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1. The Order-Sorted Nelson-Oppen Combination Theorem

We first need the following

<u>Definition</u>. Let T be an OS-FO theory with $sig(T) = \Sigma = ((S, \leq), F, P)$, $s_1, \ldots, s_n \in S$, and $\mathcal{F} \subseteq FO(\Sigma)$ a recursive set. T is called <u>stably infinite</u> <u>in</u> <u>sorts</u> $s_1, \ldots, s_n$ <u>for</u> $\mathcal{F}$-<u>satisfiability</u> iff for each $\varphi \in \mathcal{F}$ if φ is T-satisfiable, then it is satisfiable in a model $\mathcal{B} = (B, \_{\mathcal{B}}) \in mod(T)$ such that $|B_{s_i}| \geq \chi_0$, $1 \leq i \leq n$.

The OS Nelson-Oppen Combination Problem

<u>Given</u>: T_1, T_2 OS-FO theories with $\Sigma_i = sig(T_i)$, $1 \leq i \leq 2$, with Σ_1, Σ_2 optimally intersectable in many-sorted set of sorts $(S_0, =_{S_0})$, with $S_0 = \{s_1, \ldots, s_n\}$, with T_i-satisfiability of $QF(\Sigma_i)$-formulas decidable, $1 \leq i \leq 2$.

<u>Wanted</u>: A decision procedure for $T_1 \cup T_2$-satisfiability of $QF(\Sigma_1 \cup \Sigma_2)$-formulas

<u>Enough to have</u>: A decision procedure for $T_1 \cup T_2$-satisfiability of $\bigwedge Lit(\Sigma_1) \wedge \bigwedge Lit(\Sigma_2)$-formulas, because of the formula transformations

$$(T_1 \cup T_2,\ QF(\Sigma_1 \cup \Sigma_2)) \underset{1_{\Sigma_1 \cup \Sigma_2}}{\overset{dnfcnj}{\longleftrightarrow}} (T_1 \cup T_2, \bigwedge Lit(\Sigma_1 \cup \Sigma_2)) \underset{1_{\Sigma_1 \cup \Sigma_2}}{\overset{pur}{\longleftrightarrow}} (T_1 \cup T_2, \bigwedge Lit(\Sigma_1) \wedge \bigwedge Lit(\Sigma_2)).$$

<u>Some Basic Notions</u>

1. First note that in choosing $\varphi_1 \wedge \varphi_2 \in \bigwedge Lit(\Sigma_1) \wedge \bigwedge Lit(\Sigma_2)$ there is some ambiguity about literals of the form $x:s = x':s$, or $x:s \neq x':s$ with $s \in S_0$, since they could go into φ_1 or φ_2. We assume that which S_0-literal belongs to φ_1 or φ_2 is <u>syntactically identifiable</u> by: $(\varphi_1) \wedge (\varphi_2) \in \bigwedge Lit(\Sigma_1) \wedge \bigwedge Lit(\Sigma_2)$
2. Define $X^{1,2} = vars(\varphi_1) \cap vars(\varphi_2) \subseteq X_{S_0}$ where

$$X_{S_0} = \{X_s\}_{s \in S_0},$$ and

$$X^{1,2}_s = vars(\varphi_1) \cap vars(\varphi_2) \cap X_s, \quad s \in S_0$$

Note that $X^{1,2}$ are the <u>only</u> variables that can be <u>shared</u> between φ_1 and φ_2 because of the definition of optimally intersectable Σ_1 and Σ_2, with $S_1 \cap S_2 = S_0$.

3. Define $Equiv(X^{1,2})$ to the the set of S_0-sorted equivalence relations on the S_0-sorted set $X^{1,2}$, i.e.,

$\equiv = \{\equiv_s\}_{s \in S_0} \in Equiv(X^{1,2})$ iff $\equiv_s \in Equiv(X^{1,2}_s)$, $s \in S_0$.

