J. Meseguer

1. Submodels and Satisfiability

Recall from Lecture 19, (3/14/17), the following notions:

1. <u>submodel</u> $\mathcal{M} \subseteq \mathcal{M}'$ of a model $\mathcal{M}$
2. <u>closure under submodels</u> $Sub(\mathbb{A})$ of a given class $\mathbb{A}$ of Σ-models
3. <u>finitely generated Σ-model</u>, and subclass $f.g.(\mathbb{A}) \subseteq \mathbb{A}$ of the finitely generated models of a class $\mathbb{A}$.

Recall also that in general $Sub(\mathbb{A}) \supsetneq \mathbb{A}$. <u>Example</u>: $\Sigma = \{+, 0\}$, $\mathbb{A}$ = Abelian groups. Then $\mathbb{Z}_{+,0} \in \mathbb{A}$ and $\mathbb{N}_{+,0} \subseteq \mathbb{Z}_{+,0}$, but $\mathbb{N}_{+,0} \notin \mathbb{A}$.

<u>Def.</u> For Σ a FO order-sorted signature, a homomorphism $r : \mathcal{M}' \to \mathcal{M}$ is called a <u>retract</u>, where $\mathcal{M}' = (M', \_{\mathcal{M}'})$ and $\mathcal{M} = (M, \_{\mathcal{M}})$, iff:

1) $M \subseteq M'$ as S-sorted sets
2) $\forall s \in S, \forall a \in M_s \quad r_s(r_s(a)) = r_s(a)$.

<u>Lemma</u>. If $r : \mathcal{M}' \to \mathcal{M}$ is a retract, then $\mathcal{M}$ is a subalgebra of $\mathcal{M}'$ ($\mathcal{M} \subseteq \mathcal{M}'$).

<u>Proof</u>. Exercise. $\square$

For satisfiability purposes we have the following useful lemma:

<u>Lemma 1</u>. Let Σ be a FO-OS-signature and $\mathcal{B}$ a Σ-model. Then for each $\varphi \in QF(\Sigma)$, if φ satisfiable in a submodel $\mathcal{A} \subseteq \mathcal{B}$, then φ is satisfiable in $\mathcal{B}$.

<u>Proof</u>. We can without loss of generality assume that all predicate symbols have been turned into function symbols of sort Pred. The proof is an easy structural induction on the structure of QF formulas and the definition of the satisfaction relation $\mathcal{A}, a \models \varphi$, since all hinges upon the base case. Let $\mathcal{A} = (A, \_{\mathcal{A}})$, $\mathcal{B} = (B, \_{\mathcal{B}})$, and let $Y = fvars(\varphi)$, $a \in [Y \to A]$, and $u = v$ a Σ-equation. Then,

$$\mathcal{A}, a \models u - v \iff \mathcal{B}, a \models u = v$$

This is so because: (i) since $A \subseteq B$, $[Y \to A] \subseteq [Y \to B]$, and (ii) since $\mathcal{A} \subseteq \mathcal{B}$ is a subalgebra, for each $t \in T_\Sigma(Y)$ and $a \in [Y \to A]$ we have $t a_{\mathcal{A}} = t a_{\mathcal{B}}$ for $a_{\mathcal{A}} : T_\Sigma(Y) \to \mathcal{A}$, resp. $a_{\mathcal{B}} : T_\Sigma(Y) \to \mathcal{B}$ the unique Σ-homomorphism induced by $a \in [Y \to A] \subseteq [Y \to B]$. $\Box$

Note that, in general, we have the following strict containments for a class $\mathbb{A}$ of Σ-models:

$$f.g.(Sub(\mathbb{A})) \subsetneq Sub(\mathbb{A}) \supsetneq \mathbb{A}$$

However, for satisfiability of QF formulas we have:

<u>Proposition 1</u>. Let $(\Sigma, \mathbb{A})$ be a semantic theory. Then for each $\varphi \in QF(\Sigma)$ the following are equivalent:

1. φ is $(\Sigma, \mathbb{A})$-satisfiable
2. φ is $(\Sigma, Sub(\mathbb{A}))$-satisfiable
3. φ is $(\Sigma, f.g.(Sub(\mathbb{A})))$-satisfiable

