CS576 Topics in Automated Deduction

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Sets

Type `'a set` gives sets over type `'a`

- $\{\}$, $\{e_1, \ldots, e_n\}$, {x. P x}, {f(x, y) | x y. P x y}
- $e \in A$, $A \subseteq B$
- $A \cup B$, $A \cap B$, $A - B$, $-A$
- $\bigcup_{x \in A} B\ x$, $\bigcap_{x \in A} B\ x$
- $\{a,\ b,\ c\}$, $\{i.\ j\}$
- insert ::' a ⇒' a set ⇒' a set
- f ' $A \equiv \{y.\ \exists x \in A.\ y = f\ x\}$
- ...

Proofs about Sets

Natural deduction proof rules:

- equalityI: ⟦A ⊆ B; B ⊆ A⟧ ⟹ A = B
- equalityE: ⟦A = B; ⟦A ⊆ B; B ⊆ A⟧ ⟹ P⟧ ⟹ P
- subsetI: (⋀x. x ∈ A ⟹ x ∈ B) ⟹ A ⊆ B
- sutsetD: ⟦A ⊂ B; c ∈ A⟧ ⟹ c ∈ B
- IntI: ⟦c ∈ A; c ∈ B⟧ ⟹ c ∈ A ∩ B
- IntD1: $c \in A \cap B \Longrightarrow c \in A$
- IntD2: $c \in A \cap B \Longrightarrow c \in B$
- set_eqI: $(\bigwedge x.\ (x \in A) = (x \in B)) \Longrightarrow A = B$
- mem_Collect_eq: $(a \in \{x.\ P\ x\}) = P\ a$
- Collect_mem_eq: $\{x.\ x \in A\} = A$
- ... (see Tutorial)

Bounded Quantification

- ∀x ∈ A. P x ≡ ∀x. x ∈ A ⟶ P x
- ∃x ∈ A. P x ≡ ∃x. x ∈ A ∧ P x
- ballI: (⋀x. x ∈ A ⟹ P x) ⟹ ∀x ∈ A. P x
- bspec: ⟦∀x ∈ A. P x; x ∈ A⟧ ⟹ P x
- bexI: ⟦P x; x ∈ A⟧ ⟹ ∃x ∈ A. P x
- bexE: ⟦∃x ∈ A. P x; ⋀x. ⟦x ∈ A; P x⟧ ⟹ Q⟧ ⟹ Q

Demo: Some Set Theory

Format for Inductive Set Definitions

inductive_set S :: "τ set" where

⟦$a_{1,1} \in S; \ldots;\ a_{1,n} \in S; A_{1,1};\ \ldots;\ A_{1,k}$⟧ ⟹ $a_1 \in S$ |

... |

⟦$a_{m,1} \in S; \ldots;\ a_{m,l} \in S; A_{m,1};\ \ldots;\ A_{m,j}$⟧ ⟹ $a_m \in S$

where $A_{i,j}$ are side conditions not involving S.

Example: Finite Sets

Informally

- The empty set is finite
- Adding an element to a finite set yields a finite set
- These are the only finite sets

Example: Finite Sets

In Isabelle/HOL:

```
inductive_set Finites :: 'a set set
```

– The set of all finite sets

```
{ } ∈ Finites |
A ∈ Finites ⟹ insert a A ∈ Finites
```


Example: Even Numbers

Informally

- 0 is even
- If n is even, then so is $n + 2$
- These are the only even numbers

Example: Even Numbers

In Isabelle/HOL:

```
inductive_set Ev :: nat set
```

— The set of all even numbers

```
0 ∈ Ev |
n ∈ Ev ⟹ n + 2 ∈ Ev
```


Proving Properties of Even Numbers

Easy: $4 \in Ev$

$$0 \in Ev \Longrightarrow 2 \in Ev \Longrightarrow 4 \in Ev$$

Trickier: $m \in Ev \Longrightarrow m + m \in Ev$

Idea: induct on the length of the derivation of $m \in EV$

Better: induct on the *structure* of the derivation

Proving Properties of Even Numbers

Induction leads to two cases:

- **rule:** $0 \in Ev$

  $1.\ 0 + 0 \in Ev$ case $m = 0$

- **rule:** $n \in Ev \Longrightarrow n + 2 \in Ev$

  $z\ 2.\ \bigwedge n.\llbracket n \in Ev;\ n + n \in Ev \rrbracket \Longrightarrow Suc(Sucn) + Suc(Sucn) \in Ev$

  case $m = n + 2$

Rule Induction for Ev

To prove

$$n \in Ev \Longrightarrow P\ n$$

by *rule induction* on $\mathtt{n} \in \mathtt{Ev}$ we must prove

- $\mathtt{P}\ 0$
- $P\ n \Longrightarrow P(n+2)$

Uses rule `Ev.induct`:

$$\llbracket \mathtt{n} \in \mathtt{Ev};\ \mathtt{P}\ 0;\ \bigwedge \mathtt{n}.\ \mathtt{P}\ \mathtt{n} \Longrightarrow \mathtt{P}(\mathtt{n}+2) \rrbracket \Longrightarrow \mathtt{P}\ \mathtt{n}$$

An elimination rule

Rule Induction in General

Set S is defined inductively. To prove

$$\mathtt{x} \in \mathtt{S} \Longrightarrow \mathtt{P}\ \mathtt{x}$$

by *rule induction* on $x \in S$ we must prove for every rule

$$\llbracket \mathtt{a}_1 \in \mathtt{S};\ \ldots;\ \mathtt{a}_n \in \mathtt{S} \rrbracket \Longrightarrow \mathtt{a} \in \mathtt{S}$$

that P is preserved:

$$\llbracket \mathtt{P}\ \mathtt{a}_1;\ \ldots;\ \mathtt{P}\ \mathtt{a}_n \rrbracket \Longrightarrow \mathtt{P}\ \mathtt{a}$$

In Isabelle/HOL: `apply(erule S.induct)`

Demo: Inductive Set Definition

Demo: Evens are infinite