Assume a linear order on $X^{1,2}$. Then for each $\equiv \in Equiv(X^{1,2})$ define its arrangement as the formula:

$$\varphi_{\equiv} = \bigwedge_{s \in S} \left(\bigwedge_{\substack{(x_i:s,\, x_j:s) \in \equiv_s \\ i<j}} x_i : s_i = x_j : s_j \;\wedge \bigwedge_{\substack{(x_l:s,\, x_k:s) \notin \equiv_s \\ l<k}} x_l : s \neq x_k : s \right)$$

Proposition 1. For Σ_1 and Σ_2 optimally intersectable in $(S_0, =_{S_0})$, the function

$$arr : \bigwedge Lit(\Sigma_1) \wedge \bigwedge Lit(\Sigma_2) \longrightarrow \mathcal{P}_{fin}\left(\bigwedge Lit(\Sigma_1) \wedge Lit(S_0) \wedge \bigwedge Lit(\Sigma_2)\right)$$

$$\varphi_1 \wedge \varphi_2 \longmapsto \{(\varphi_1) \wedge \varphi_{\equiv} \wedge (\varphi_2) \mid \equiv \in Equiv(X^{1,2})\}$$

is a correct formula transformation where $X^{1,2} = vars(\varphi_1) \cap vars(\varphi_2)$

$$(T_1 \cup T_2,\ \bigwedge Lit(\Sigma_1) \wedge \bigwedge Lit(\Sigma_2)) \xleftrightarrow[\Sigma_1 \cup \Sigma_2]{arr} (T_1 \cup T_2,\ \bigwedge Lit(\Sigma_1) \wedge Lit(S_0) \wedge Lit(\Sigma_2))$$

Proof. $(\Leftarrow)$ Let $[M_1, M_2] \in mod(T_1 \cup T_2)$. Obviously, if $[M_1, M_2]$ satisfies some $\varphi_1 \wedge \varphi_{\equiv} \wedge \varphi_2$, a fortiori it satisfies $\varphi_1 \wedge \varphi_2$.

$(\Leftarrow)$ Let $[M_1, M_2], a \models \varphi_1 \wedge \varphi_2$ for $a \in [Y \to [M_1, M_2]]$ with $Y = vars(\varphi_1 \wedge \varphi_2)$. Let $\equiv_a \in X^{1,2}$ be defined by the equivalence $(x:s, x':s) \in \equiv_s \Leftrightarrow a(x:s) = a(x':s)$, $s \in S$. Then $[M_1, M_2], a \models \varphi_1 \wedge \varphi_{\equiv_a} \wedge \varphi_2$. □

1.2 Reduction Families

The notion of reduction map can be further extended to that of <u>reduction family</u>, i.e., a family of maps that <u>jointly</u> provide a reduction.

<u>Definition</u>. Given OS-FO theories $T, T_1, \ldots, T_n$, $n \geq 1$, and recursive sets $\mathcal{F} \subseteq FO(sig(T))$, $\mathcal{G}_i \subseteq FO(sig(T_i))$, $1 \leq i \leq n$, a <u>reduction family</u> is a sequence of recursive functions $\rho_1, \ldots, \rho_n$, where $\rho_i : \mathcal{F} \to \mathcal{P}_{fin \neq \emptyset}(\mathcal{G}_i)$, denoted $(T, \mathcal{F}) \xrightarrow{(\rho_1, \ldots, \rho_n)} ((T_1, \mathcal{G}_1), \ldots, (T_n, \mathcal{G}_n))$ such that for each $\varphi \in \mathcal{F}$, φ is T-satisfiable iff $\exists \psi_i \in \rho_i(\varphi)$, $1 \leq i \leq n$ such that ψ_i is T_i-satisfiable. $(\rho_1, \ldots, \rho_n)$ is called <u>effective</u> if from $\mathcal{M}_i, a_i \models \psi_i$, $\mathcal{M}_i \in mod(T_i)$ we can reconstruct a corresponding $\mathcal{M}, a \models \varphi$, $\mathcal{M} \in mod(T)$.

