CS 576 Lecture 29, 4/25/17

J. Meseguer

## 1. Translating Models Along Bijections

Up to isomorphism the specific details of a model's data do not matter, although they <u>may</u> matter algorithmically: for example, a binary representation of a natural number will be much more efficient that its representation by means of a Gödel number as a product of powers of primes. This means that we can translate the entire <u>structure</u> of a model to another family of sets to obtain an isomorphic model there. Specifically:

<u>Definition</u> Let $\Sigma = ((S, \leq), F, P)$ be an OS-FO-signature, and let $\mathcal{M} = (M, \_{}_{\mathcal{M}})$ be a $\Sigma$-model. For each connected component $[s] \in S/{\equiv_{\leq}}$ define

$$M_{[s]} = \bigcup_{s' \in [s]} M_{s'}.$$

Let $h = \{h_{[s]} : M_{[s]} \to M'_{[s]}\}_{[s] \in S/\equiv_{\leq}}$ be an $S/{\equiv_{\leq}}$-sorted family of <u>bijections</u> to the $S/{\equiv_{\leq}}$-sorted set $\{M'_{[s]}\}_{[s] \in S/\equiv_{\leq}}$. We then define the $\Sigma$-model $\mathcal{M}_h$, called the <u>translation</u> of $\mathcal{M}$ <u>along</u> $h$ as follows: $\mathcal{M}_h = (M', \_{}_{\mathcal{M}_h})$, where:

1. $M'_s = h_{[s]}[M_s]$, $s \in S$ (so that $h_{[s]}[M_s] \subseteq M'_{[s]}$)

2. For each $f : s_1 \dots s_n \to s$ in $F$

$$f_{\mathcal{M}_h} : M'_{s_1} \times \dots \times M'_{s_n} \ni (b_1, .., b_n) \mapsto h_{[s]}\left(f_{\mathcal{M}}\left(h^{-1}_{[s_1]}(b_1), \dots, h^{-1}_{[s_n]}(b_n)\right)\right) \in M'_s$$

3. for each $p : s_1 \dots s_n$ in $P$, $(b_1, .., b_n) \in M'_{s_1} \times \dots \times M'_{s_n}$

$$(b_1, .., b_n) \in p_{\mathcal{M}_h} \Leftrightarrow \left(h^{-1}_{[s_1]}(b_1), \dots, h^{-1}_{[s_n]}(b_n)\right) \in p_{\mathcal{M}}.$$

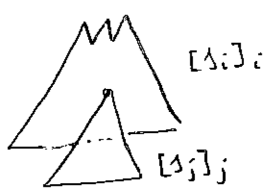
It is then easy to check that for each $s \in S$ $h_{[s]}$ restricts to a bijective function $h_s : M_s \to M'_s$, $s \in S$, and that $h : \mathcal{M} \to \mathcal{M}_h$ is a $\Sigma$-isomorphism.

## 2. Amalgamations up to Isomorphism

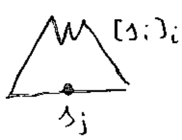
The $\mathcal{M} \mapsto \mathcal{M}_h$ construction can be useful to tackle the following problem. Suppose $\Sigma_1$ and $\Sigma_2$ are nicely or optimally intersectable. Then, if $\mathcal{M}_i \in Mod(\Sigma_i)$, $1 \leq i \leq 2$, and $\mathcal{M}_1|_{\Sigma_0} = \mathcal{M}_2|_{\Sigma_0}$, for $\Sigma_0 = \Sigma_1 \cap \Sigma_2$, we get an amalgamation $[\mathcal{M}_1, \mathcal{M}_2] \in Mod(\Sigma_1 \cup \Sigma_2)$. But what can we do if we only have an isomorphism $\mathcal{M}_1|_{\Sigma_0} \cong \mathcal{M}_2|_{\Sigma_0}$? We can build isomorphic models $\mathcal{M}'_i \simeq \mathcal{M}_i$, $1 \leq i \leq 2$ such that $\mathcal{M}'_1|_{\Sigma_0} = \mathcal{M}'_2|_{\Sigma_0}$. Although a more general theorem can be proved, the following theorem will suffice for our present purposes:

<u>Theorem 1 (Amalgamation up to Isomorphism)</u>. Let $T_1, T_2$ be OS-FO theories with $\Sigma_i = sig(T_i)$, $1 \leq i \leq 2$ such that $\Sigma_1, \Sigma_2$ are optimally intersectable in the broader sense of Lecture 28 and $\Sigma_0 = \Sigma_1 \cap \Sigma_2$ is many-sorted and has no function or predicate symbols, just sorts $s_1, .., s_n$, each as a separate singleton connected component $\{s_i\}$. Let $\mathcal{M}_i \in Mod(T_i)$, $1 \leq i \leq 2$ be such that $|M_{1,s_j}| = |M_{2,s_j}|$, $1 \leq j \leq n$ (which of course means $\mathcal{M}_1|_{\Sigma_0} \cong \mathcal{M}_2|_{\Sigma_0}$). Then there exist $\mathcal{M}'_i$ such that: (i) $\mathcal{M}'_i \cong \mathcal{M}_i$, $1 \leq i \leq 2$, and (ii) $\mathcal{M}'_1|_{\Sigma_0} = \mathcal{M}'_2|_{\Sigma_0}$, so that $[\mathcal{M}'_1, \mathcal{M}'_2] \in Mod(T_1 \cup T_2)$.

<u>Proof</u>. First of all note that if $[s_i]_i \cap [s_j]_j \neq \emptyset$, $\{i,j\} = \{1,2\}$, since $\Sigma_1$ and $\Sigma_2$ are optimally intersectable and $[s_i]_i \cap [s_j]_j = \{s_\ell\}$, $1 \leq \ell \leq n$, the only possible relationships between $[s_i]$ and $[s_j]$ are as follows:

(i) 


(ii) 


(iii) 


(iv) $s_i \bullet = \bullet s_j$

Where each subsequent case is a <u>special case</u> of the previous one, and where in cases (ii) − (iv) we have actual <u>containments</u> $[s_i] \subseteq [s_j]$, but $[s_i] = \{s_\ell\}$. Define

$h = \{h_{s_\ell}\}_{1 \leq \ell \leq n}$ as follows:

In cases (i) - (iii) as a bijection $h_{s_\ell} : M_{j,s_\ell} \to M_{i,s_\ell}$

In case (iv) chose one direction $h_{s_\ell} : M_{j,s_\ell} \to M_{i,s_\ell}$.

By abuse of language, call also $h_{s_\ell}$ to the translation

$$h_{s_\ell} = \{ h_{s_\ell [s]_j} : M_{[s]_j} \to M'_{[s]_j} \}_{[s]_j \in S_j / \equiv_{\leqslant_j}}$$ where:

(i) if $[s]_j = [s_\ell]_j$ then $M'_{[s]_j} = M_{i,s_\ell}$ and

$h_{s_\ell [s_\ell]_j} = h_{s_\ell}$ ; (ii) otherwise $M_{[s]_j} = M'_{[s_j]}$ and

$h_{s_\ell [s]_j} = 1_{M_{[s]_j}}$. That is, only connected component

$[s_\ell]_j$ is translated.

Call such translations <u>single-kind</u> translations. Note that the notation still makes sense after having performed several such single-kind translations.

That is, if $[s]_i \cap [s']_i = \emptyset$ with $s$, resp. $s'$ the top element of $([s]_i, \leq_i)$, resp. $([s']_i, \leq_i)$ we can perform translations:

$$\mathcal{M}_i \simeq \mathcal{M}_{i\, h_s} \simeq \mathcal{M}_{i\, h_s h_{s'}}$$

Now note that the maps $\{h_{s_\ell}\}_{1 \leq \ell \leq n}$ split into maps

$$h_{s_{i_1}}, \ldots, h_{s_{i_{k_1}}} \qquad i_1 < i_2 < \cdots < i_{k_1} \quad \text{for } \mathcal{M}_1$$

$$h_{s_{j_1}}, \ldots, h_{s_{j_{k_2}}} \qquad j_1 < j_2 < \cdots < j_{k_2} \quad \text{for } \mathcal{M}_2$$

with $k_1 + k_2 = n$. Then observe that

$$\mathcal{M}_1 \cong \mathcal{M}_{1\, h_{s_{i_1}} \cdots h_{s_{i_{k_1}}}}, \quad \mathcal{M}_2 \cong \mathcal{M}_{2\, h_{s_{j_1}} \cdots h_{s_{j_{k_2}}}}$$

and we have $\mathcal{M}_{1\, h_{s_{i_1}} \cdots h_{s_{i_{k_1}}}}\big|_{\Sigma_0} = \mathcal{M}_{2\, h_{s_{j_1}} \cdots h_{j_{k_2}}}\big|_{\Sigma_0}$

as desired. $\square$

## 3. Excursus: Order-Sorte Tarski-Vaught and Downard Löwenheim-Skolem Theorems

To obtain the desired order-sorted Nelson-Oppen combination result we first need a detour to prove order-sorted generalizations of the Tarski-Vaught test for elementary submodels and the downward Löwenheim-Skolem theorem.

