# Complexity Homework 1

Released: January 27, 2008
Due: February 10, 2008

For problems that involve nondeterministic complexity classes, the solutions maybe simpler when phrased in terms of "certificates" (instead of non-determinism).

**Problem 1:**

(a) Let $L_1, L_2$ be languages in $\mathbf{NP}$. Are $L_1 \cup L_2$ and $L_1 \cap L_2$ necessarily in $\mathbf{NP}$?

(b) Let $L_1, L_2$ be languages in $\mathbf{NP}$. Show that $L_1 L_2$ and $L_1^*$ are in $\mathbf{NP}$.

(c) Let $L_1, L_2$ be languages in $\mathbf{P}$. Show that $L_1 L_2$ and $L_1^*$ are in $\mathbf{P}$.

(d) Let $L_1, L_2$ be languages in $\mathbf{NP} \cap \mathbf{co\text{-}NP}$. Show that their symmetric difference

$$L_1 \oplus L_2 \stackrel{\text{def}}{=} \{x \mid x \text{ is in exactly one of } L_1, L_2\}$$

is also in $\mathbf{NP} \cap \mathbf{co\text{-}NP}$.

**Problem 2:**

(a) Show that the halting problem is $\mathbf{NP}$-hard. Is it $\mathbf{NP}$-complete?

(The halting problem is given by the language $H = \{(\langle M \rangle, x) \mid M \text{ is a TM that halts on input } x\}$. You may recall that $H$ is *undecidable*.)

(b) Show that $\overline{\mathsf{SAT}}$ (the complement of $\mathsf{SAT}$) is $\mathbf{NP}$-hard *under Cook reductions*. That is, every language in $\mathbf{NP}$ reduces to $\overline{\mathsf{SAT}}$ via a Cook reduction.

**Problem 3:**

Show that the following two statements are equivalent (we don't know if they are true):

(a) Every unary[1] language in $\mathbf{NP}$ is also in $\mathbf{P}$.

(b) $\mathbf{DTIME}(2^{O(n)}) = \mathbf{NTIME}(2^{O(n)})$ (these classes are called $\mathbf{E}$ and $\mathbf{NE}$, respectively).

*Hint: It takes $\Theta(\log n)$ bits to encode the number "$n$" in binary.*

**Problem 4:**

Give a parsimonious Karp reduction from $\mathsf{SAT}$ to $\mathsf{3SAT}$.

**Problem 5:**

In this problem, we analyze a reduction from $\mathsf{3SAT}$ to the following language:

$$\mathsf{MAX\text{-}2SAT} = \{(\phi, k) \mid \phi \text{ is a 2-CNF formula, and there is an assignment that satisfies at least } k \text{ clauses}\}$$

Our reduction is the following: Given a $\mathsf{3SAT}$ instance $\phi$, we will output a $\mathsf{MAX\text{-}2SAT}$ instance $(\phi', k)$, where $\phi'$ is a 2-CNF formula. To construct $\phi'$, do the following: for each clause $(x \vee y \vee z)$ in $\phi$, add the following 10 clauses to $\phi'$ (where $w$ is a fresh variable for each clause):

$$(x), (y), (z), (\neg x \vee \neg y), (\neg y \vee \neg z), (\neg x \vee \neg z), (w), (x \vee, \neg w), (y \vee \neg w), (z \vee \neg w)$$

Find a value of $k$ such that $(\phi', k) \in \mathsf{MAX\text{-}2SAT}$ if and only if $\phi \in \mathsf{3SAT}$. Prove the correctness of the reduction.

[1] A language is *unary* if it is a subset of $\{1\}^*$ — that is, it only uses one symbol of the alphabet.

**Problem 6 (Extra credit):**

Show that $\mathsf{2SAT}$ is in $\mathbf{P}$.