Note that when $n = 1$ this is just a reduction map. Therefore, reduction families <u>generalize reduction maps</u>. Furthermore, they can be <u>composed</u> as follows:

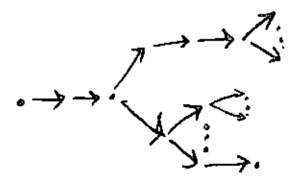
<u>Lemma 1</u> Let $(T, \mathcal{F}) \xrightarrow{(\rho_1, \ldots, \rho_n)} ((T_1, \mathcal{G}_1), \ldots, (T_n, \mathcal{G}_n))$, and $(T_i, \mathcal{G}_i) \xrightarrow{(\gamma^i_1, \ldots, \gamma^i_{m_i})} ((T^i_1, K^i_1), \ldots, (T^i_{m_i}, K^i_{m_i}))$, $1 \leq i \leq n$ be reduction families. Then

$$(T, \mathcal{F}) \xrightarrow{(\rho_1;\gamma^1_1, \ldots, \rho_1;\gamma^1_{m_1}, \ldots, \rho_n;\gamma^n_1, \ldots, \rho_n;\gamma^n_{m_n})} ((T^1_1, K^1_1), \ldots, (T^n_{m_n}, K^n_{m_n}))$$

is a reduction family. $\square$

Pictorially, we can visualize such compositions as trees

Note that, in particular, since each reduction map is a reduction family we also get trees of the form:

 or, more generally, 

1.3 Order-Sorted Nelson-Oppen

The order-sorted Nelson-Oppen Combination Theorem has now a succint statement and proof in terms of reduction families.

<u>Theorem (Order-Sorted Nelson-Oppen Combination)</u>. Let T_1, T_2 have optimally intersectable with $T_1 \cap T_2 = (S_0, =_{S_0})$, $S_0 = \{s_1, .., s_n\}$, and $|\Sigma_i| \leq \aleph_0$, $1 \leq i \leq 2$. Assume, furthermore, that T_i is stably infinite in sorts $s_1, .., s_n$ for $QF(\Sigma_i)$, $1 \leq i \leq n$. Then the family obtained by the composition:

$$(T_1 \cup T_2, QF(\Sigma_1 \cup \Sigma_2)) \underset{1_{\Sigma_1 \cup \Sigma_2}}{\overset{\text{dnf cnjs purian}}{\longleftrightarrow}} (T_1 \cup T_2, \bigwedge Lit(\Sigma_1) \wedge \bigwedge Lit(S_0) \wedge Lit(\Sigma_2)) \xrightarrow{(\pi_1, \pi_2)} \begin{matrix} (T_1, \bigwedge Lit(\Sigma_1) \wedge Lit(S_0)), \\ (T_2, Lit(\Sigma_2) \wedge Lit(S_0)) \end{matrix}$$

where $\pi_1((\varphi_1) \wedge (\varphi_0) \wedge (\varphi_2)) = \varphi_1 \wedge \varphi_0$, $\pi_2((\varphi_1) \wedge (\varphi_0) \wedge (\varphi_2)) = \varphi_2 \wedge \varphi_0$

is an effective reduction family.

Proof. First of all note that we do not claim that (π_1, π_2) itself is a reduction family, but that the composition (dfn cnjs pur arr π_1, dfn cnjs pur arr π_2) is so.

$a \in [Y \to [M_1, M_2]]$ s.t. $[M_1,M_2], a \models \varphi$.