First of all, note that we can view a poset of sorts $(S, \leq)$ as an OS-FO signature $(S, \leq) = ((S, \leq), \emptyset, \emptyset)$. In particular, a set $S$ of sorts can be view as the signature $S = ((S, =_S), \emptyset, \emptyset)$. Therefore, if $\Sigma = ((S,\leq), F, P)$ we have signature inclusions

$$S \xhookrightarrow[J]{} (S, \leq) \xhookrightarrow[K]{} \Sigma$$

and since all these signatures can be viewed as Horn theories with empty set of axioms we have left adjoints to reducts:

$$\begin{tikzcd}
\mathbf{Set}_S \arrow[r, dashed, bend left, "\mathbb{F}_J"] \arrow[d, equal] & \mathbf{Set}_{(S,\leq)} \arrow[l, "-|_S"] \arrow[r, dashed, bend left, "\mathbb{F}_K"] \arrow[d, equal] & \mathbf{Mod}(\Sigma) \arrow[l, "-|_{(S,\leq)}"] \\
\mathbf{Mod}(S) & \mathbf{Mod}(S,\leq) &
\end{tikzcd}$$

and therefore, also $\mathbb{F}_{J;K} = \mathbb{F}_J ; \mathbb{F}_K$. $\mathbb{F}_J$ is quite easy to describe on objects. It maps an $S$-sorted set $A = \{A_s\}_{s \in S}$ to $\hat{A} = \{\hat{A}_s\}_{s \in S}$ with $\hat{A}_s = \bigcup_{s' \leq s} A_{s'}$

so that if $s \leq s'$, then $\hat{A}_s \subseteq \hat{A}_{s'}$, which is the essential requirement for $\hat{A} \in Mod(S, \leq)$. So, $\mathbb{F}_J = (\hat{-})$.

$\mathbb{F}_K$ is also easy to describe. Given $A \in Set_{(S,\leq)} = Mod(S, \leq)$, assume WLOG $\begin{cases} \bullet\ A_{[s]} \cap A_{[s']} = \emptyset \text{ if } [s] \neq [s'] \\ \bullet\ \bigcup_{s \in S} A \cap fun(\Sigma) = \emptyset \end{cases}$; otherwise translate $A$ to some $A'$ disjoint from $fun(\Sigma)$ by some bijection $h$ as done in §1. Then we can define $\Sigma(A) = ((S, \leq), F_A, P)$ where $F_A : \bigcup_{s \in S} A_s \dot{\cup} fun(\Sigma) \longrightarrow \mathcal{P}(S^* \times S)$ is such that $F_A|_{fun(\Sigma)} = F$, and $F_A : \bigcup_{s \in S} A_s \ni a \longmapsto \{(\varepsilon, s) \mid a \in A_s\} \in \mathcal{P}(S^* \times S)$.

Then $\mathbb{F}_K(A) = \mathcal{T}_{\Sigma(A)}|_\Sigma$, where, as before, we use the notation $\mathcal{T}_{\Sigma(A)}|_\Sigma = \mathcal{T}_\Sigma(A)$, and where $\mathcal{T}_{\Sigma(A)}$ is just the initial $F(A)$-algebra, extended to a $\Sigma$-model by empty predicates, i.e. for each $p : s_1 \ldots s_n$, $p_{\mathcal{T}_{\Sigma(A)}} = \emptyset$.

Note that for $\Sigma = ((S, \leq), F, \emptyset)$, the previous notation $\mathcal{T}_\Sigma(X)$ for $X$ an $S$-sorted set of variables now becomes $\mathcal{T}_\Sigma(\hat{X})$, i.e., $\mathcal{T}_\Sigma(X) = \mathbb{F}_K(\mathbb{F}_J(X))$.