*Hint: Consider a directed graph with all the literals as nodes, and edges as implications ( $(x \vee y)$ corresponds to $(\neg x \Rightarrow y)$ and $(\neg y \Rightarrow x)$). Look to derive contradictions of the form $(\neg x \Rightarrow x)$ and $(x \rightarrow\Rightarrow x)$. What do such contradictions tell you about a possible satisfying assignment?*

**Problem 7 (Extra credit) [See Arora-Barak Chapter 2, Exercise #13]:**

Show that if there is a unary language that is **NP**-complete, then $\mathbf{P} = \mathbf{NP}$.

**Problem 8 (Extra credit):**

Consider the following language:

$$\mathsf{MAX\text{-}CUT} = \{(G, k) \mid G \text{ is a multigraph with a cut of size at least } k\}$$

A *cut* in a graph is a partition of its vertices into two parts. The size of the cut is the number of edges which "cross" the cut (whose endpoints are in opposite parts). A multigraph means we allow duplicate edges.

We now analyze a reduction from $\mathsf{MAX\text{-}2SAT}$ to $\mathsf{MAX\text{-}CUT}$. Given an instance $(\phi, k)$ of $\mathsf{MAX\text{-}2SAT}$, let $n$ be the number of variables occuring in $\phi$, and $m$ the number of clauses. Consider the following graph:

$G_\phi$ is a graph with a vertices labeled $x_i$ and $\neg x_i$ for each variable $x$ occuring in $\phi$, and two special vertices labeled $T$ and $F$. We add $5m$ edges between $T$ and $F$, and $5m$ edges between each pair $(x_i, \neg x_i)$ — see Figure 1. Then, for each clause $(x \vee y) \in \phi$, where $x$ and $y$ are literals, we add the following 7 edges (see Figure 2):

- $(x, y), (T, x), (T, y)$.
- Two copies of the edges $(x, F)$ and $(y, F)$.

(a) Show that in the largest cut in $G_\phi$, $T$ and $F$ must be in opposite parts.

(b) Show that in the largest cut in $G_\phi$, the vertices corresponding to $x$ and $\neg x$ must be in opposite parts.

(c) Argue that $(\phi, k) \in \mathsf{MAX\text{-}2SAT}$ if and only if $(G_\phi, 5m + 5mn + 4k + 2(m - k)) \in \mathsf{MAX\text{-}CUT}$.

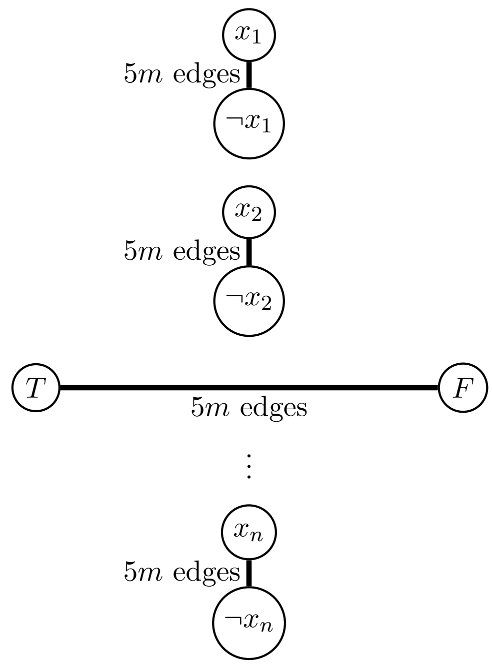


Figure 1: Starting graph for $G_\phi$, where $\phi$ has $n$ variables, $x_1, \ldots, x_n$.

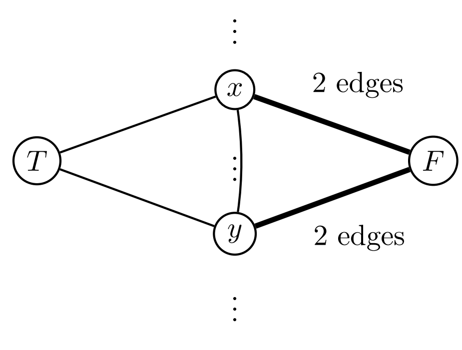


Figure 2: Edges to add for a clause of the form $(x \vee y)$