To see the ($\Rightarrow$) part, let $\varphi \in QF(\Sigma_1 \cup \Sigma_2)$, $[M_1,M_2] \in mod(T_1 \cup T_2)$

Since we have proved that dfn cnjs pur arr is a formula transformation, this means that there is $\varphi_1 \wedge \varphi_2 \in$ (dfn cnjs pur)(φ) such that $[M_1, M_2], a \models \varphi_1 \wedge \varphi_2$ and therefore $\varphi_1 \wedge \varphi_{\equiv_a} \wedge \varphi_2 \in$ (dfn cnjs pur arr)(φ) such that $[M_1, M_2], a \models \varphi_1 \wedge \varphi_{\equiv_a} \wedge \varphi_2$, which forces $M_1, a|_{X_{S_1}} \models \varphi_1 \wedge \varphi_{\equiv a}$, and $M_2, a|_{X_{S_2}} \models \varphi_2 \wedge \varphi_{\equiv a}$, as desired.

To see the ($\Leftarrow$) part it is enough to show that if for some $\varphi_1 \wedge \varphi_{\equiv} \wedge \varphi_2$ with $X^{1,2} = vars(\varphi_1) \cap vars(\varphi_2)$, $\equiv \in Equiv(X^{1,2})$ in (dfn cnjs pur arr)(φ) we have $M_i \in Mod(T_i)$, $a_i \in [Y_i \to M_i]$ with $Y_i = vars(\varphi_i \wedge \varphi_{\equiv})$, such that $M_i, a_i \models \varphi_i \wedge \varphi_{\equiv}$, $1 \le i \le 2$, then there is $[M_1''', M_2'''] \in mod(T_1 \cup T_2)$ and $a \in [Y \to [M_1''', M_2''']]$ with $Y = vars(\varphi_1 \wedge \varphi_{\equiv} \wedge \varphi_2)$ such that $[M_1''', M_2'''], a \models \varphi_1 \wedge \varphi_{\equiv} \wedge \varphi_2$, since then $[M_1''', M_2'''], a \models \varphi$. But by stable infinity we have models M_i', $1 \le i \le 2$ satisfying $\varphi_i \wedge \varphi_{\equiv}$ and such that $|M_{i,s}'| \geq \aleph_0$, $s \in S_0$. But this exactly means $M_i' \models \exists\, \varphi_i \wedge \varphi_{\equiv}$, $1 \le i \le 2$. And by the downward Löwenheim-

Skolem Theorem we then have elementary submodels $\mathfrak{M}''_i \preccurlyeq \mathfrak{M}'_i$ such that $\mathfrak{M}''_i \models \exists \varphi_i \wedge \varphi_\equiv$, $1 \le i \le 2$, and $|M''_{i,s}| = \aleph_0$, $s \in S_0$. But this exactly means that we have $a_i \in [Y_i \to M''_i]$ such that $\mathfrak{M}''_i, a_i \models \varphi_i \wedge \varphi_\equiv$, $1 \le i \le 2$, and since, by $\varphi_\equiv$, for each $x{:}s, x'{:}s \in X^{1,2}_s$, $s \in S_0$ we have $a_1(x{:}s) = a_1(x'{:}s)$ $\Leftrightarrow$ $a_2(x{:}s) = a_2(x'_s)$ it follows that we can find a bijection h^0_s such that

$$\begin{tikzcd}
 & X^{1,2}_s \arrow[dl, "a_{1,s}"'] \arrow[dr, "a_{2,s}"] \arrow[d, phantom, "="] & \\
a_{1,s}[X^{1,2}_s] \arrow[rr, "\cong", "h^0_s"'] & {} & a_{2,s}[X^{1,2}_s]
\end{tikzcd}$$

for each $s \in S_0$

and since $M''_{1,s} - a_{1,s}[X^{1,2}_s]$ and $M''_{2,s} - a_{2,s}[X^{1,2}_s]$ are countable, they can also be related by a bijection, so that for each $s \in S_0$ we get a bijection $h_s : M''_{1,s} \xrightarrow{\cong} M''_{2,s}$, and therefore an isomorphism $h : \mathfrak{M}''_1|_{S_0} \cong \mathfrak{M}''_2|_{S_0}$ such that $a_2 = a_1 ; h$. But by the amalgamation up to isomorphism theorem (Theorem 1, Lecture 29) and its proof (which uses the above h_s or its inverse h_s^{-1} for each $s \in S$) we then get isomorphisms $g_i : \mathfrak{M}''_i \xrightarrow{\cong} \mathfrak{M}'''_i$, $1 \le i \le n$ such that $\mathfrak{M}'''_1|_{S_0} = \mathfrak{M}'''_2|_{S_0}$, $g_{i,s}$ is either h_s or h_s^{-1}, and $\quad a_1|_{X^{1,2}} ; g_1 = a_2|_{X^{1,2}} ; g_2$