In summary, $\mathbb{F}_J = (\hat{-})$, and $\mathbb{F}_K = \mathcal{T}_\Sigma(-)$.

Note that $\mathcal{T}_\Sigma(-)$ left adjoint just means:

$$\begin{tikzcd} A \arrow[r, hook, "\eta_A"] \arrow[dr, "\forall\, h"'] \arrow[r, phantom, shift right=6, "="] & T_\Sigma(A) \arrow[d, dashed, "-h"] \\ & B \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} \mathcal{T}_\Sigma(A) = (T_\Sigma(A), -_{\mathcal{T}_\Sigma(A)}) \arrow[d, dashed, "-h", "\exists!"'] \\ \mathcal{B} = (B, -_{\mathcal{B}}) \end{tikzcd}$$

$\underline{Set}_{(S,\leq)}$ $\qquad\qquad\qquad\qquad$ $\underline{Mod}(\Sigma)$

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In particular, for a $\mathcal{A} = (A, -_{\mathcal{A}}) \in Mod(\Sigma)$ we get:

$$\begin{tikzcd} A \arrow[r, hook, "\eta_A"] \arrow[dr, "1_A"'] & T_\Sigma(A) \arrow[d, dashed, "-1_A"] \\ & A \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} \mathcal{T}_\Sigma(A) \arrow[d, dashed, "-1_A"] \\ \mathcal{A} \end{tikzcd}$$

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Let us now consider elementary submodels

<u>Definition</u> A $\Sigma$-submodel $\mathcal{A} \subseteq \mathcal{B}$ is called an <u>elementary submodel</u>, denoted $\mathcal{A} \preccurlyeq \mathcal{B}$ iff $\mathcal{A} \equiv \mathcal{B}$.

The following criterion by Tarski and Vaught gives a useful way to check that $\mathcal{A} \subseteq \mathcal{B}$ is an elementary submodel of $\mathcal{B}$. Note that, since $\Sigma \subseteq \Sigma(A)$, we have $FO(\Sigma) \subseteq FO(\Sigma(A))$.

Recall also from the above discussion that a $\Sigma(A)$-model is a pair $(\mathcal{B}, h)$ with $\mathcal{B} = (B, -_{\mathcal{B}}) \in Mod(\Sigma)$, and $h : A \to B$.

$\Sigma = ((S, \leq), F, P)$

<u>Theorem</u> (Tarski-Vaught Test). Let $\Sigma$ be an OS-FO signature, and $\mathcal{B} = (B, \_{}_{\mathcal{B}}) \in \underline{Mod}(\Sigma)$. Let $A \in \underline{Set}_{(S,\leq)}$ be such that $A \overset{j}{\hookrightarrow} B$ and for each $\varphi \in FO(\Sigma(A))$ with $fvars(\varphi) = \{x:s\}$ the following equivalence holds:

(TV) $(\mathcal{B}, j) \models (\exists x:s)\,\varphi \Leftrightarrow (\exists a \in A_s)\,(\mathcal{B}, j) \models \varphi\{x:s \mapsto a\}$

Then there is an elementary submodel $\mathcal{A} \preccurlyeq \mathcal{B}$ with $\mathcal{A} = (A, \_{}_{\mathcal{A}})$.

<u>Proof</u>. First of all note that for each $s \in S$, $A_s \neq \emptyset$. This is because, since $\mathcal{B} \in Mod(\Sigma)$, $\forall s \in S$ $B_s \neq \emptyset$, and therefore for each $s \in S$ we have $(\mathcal{B}, j) \models (\exists x:s)\; x:s = x:s$, which by (TV) forces $\exists a \in A_s$. Second, note that for each $f : s_1 \ldots s_n \to s$ in $F$ and $(a_1,.., a_n) \in A_{s_1} \times \cdots \times A_n$,
$(\mathcal{B}, j) \models (\exists x:s)\; f(a_1,.., a_n) = x$, which forces $f_{\mathcal{B}}(a_1,.., a_n) \in A_s$.
Therefore, we indeed have a $\Sigma$-submodel $\mathcal{A} \subseteq \mathcal{B}$ with $\mathcal{A} = (A, \_{}_{\mathcal{A}})$, for each $f : s_1 \ldots s_n \to s$ $f_{\mathcal{A}} = f_{\mathcal{B}} \cap A_{s_1} \times \cdots \times A_n \times A$, and for each $p : s_1 \ldots s_n$, $p_{\mathcal{A}} = p_{\mathcal{B}} \cap A_{s_1} \times \cdots \times A_{s_n}$. We just need to prove $\mathcal{A} \equiv_{\Sigma} \mathcal{B}$. Since $FO(\Sigma) \subseteq FO(\Sigma(A))$, it is enough to prove $(\mathcal{A}, 1_A) \equiv_{\Sigma(A)} (\mathcal{B}, j)$, that is, that for each $\varphi \in Sen(\Sigma(A))$ we have:

$$(\mathcal{A}, 1_A) \models \varphi \Leftrightarrow (\mathcal{B}, j) \models \varphi.$$

We prove this by induction on the following measure $\|-\| : FO(\Sigma(A)) \to \mathbb{N}$, where, by logical equivalence, we may assume WLOG that formulas are built from atomic ones by $\neg$, $\vee$, and $\exists$ only:

- $\| u = v \| = 1$ for any atomic sentence $u = v$
- $\| \varphi \vee \psi \| = \| \varphi \| + \| \psi \|$
- $\| \neg \varphi \| = \| (\exists x : 1)\, \varphi \| = \| \varphi \| + 1$

Base case. $\varphi \equiv u = v$, ground equation. We have

$$(\mathcal{A}, 1_A) \models u = v \Leftrightarrow (\mathcal{B}, j) \models u = v$$

because we have

$$\begin{tikzcd} T_{\Sigma(A)} \arrow[r, "-1_A"] \arrow[rr, bend right, "-j"'] & (\mathcal{A}, 1_A) \arrow[r, hook, "j"] \arrow[d, phantom, "="] & (\mathcal{B}, j) \\ & {} & \end{tikzcd}$$

so that for each $u_{1_A} = u_j$, and $v_{1_A} = v_j$ and, by definition, $(\mathcal{A}, 1_A) \models u = v$ iff $u_{1_A} = v_{1_A}$, and $(\mathcal{B}, j) \models u = v$ iff $u_j = v_j$.

Induction Step

- $(\mathcal{A}, 1_A) \models \varphi \vee \psi \Leftrightarrow (\mathcal{A}, 1_A) \models \varphi$ or $(\mathcal{A}, 1_A) \models \psi \Leftrightarrow$ (I. H.) $(\mathcal{B}, j) \models \varphi$ or $(\mathcal{B}, j) \models \psi \Leftrightarrow (\mathcal{B}, j) \models \varphi \vee \psi$
- $(\mathcal{A}, 1_A) \models \neg\varphi \Leftrightarrow (\mathcal{A}, 1_A) \not\models \varphi \Leftrightarrow$ (I.H.) $(\mathcal{B}, j) \not\models \varphi \Leftrightarrow (\mathcal{B}, j) \models \neg\varphi$

- $(\mathcal{A}, 1_A) \models (\exists x:s)\varphi \Leftrightarrow (\exists a \in A_s)(\mathcal{A}, 1_A) \models \varphi\{x:s \mapsto a\} \Leftrightarrow$

  (I.H., since $\|\varphi\| = \|\varphi\{x:s\mapsto a\}\|$) $(\exists a \in A_s)(\mathcal{B}, j) \models \varphi\{x:s\mapsto a\}$

  $\Leftrightarrow$ (TV) $(\mathcal{B}, j) \models (\exists x:s)\varphi$. $\square$

We are now ready to state and prove the order-sorted generalization of the downward Löwenheim-Skolem theorem. We just need to define for $\Sigma = ((S, \leq), F, P)$, $|\Sigma| = max\{|S|, |F|, |P|\}$, where for any function $f$ like $F$ or $P$, $|f|$ is the cardinality of $f$ as a set of pairs.

<u>Theorem</u> (Order-Sorted Downward Löwenheim-Skolem Theorem). Let $\Sigma = ((S, \leq), F, P)$, $s_i \in S$, $1 \leq i \leq n$, and $\mathcal{B} = (B, \_{\mathcal{B}})$ $\in \underline{Mod}(\Sigma)$ such that $\aleph_0, |\Sigma| \leq |B_{s_i}| = \alpha_i$, $1 \leq i \leq n$. Then for each $\gamma$ such that $\aleph_0, |\Sigma| \leq \gamma \leq min\{\alpha_i\}_{1 \leq i \leq n}$ there exists an elementary submodel $\mathcal{A}_\gamma \preceq \mathcal{B}$, with $\mathcal{A}_\gamma = (A_\gamma, \_{\mathcal{A}_\gamma})$ such that $|A_{\gamma, s_i}| = \gamma$, $1 \leq i \leq n$, and $|A_{\gamma, s}| \leq \gamma$, for $s \in S$ such that $s \neq s_i$, $1 \leq i \leq n$.

