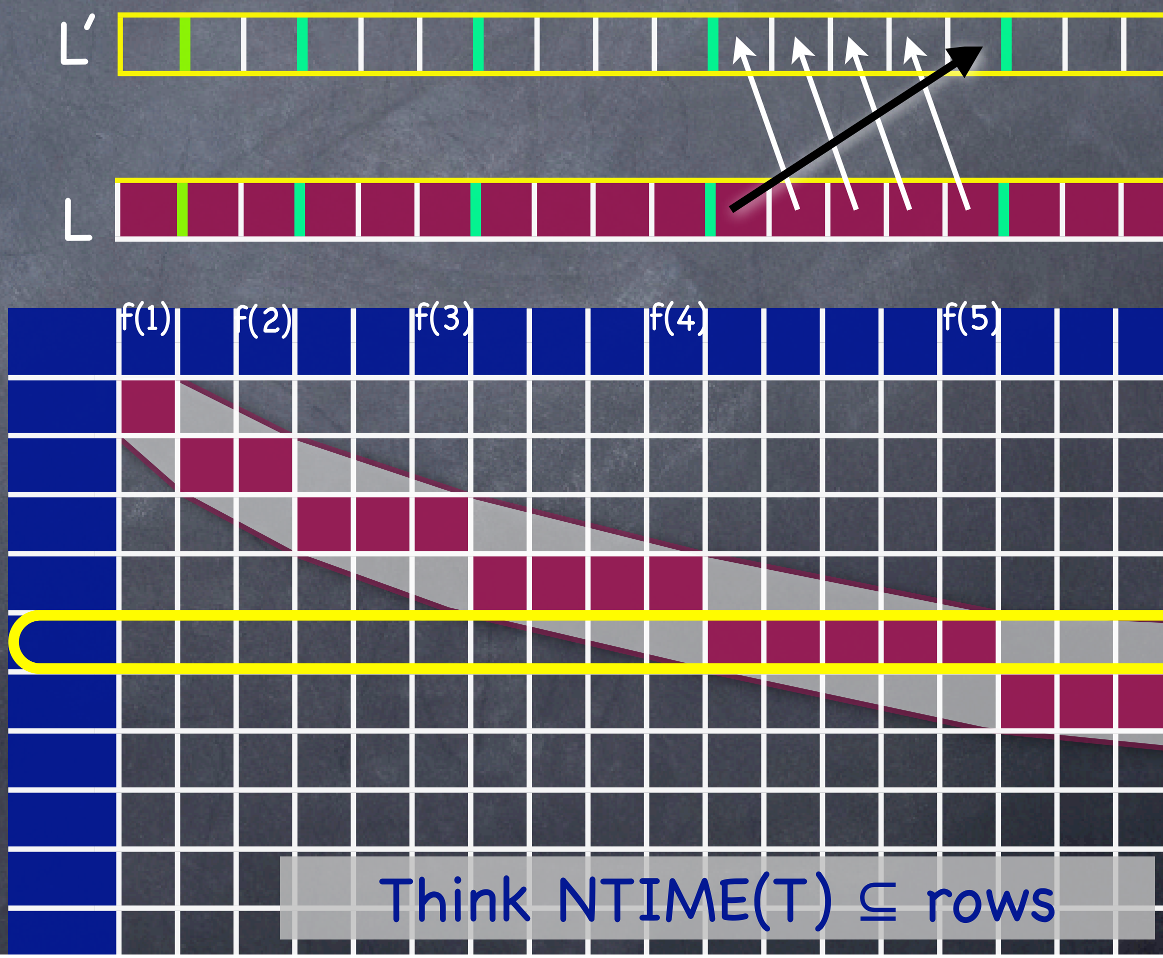

NTIME Hierarchy

- "Delayed flip" on a "rapidly thickening diagonal"
  - $f(i+1)=\exp(f(i))$
  - Let L be the "diagonal" language
  - $L'(j)=L(j+1)$
    - except if $j=f(i)$, then $L'(j) = 1 - L(f(i-1)+1)$