But since $X^{1,2}$ are the only variables shared by φ_1 and φ_2 this means that $a = (a_1 ; g_1 \cup a_2 ; g_2) \in [Y \to [M_1''', M_2''']]$ is well-defined, and we have

$[\mathcal{M}_1''', \mathcal{M}_2'''], a \models \varphi_1 \wedge \varphi_\equiv \wedge \varphi_2$, since

$\mathcal{M}_1''', a|_{X_{S_1}} \models \varphi_1 \wedge \varphi_\equiv$, and $\mathcal{M}_2''', a|_{X_{S_2}} \models \varphi_2 \wedge \varphi_\equiv$,

as desired. Note that $[\mathcal{M}_1''', \mathcal{M}''']$ and a have been "effectively" found in terms of a mathematical construction, not in a computational sense. □

<u>Corollary</u>. Suppose that in the above theorem satisfiability of QF formulas in T_i is decidable, $1 \leq i \leq 2$. Then satisfiability of $QF(\Sigma_1 \cup \Sigma_2)$-formulas in $T_1 \cup T_2$ is also decidable. □

<u>Remark</u>. Since we know that $(T_1 \cup T_2, QF(\Sigma_1 \cup \Sigma_2)) \xleftrightarrow[1_{\Sigma_1 \cup \Sigma_2}]{\text{dnf cnjs fun}} (T_1 \cup T_2, \bigwedge Lit(\Sigma_1) \cup \bigwedge Lit(\Sigma_2))$ is a formula transformation, the statement (and proof) of the OS Nelson-Oppen Combination Theorem can be simplified to the claim that:

$$(T_1 \cup T_2, \bigwedge Lit(\Sigma_1) \wedge \bigwedge Lit(\Sigma_2)) \xrightarrow{(ans\,\pi_1, ans\,\pi_2)} ((T_1, \bigwedge Lit(\Sigma_1) \wedge Lit(S_0)), (T_2, \bigwedge Lit(\Sigma_2) \wedge Lit(S_0)))$$

is an effective reduction family under the Theorem's assumptions. Then we obtain the original statement in the theorem by applying Lemma 1.

Remark. The OS version of the Nelson-Oppen procedure can be much more efficient than the unsorted one, since typically there will be much fewer S_0-sorted equivalence classes for $X^{1,2}$ than unsorted ones. However, if $X^{1,2}$ is large the size of $Equiv(X^{1,2})$ may be very large even in the S_0-sorted case. One first optimization is to decompose $\varphi_1 \wedge \varphi_2$ into $\varphi_1' \wedge \varphi_0 \wedge \varphi_2$, with φ_0 the S_0-literals in the conjunction. Then we can automatically discard those $\equiv \in Equiv(X^{1,2})$ such that $\varphi_0 \wedge \varphi_\equiv$ is S_0-unsatisfiable, which can be easily checked by S_0-congruence closure, which boils down to a union-find computation in this case. Much greater efficiency can be gained by additional methods explained later.