<u>Proof</u>. Because of the above containments we trivially have $(1) \Rightarrow (2)$ and $(3) \Rightarrow (2)$. And because of Lemma 1 we have $(2) \Rightarrow (1)$. Therefore we will be done if we show $(3) \Rightarrow (1)$. Suppose $Y = vars(\varphi)$, $\mathcal{B} = (B, -_{\mathcal{B}}) \in \mathbb{A}$, $b \in [Y \to B]$, and $\mathcal{B}, b \models \varphi$. Let $\mathcal{A}$ be the image of the unique Σ-homomorphism $-b : T_\Sigma(Y) \to \mathcal{B}$, i.e., $\mathcal{A} = (A, -_{\mathcal{A}})$ with:

1. $A_s = h_s[T_\Sigma(X)_s] \subseteq B_s$
2. $\forall f : s_1 \ldots s_n \to s$ in F, $f_{\mathcal{A}} = f_{\mathcal{B}} \cap A_{s_1} \times \cdots \times A_{s_n} \times A_s$

   which is well-defined by h_s Σ-homomorphism
3. $\forall p : s_1 \ldots s_n$ in P, $p_{\mathcal{A}} = p_{\mathcal{B}} \cap A_{s_1} \times \cdots \times A_{s_n}$

Then $\mathcal{A} \subseteq \mathcal{B}$ is a subalgebra, by definition of $\mathcal{A}$ we have $b \in [Y \to A]$ and, by the inductive argument in Lemma 1 we have $\mathcal{A}, b \models \varphi$, as desired. $\square$

Note that in Proposition 1 the requirement that $\varphi \in QF(\Sigma)$ is essential. For a counterexample when $\varphi \notin QF(\Sigma)$, let $\Sigma = \{-\cdot-\}$, $\varphi = (\forall x, y)\ x \cdot y = y \cdot x$, and consider the Σ-subalgebra $\{1\}^* \subseteq \{0,1\}^*$, with $-\cdot-$ interpreted as string concatenation. Then $\{1\}^* \models (\forall x, y)\ x \cdot y = y \cdot x$, but $\{0,1\} \not\models (\forall x, y)\ x \cdot y = y \cdot x$.

Although Proposition 1 does not require $\mathbb{A} = Sub(\mathbb{A})$ (that's the whole point), nevertheless, it is useful to know about cases when $\mathbb{A} = Sub(\mathbb{A})$. Here are some:

Proposition 2. The following kinds of theories:

1. $T = (\Sigma, \Gamma)$, with Γ a set of universal formulas (universal theory)
2. $T = (\Sigma, Init(\Sigma, \Gamma))$, with (Σ, Γ) Horn theory

Proof. (1) was an exercise in Lecture 19. (2) follows from the fact that the initial model $T_{\Sigma/\Gamma}$ has no submodels except itself, since if $A \underset{j}{\hookrightarrow} T_{\Sigma/\Gamma}$ is a submodel, and $-_A : T_{\Sigma/A} \to A$ the unique Σ-homomorphism, then initiality forces $-_A ; j = 1_{T_{\Sigma/A}}$, which forces j to be surjective, and therefore $A = T_{\Sigma/\Gamma}$. $\square$

Definition Let $T = (\Sigma, \mathbb{A})$ be a semantic theory with Σ having a finite set of sorts, and let $\mathcal{F} \subseteq FO(\Sigma)$. T has the finite model property for satisfiability of $\mathcal{F}$ formulas iff for each $\varphi \in \mathcal{F}$,

φ is T-satisfiable iff φ is $(\Sigma, \mathbb{A}_{<\omega})$-satisfiable.

Here are some sufficient conditions:

Proposition 3. Let T have $\Sigma = sig(T)$ finite and be:

1. a universal theory with $T_\Sigma(Y)$ finite for each finite S-sorted set of variables
2. a Horn theory $T = (\Sigma, \Gamma)$ where $\mathcal{T}_{\Sigma/\Gamma}(Y)$ is finite for each finite set Y of S-sorted variables
3. the theory $(\Sigma, \{\mathcal{T}_{\Sigma/\Gamma}\})$ for (Σ, Γ) a Horn theory with $\mathcal{T}_{\Sigma/\Gamma}$ finite
4. A parameterized theory $(\Sigma_B, \mathbb{F}_J)$ where $J : P \hookrightarrow B$ is such that P is the disjoint union of n copies of the TRIV theory, so that $\mathbb{F}_J = \{\mathbb{F}_J[A_1,.., A_n] \mid A_i \in Set_{T\emptyset},\ 1 \leq i \leq n\}$ and so that if $B_1,.., B_n$ finite, then $\mathbb{F}_J[B_1,.., B_n]$ finite.