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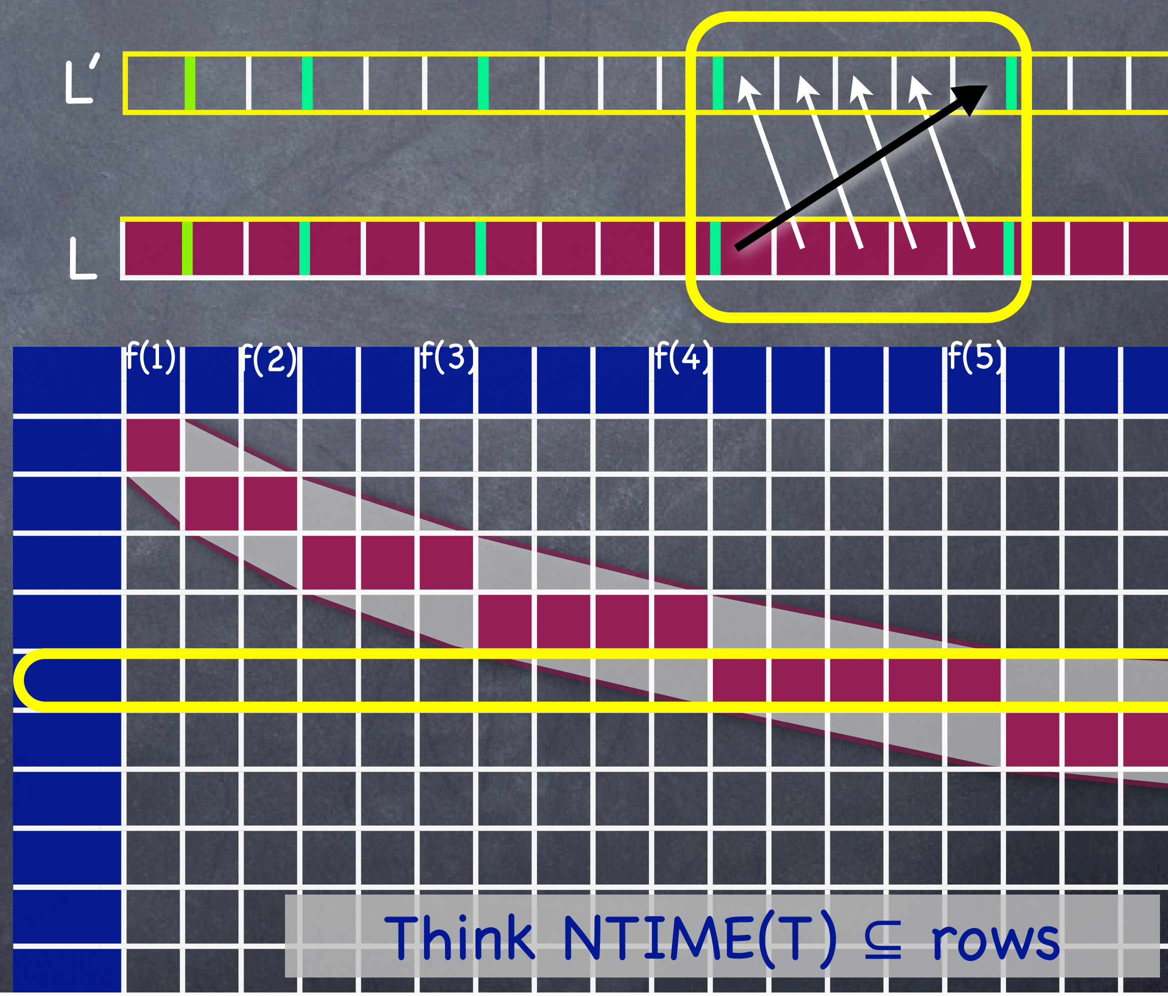