Example. It may be useful to see the Nelson-Oppen (NO) combination method in action through a single example. Let $T_1 = List[X]*(X \text{ to } Rat)$, and $T_2 = (\{0,1,+,\geq\}, \{\mathbb{Q}_{+,\geq}\})$ the theory of linear arithmetic. I claim that the following is a theorem in $T_1 \cup T_2$ for L, L' variables of sort NeList:

$$(\ast) \quad head(L) > head(L') \Rightarrow L \neq L' \wedge 1/2 \cdot head(L) > 1/2\, head(L')$$

which gets desugared as:

$$head(L) + head(L) > head(L') + head(L') \Rightarrow L \neq L' \wedge head(L) > head(L')$$

It negation becomes the disjunction of conjunctions:

$$head(L) + head(L) > head(L') + head(L') \wedge L = L'$$

$$\vee$$

$$head(L) + head(L') > head(L') + head(L') \wedge head(L) \not> head(L')$$

which become purified as:

$$L = L' \wedge head(L) = x \wedge head(L') = y \wedge x + x > y + y$$

$$head(L) = x \wedge head(L') = y \wedge x + x > y + y \wedge x \not> y$$

$X^{1,2} = \{x, y\}$ so we have two arrangements:

$$\varphi_{\equiv_1} = x \neq y$$

$$\varphi_{\equiv_2} = x = y$$

But $L = L' \wedge head(L) = x \wedge head(L') = y \wedge x \neq y$ is unsatisfiable in List[X]∗(X to Rat), and

$x + x > y + y \wedge x > y \wedge x = y$ is unsatisfiable in T_2.

Therefore, (✡) is a theorem of $T_1 \cup T_2$, since both List[X]∗(X to Rat) is stably infinite, in Rat thanks to its generic model, and T_2 is stably infinite in Rat because T_2 is a complete theory and therefore $\mathbb{Q}_{+,>}$ is a generic model for it.

1.4 An Efficient NO Inference System

As pointed out above, the number of S_0-sorted equivalence relations on $X^{1,2}$, although usually much smaller than that of unsorted equivalence relations on the same set of variables can grow exponentially. As shown in Fig. 1 of the Manna-Zarba paper on "Combining Decision Procedures", for 12 variables of the same sort there are 4,213,597 Equivalence relations!

Rather than trying them all, a much better approach is a *Darwinian* one, where we allow T_1 and T_2 to exchange information about *what* equalities between variables in $X^{1,2}$ are *logical consequences* of φ_1, (resp. φ_2) to see if any of them already "kills" the other procedure because it renders φ_i unsatisfiable. The only source of non-determinism is that it is not enough to consider single equalities $x = y$ such that

$$T_i \models \varphi_i \wedge \bigwedge E \Rightarrow x = y$$

where the E are previously tested equalities. Unless some additional property of *convexity* (more on this to come) can be assumed of T_i, we also need to consider entailments of the form:

$$T_i \models \varphi_i \wedge \bigwedge E \Rightarrow \bigvee_{i \in I} x_i = y_i$$

The inference system has an easy description as a <u>rewrite theory</u> at the meta-level of OS-FO logic. It has just <u>two equations</u> and a <u>single rewrite rule</u>. The operators in the rewrite theory are:

1. A <u>commutative</u> pairing operator $\{-,-\}$, where we will keep $\{\varphi_i, \varphi_j\}$, $\{i,j\}=\{1,2\}$, $\varphi_i \in \bigwedge Lit(\Sigma_i)$, $1 \leq i \leq 2$.
2. A <u>marking</u> operator $[-]$ to mark $[\varphi_i]$ as needing to be check for satisfiability with equations on shared variables $X^{1,2}$.
3. A <u>triple operator</u> $[-,-,-]$. to represent $T_1 \cup T_2$-satisfiability problems as triples $[\{\Gamma_i, \Gamma_j\}, E, X^{1,2}]$ with E equations among shared variables $X^{1,2}$
4. An associative-commutative disjunction operator $-\vee-$ with identity $\perp$, to represent disjunctions of satisfiability problems, say, $[\{\varphi_i,\varphi_j\},E_1,X^{1,2}] \vee \cdots \vee [\{\varphi_i,\varphi_j\},E_n,X^{1,2}]$

Given a $T_1 \cup T_2$-satisfiability problem $\varphi_1 \wedge \varphi_2$, we begin in the initial state $[\{\varphi_1,\varphi_2\},\emptyset,X^{1,2}]$ and perform R/E-rewriting searching for a terminating state of the form $[\{\varphi_1,\varphi_2\},E,X^{1,2}]$ which will solve the satisfiability problem.