Then any of the theories (1) - (4) have the finite model property for $QF(\Sigma)$.

Proof. (3) is trivial, since $\underbrace{\{\mathcal{T}_{\Sigma/\Gamma}\}}_{\text{then}} = \{\mathcal{T}_{\Sigma/\Gamma}\}_{<\omega}$.

(1) follows from Propositions (1) and (2), since
$$f.g.(Sub(mod(T)) = f.g.(mod(T)) = mod(T)_{<\omega}.$$

(3) follows from the fact that for any

finite set of variables Y, T-model $\mathcal{A} = (A, -_{\mathcal{A}})$, and $a \in [Y \to A]$ the unique Σ-homomorphism $-a : \mathcal{T}_\Sigma(Y) \longrightarrow \mathcal{A}$ factors as

$$\begin{tikzcd}
\mathcal{T}_\Sigma(Y) \arrow[rr, "-a"] \arrow[dr, two heads] & & \mathcal{A} \\
& \mathcal{T}_{\Sigma/\Gamma}(Y) \arrow[ur, "-^{\Gamma}_{a}"'] &
\end{tikzcd}$$

which forces the image of $-a$ to be a finite submodel of $\mathcal{A}$, and therefore forces:
$f.g.(Sat(mod(T))) = f.g.(mod(T)) = mod(T)_{<\omega}$,
yielding the desired result by Proposition 1.

To see (4) let $\varphi \in QF(\Sigma)$, $Y = vars(\varphi)$, $\mathbb{F}_J[A_1,, A_n] = (F_J[A_1,, A_n], -_{\mathbb{F}_J[A_1 .. A_n]})$, and $a \in [Y \to F_J[A_1,, A_n]]$. Then define for $Elt_1, .., Elt_n$ the sorts of P, $[B_1, .., B_n] \overset{j}{\hookrightarrow} [A_1,, A_n]$ as follows: (i) if $Y_{Elt_j} \neq \emptyset$, then $a[Y_{Elt_j}] = B_j$; otherwise, $B_j = \{b_j\}$, for some chosen $b_j \in A_j$. Note that we then can define an indexed family of functions $r = \{r_j : A_j \to B_j\}$ such that $j ; r = 1_{[B_1,, B_n]}$. Then note that $\mathbb{F}_J$ in a functor. Therefore $\mathbb{F}_J(j); \mathbb{F}_J(r) = 1_{\mathbb{F}_J[B_1,..,B_n]}$, so that $\mathbb{F}_J[B_1,..,B_n] \subseteq \mathbb{F}_J[A_1,..,A_n]$ is a finite submodel. Suppose now that $\mathbb{F}_J[A_1,..,A_n], a \models \varphi$. Then $\mathbb{F}_B[B_1,..,B_n], a \models \varphi$ because we have, by the way the $B_1,..,B_n$ have been defined, a factorization

$$\begin{tikzcd}
J_\Sigma(Y) \arrow[rr, "-a"] \arrow[dr, dashed, "-a"'] & & \mathbb{F}_J[A_1,..,A_n] \\
& \mathbb{F}_J[B_1,..,B_n] \arrow[ur, hook] &
\end{tikzcd}$$

and this shows that $(\Sigma_B, \mathbb{F}_J)$ has the finite model property for $QF(\Sigma_B)$. □

Examples For:

(1) the theories POSET and TOSET of partial, resp. total, order are good examples.

(2) the LOOKUP theory is a good example of (1) and (2)

(3) the theory fmod BOOL enfm in Maude is a good example

(4) the theory Set [X] in the SatVar paper is a good example.

More generally, if Σ has only constants and predicate symbols and is finite, theories of type (1)-(2) will always have the finite model property.