<u>Proof</u>. We first define a simple construction, given a $\Sigma$-model $\mathcal{B} = (B, \_{\mathcal{B}})$ and $A \in \underline{Set}_{(S,\leq)}$ with $A \overset{j_A}{\hookrightarrow} B$ to obtain another $A^+ \in Set_{(S,\leq)}$ with $A \subseteq A^+ \subseteq B$ and satisfying some existential closure properties. $A^+ = \widehat{A^\circ}$, where

for each $(\exists x:s)\varphi \in Sen(\Sigma(A))$ such that $(\mathcal{B}, j_A) \models (\exists x:s)\varphi$, $s \in S$,
we choose an element $b_\varphi \in B_s$ such that $(\mathcal{B}, j_A), \{(x:s, b_\varphi)\} \models \varphi$.
Then $A^\bullet_s = A_s \cup \{ b_\varphi \mid \exists(x:s)\varphi \in Sen(\Sigma(A)) \wedge (\mathcal{B}, j_A) \models (\exists x:s)\varphi \}$

and $A^+ = \widehat{A^\bullet}$.

Now define $A_0 = \widehat{A^\bullet_0}$, where $A^\bullet_{0,s_i} = A_{0,s_i} \subseteq B_{s_i}$

with $|A^\bullet_{0,s_i}| = \gamma$, $1 \leq i \leq n$, and $A^\bullet_{0,s} = \{b_s\} \subseteq B_s$

for $s \in S$, $s \neq s_i$, $1 \leq i \leq n$. Note that $|A_{0,s_i}| = \gamma$,

$1 \leq i \leq n$, and $|A_{0,s}| \leq \gamma$, if $s \in S$, $s \neq s_i, 1 \leq i \leq n$.

We now define a chain

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \cdots$$

with $A_{n+1} = A^+_n$. Note that, since given any

$C \subseteq B$, $C \in \underline{Set}_{(S,\leq)}$ for each $s \in S$,

$|C^+_s| \leq \max(|\Sigma|, \max(\chi_0, \max(\{|C_s| \mid s \in S\}))$,

we do have $|A_{j,s_i}| = \gamma$, $|A_{j,s}| \leq \gamma$, $s \neq s_i$, $j \in \mathbb{N}$, $1 \leq i \leq n$.

Consider now $A^\infty \subseteq B$, where $A^\infty_s = \bigcup_{i \in \mathbb{N}} A_i$.

Obviously, since $A_i \in \underline{Set}_{(S,\leq)}$, $A^\infty \in \underline{Set}_{(S,\leq)}$, and

$|A^\infty_{s_i}| = \gamma$, $|A^\infty_s| \leq \gamma$, $s \neq s_i$, $1 \leq i \leq n$. Furthermore, by construction $A^\infty$ satisfies the TV condition, since

$$FO(\Sigma(A^\infty)) = \bigcup_{i \in \mathbb{N}} FO(\Sigma(A_i))$$ , so that for each

$(\exists x:s)\varphi \in Sen(\Sigma(A^\infty))$ $\exists\, i \in \mathbb{N}$ with $(\exists x:s)\varphi \in Sen(\Sigma(A_i))$, and

therefore $b_\varphi \in A_{i+1,s} \subseteq A^\infty_{,s}$. Therefore our desired

model $\mathcal{A}_\gamma$ is precisely $\mathcal{A}_\gamma = (A^\infty, -_{\mathcal{A}_\gamma})$. $\square$

<u>Corollary</u>. If $|\Sigma| \leq \aleph_0$, we can choose $\gamma = \aleph_0$

and get $\mathcal{A}_\gamma$, with $|A^\infty_{s_i}| = \aleph_0$, $1 \leq i \leq n$, and

$|A^\infty_s| \leq \aleph_0$, $s \in S$, $s \neq s_i$, $1 \leq i \leq n$.