In Maude we would do this with the command:

$$\underline{\text{search}}\quad [\{\varphi_1, \varphi_2\}, \emptyset, X^{1,2}] \Rightarrow! \ [\{\varphi_1, \varphi_2\}, E, X^{1,2}]$$

where E is a variable of sort Equation Set.

The two equations E are:

equality-propagation

$$[\{\varphi_i, \varphi_j\}, E, X^{1,2}] = [\{\varphi_i, [\varphi_j]\}, E \cup \{x = y\}, X^{1,2}]$$

if $\exists s \in S_0,\ x, y \in X^{1,2}_s,\ x \not\equiv y,\ x = y \notin E,\ T_i \models \varphi_i \wedge \bigwedge E \Rightarrow x = y$

Note: $\_=\_$ is a comutative symbol

sat-check

$$[\{\varphi_i, [\varphi_j]\}, E, X^{1,2}] = \underline{\text{if}}\ \ \varphi_j \wedge \bigwedge E \ \ T_j\text{-unsatisfiable}$$
$$\underline{\text{then}}\ \ \bot$$
$$\underline{\text{else}}\ \ [\{\varphi_i, \varphi_j\}, E, X^{1,2}]$$
$$\underline{\text{fi}}$$

The only rule in R is:

split

$$[\{\varphi_i, \varphi_j\}, E, X^{1,2}] \longrightarrow [\{\varphi_i, \varphi_j\}, E \cup \{x_1 = y_1\}, X^{1,2}] \vee \ldots \vee [\{\varphi_i, \varphi_j\}, E \cup \{x_n = y_n\}, X^{1,2}]$$

if $\exists s_1, \ldots, s_n \in S,\ x_i, y_i \in X^{1,2}_{s_i},\ 1 \leq i \leq n,\ x_i \not\equiv y_i,\ x_i = y_i \notin E,$

$T_i \models \varphi_i \wedge \bigwedge E \Rightarrow x_1 = y_1 \vee \ldots \vee x_n = y_n.$

The main theorem about this terminating! rewrite theory / inference sistem is:

Theorem (Soundness and Completeness). Given T_1, T_2 satisfyp the assumptions in the NO theorem and $\varphi_i \in \bigwedge Lit(sig(T_i))$, $\varphi_1 \wedge \varphi_2$ is $T_1 \cup T_2$-satisfiable iff

$$[\{\varphi_1, \varphi_2\}, \emptyset, X^{1,2}] \xrightarrow[R/E]{}! [\{\varphi_1, \varphi_2\}, E, X^{1,2}]$$

for some set E of equations among the shared variables $X^{1,2}$ between φ_1 and φ_2.

Proof. For soundness, we need to prove that the two equations and the rule preserve equi-satisfiability in the sense that $\varphi_1 \wedge \varphi_2 \wedge \bigwedge E$ from the left-hand side is $T_1 \cup T_2$-satisfiable iff the (possibly disjunctive) corresponding formula in the righthand side is $T_1 \cup T_2$-satisfiable. Specifically:

equality propagation: If $[M_1, M_2], a \models \varphi_1 \wedge \varphi_2 \wedge \bigwedge E$ since by $T_i \models \varphi_i \wedge \bigwedge E$ we have $M_i, a|_{X_{S_i}} \models x = y$ and therefore $M_i|_{S_0}, a|_{X^{1,2}} \models x = y$, so that $M_j|_{S_0}, a|_{X^{1,2}} \models x = y$ and therefore $M_j, a|_{S_j} \models x = y$, which forces $[M_1, M_2], a \models \varphi_1 \wedge \varphi_2 \wedge E \wedge x = y$.