2. The Category Red of Reduction Maps

We have seen how ascent and descent maps are very useful to:

1. obtain decidable satisfiability for new theories
2. structure proof of such decidability as composition of such maps
3. reuse and extend such decidability results, and also implement them by developing a library of such maps.

Obviously, the more general notions of this kind can be, the more extensible will SMT solving methods become.

This brings us to the natural question:

Can we further generalize ascent and descent maps to an even more general and comprehensive notion?

The answer is: Yes!

Definition. Given theories T, T', and recursive sets of formulas $\mathcal{F} \subseteq FO(sig(T))$, $\mathcal{G} \subseteq FO(sig(T'))$, a reduction map from $(T, \mathcal{F})$ to $(T', \mathcal{G})$ is either:

1. an ascent map $(T, \mathcal{F}) \overset{\alpha}{\underset{H}{\Longrightarrow}} (T', \mathcal{G})$
2. a descent map, $(T, \mathcal{F}) \overset{\delta}{\underset{H}{\rightleftarrows}} (T', \mathcal{G})$, or
3. a recursive function, called a plain reduction map, of the form:
$\rho : \mathcal{F} \to \mathcal{P}_{fin \neq \emptyset}(\mathcal{G})$, denoted $(T, \mathcal{F}) \xrightarrow{\rho} (T', \mathcal{G})$.

such that $\forall \varphi \in \mathcal{F}$, φ is T-satisfiable iff $\exists \psi \in \rho(\varphi)$ s.t. ψ is T'-satisfiable.

The category $\underline{\text{Red}}$ has as objects pairs $(T, \mathcal{F})$ with T a theory and $\mathcal{F} \subseteq FO(sig(\Sigma))$ a recursive set, and as arrows reduction maps. Arrow composition is defined as follows:

1. composition of ascent maps if composable arrows are so
2. composition of descent maps if composable arrows are so
3. function composition if composable arrows are plain
4. $$\begin{tikzcd}
(T_1, \mathcal{F}_1) \arrow[r, shift left, "\delta"] \arrow[rr, bend right=40, "\delta;\alpha"'] & (T_2, \mathcal{F}_2) \arrow[l, shift left, "G"] \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "H"'] \arrow[d, phantom, "="] & (T_3, \mathcal{F}_3) \\
& {} &
\end{tikzcd}$$

provided neither (G, δ) nor (H, α) are $\underline{\text{formula transformations}}$.
i.e., neither (i) $T_1 = T_2$, $G = 1_{sig(T_1)}$, nor (ii) $T_2 = T_3$, $H = 1_{sig(T_2)}$, since formula transformations are both ascent and descent maps, so that in case (i) the composition is an ascent map, and in case (ii) it is a descent map

5. $$\begin{tikzcd}
(T_1, \mathcal{F}_1) \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "G"'] \arrow[rr, bend right=40, "\alpha;\delta"'] & (T_2, \mathcal{F}_2) \arrow[r, shift left, "\delta"] \arrow[d, phantom, "="] & (T_3, \mathcal{F}_3) \arrow[l, shift left, "H"] \\
& {} &
\end{tikzcd}$$

provided, again, neither (G, α) nor (H, δ) are formula transformations.

6. $$\begin{tikzcd}
(T_1, \mathcal{F}_1) \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "G"'] \arrow[rr, bend right=40, "\alpha;\rho"'] & (T_2, \mathcal{F}_2) \arrow[r, "\rho"] \arrow[d, phantom, "="] & (T_3, \mathcal{F}_3) \\
& {} &
\end{tikzcd}
\qquad
\begin{tikzcd}
(T_1, \mathcal{F}_1) \arrow[r, "\rho"] \arrow[rr, bend right=40, "\rho;\alpha"'] & (T_2, \mathcal{F}_2) \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "G"'] \arrow[d, phantom, "="] & (T_3, \mathcal{F}_3) \\
& {} &
\end{tikzcd}$$