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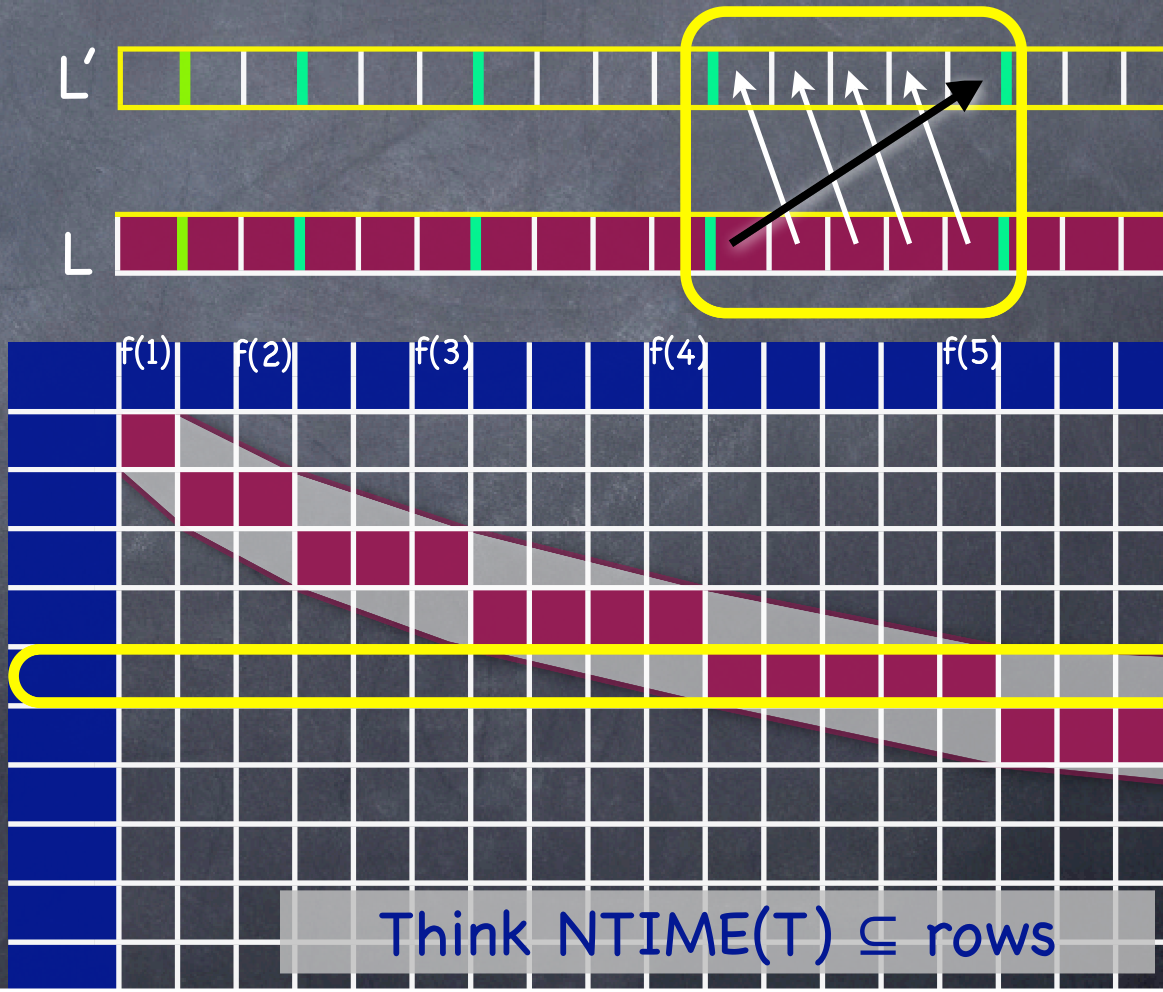
Flip, Diagonal

Delay, Rapid thickening



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  - P and NP?
    - Just diagonalization won't help (next lecture)

Today

- DTIME Hierarchy
  - $\mathrm{DTIME}(T) \subsetneq \mathrm{DTIME}(T')$ if $T \log T = o(T')$
- NTIME Hierarchy
  - $\mathrm{NTIME}(T) \subsetneq \mathrm{NTIME}(T')$ if $T = o(T')$
- Using diagonalization

Next Lecture

- Another application of diagonalization
  - Ladner's Theorem: If P≠NP, NP language which is neither in P nor NP-complete
- Limits of Diagonalization
- Starting Space Complexity