sat-check: trivially preserves $T_1 \cup T_2$-satisfiability

split: If $[\mathcal{M}_1, \mathcal{M}_2], a \models \varphi_1 \wedge \varphi_2 \wedge \bigwedge E$, we need to prove that $[\mathcal{M}_1, \mathcal{M}_2], a \models \varphi_1 \wedge \varphi_2 \wedge \bigwedge E \wedge (x_1 = y_1 \vee \cdots \vee x_n = y_n)$.
But since $T_i \models \varphi_i \wedge \bigwedge E \Rightarrow x_1 = y_1 \vee \cdots \vee x_n = y_n$ there must be an i, $1 \leq i \leq n$ such that $\mathcal{M}_i, a|_{S_i} \models x_i = y_i$, therefore $\mathcal{M}_i|_{S_0}, a|_{X^{1,2}} \models x_i = y_i$, therefore $\mathcal{M}_j, a|_{S_j} \models x_i = y_i$, therefore, $[\mathcal{M}_1, \mathcal{M}_2], a \models \varphi_1 \wedge \varphi_2 \wedge \bigwedge E \wedge (x_1 = y_1 \vee \cdots \vee x_n = y_n)$.

Completeness. Note that the only R/E-terminating states are either $\perp$, which by satisfiability preservation shows that $\varphi_1 \wedge \varphi_2$ is $T_1 \cup T_2$-unsatisfiable, or must have the form $[\{\varphi_1, \varphi_2\}, E_1, X^{1,2}] \vee \cdots \vee [\{\varphi_1, \varphi_2\}, E_m, X^{1,2}]$ where none of the triples can be rewritten by E or by R. It is enough to show that for any such triple $[\{\varphi_1, \varphi_2\}, E, X^{1,2}]$, $\varphi_1 \wedge \varphi_2 \wedge \bigwedge E$ is $T_1 \cup T_2$-satisfiable, and for this, by NO, it is enough to prove that $\varphi_i \wedge \varphi_{\equiv_E}$ is T_i-satisfiable, $1 \leq i \leq 2$.
First of all note that if we add to E all trivial equalities $x = x$, $x \in X^{1,2}_s$, $s \in S_0$, we obtain an equivalence relation, since otherwise there would be an $x = y \notin E$ such that $S_0 \models \bigwedge E \Rightarrow x = y$ and, a fortiori, $T_i \models \varphi_i \wedge \bigwedge E \Rightarrow x = y$, $1 \leq i \leq 2$, so that $[\{\varphi_1, \varphi_2\}, E, X^{1,2}]$ would not be irreducible.

But since E with the trivial identities defines an equivalence relation $\equiv_E$, we have

$$\varphi_{\equiv_E} = \bigwedge E \wedge \bigwedge \overbrace{\{x \neq y \mid \exists 1 \in S_0, x, y \in X_1^{1,2}, x \not\equiv y, x = y \notin E\}}^{D_E}$$

We now reason by contradiction. Since sat-check cannot be applied, we know that $\varphi_i \wedge \bigwedge E$ is T_i-satisfiable, $1 \leq i \leq 2$. Suppose however that $\varphi_j \wedge \varphi_{\equiv_E}$ is T_j-unsatisfiable for some $j \in \{1,2\}$. This means that for any $\mathcal{M}_j \in mod(T_j)$ and $a \in [Y|_{S_j} \to M_j]$ such that $\mathcal{M}_j, a \models \varphi_j \wedge \bigwedge E$

we must have $\mathcal{M}_j, a \not\models \bigwedge D_E$, that is,

$\mathcal{M}_j, a \models \bigvee \{x = y \mid x \neq y \in D_E\}$, which proves that

$T_j \models \varphi \wedge \bigwedge E \Rightarrow \bigvee \{x = y \mid x \neq y \in D_E\}$, so that

$[\{\varphi_1, \varphi_2\}, E, X^{1,2}]$ can be rewritten with the split rule, contradicting the R/E-irreducibility of

$[\{\varphi_1, \varphi_2\}, E, X^{1,2}]$. $\square$