7. $$\begin{tikzcd} (T_1,\mathcal{F}_1) \arrow[r, shift left, "\delta"] \arrow[rr, bend right, "="', "\delta;\rho"'] & (T_2,\mathcal{F}_2) \arrow[l, shift left, "H"] \arrow[r, "\delta"] & (T_3,\mathcal{F}_3) \end{tikzcd} \qquad \begin{tikzcd} (T_1,\mathcal{F}_1) \arrow[r, "\rho"] \arrow[rr, bend right, "="', "\rho;\delta"'] & (T_2,\mathcal{F}_2) \arrow[r, shift left, "\delta"] & (T_3,\mathcal{F}_3) \arrow[l, shift left, "H"] \end{tikzcd}$$

It is easy to check that all these compositions are associative and have maps of the form $(T,\mathcal{F}) \underset{1_\Sigma}{\overset{1_\mathcal{F}}{\underset{\longleftrightarrow}{\longrightarrow}}} (T,\mathcal{F})$ as identities, for $\Sigma = sig(T)$. Note that we have the following subcategory inclusions:

$$\begin{tikzcd}
 & & S_{exp}\underline{Ascent} \arrow[dr, hook] & & \\
 & & S_{lexp}\underline{Ascent} \arrow[r, hook] & \underline{Ascent} \arrow[dr, hook] & \\
\underline{Id} \arrow[r, hook] & \underline{FormTrans} \arrow[uur, hook] \arrow[ur, hook] \arrow[dr, hook] \arrow[ddr, hook] & & & \underline{Red} \\
 & & \underline{S_{exp}Descent} \arrow[r, hook] & \underline{Descent} \arrow[ur, hook] & \\
 & & \underline{S_{lexp}Descent} \arrow[ur, hook] & &
\end{tikzcd}$$

where what before were called _trivial_ ascent maps are now called _simple_ ascent maps, to keep the terminology consistent with that used for descent maps, and where:

1. the arrows in $\left\{\begin{matrix} S_{exp}\underline{Ascent} \\ S_{exp}\,Descent \end{matrix}\right\}$ are _expansive_ simple $\left\{\begin{matrix} \text{ascent} \\ \text{descent} \end{matrix}\right\}$ maps

2. the arrows in $\left\{\begin{matrix} \underline{S}_{lexp}\underline{Ascent} \\ \underline{S_{lexp}\,Descent} \end{matrix}\right\}$ are _literally expansive_ simple $\left\{\begin{matrix} \text{ascent} \\ \text{descent} \end{matrix}\right\}$ maps.

Examples. Besides the obvious compositions of the form $\delta;\alpha$ (4), $\alpha;\delta$ (5), $\alpha;\rho$, or $\rho;\alpha$ (6), note that, by Proposition 1, for $T_1 = (\Sigma, \mathbb{A})$, $T_2 = (\Sigma, Sub(\mathbb{A}))$, $T_3 = (\Sigma, f.g.(Sub(\mathbb{A})))$, and $\mathcal{F} \subseteq QF(\Sigma)$ any $(T_i,\mathcal{F}) \xrightarrow{1_\mathcal{F}} (T_j,\mathcal{F})$, $i,j \in \{1,2,3\}$ is a reduction map.

3. Generic Models

Let T be a theory, and $\mathcal{F} \subseteq FO(\Sigma)$ a recursive class of formulas. Then satisfiability of $\varphi \in \mathcal{F}$ may be achived by many different models of T, which may depend on the choice of φ. This is, for example, obviously the case when $T = (\Sigma, \emptyset)$, or $T = (\Sigma, AC_\Delta)$ for $\Delta \subseteq \Sigma$, since the model of, say,

$$\varphi = \bigwedge_{i \in I} u_i = v_i \wedge \bigwedge_{j \in J} u'_j \neq v'_j$$

if it exists, is the congruence closure of $\{u_i = v_i\}_{i \in I}$.

It can however be advantageous if there is a model $\mathcal{M} \in mod(T)$ such that $\varphi \in \mathcal{F}$ is T-satisfiable iff φ is satisfiable in $\mathcal{M}$.

__Definition__. Given T and recursive $\mathcal{F} \subseteq FO(sig(T))$, $\mathcal{M} \in mod(T)$ is called a __generic model__ for T-satisfaction of $\mathcal{F}$-formulas iff for each $\varphi \in \mathcal{F}$

φ is T-satisfiable iff φ is satisfiable in $\mathcal{M}$.

Let us see a useful example.

__Theorem__. $\mathbb{N}_{\leq}$, the natural numbers with the standard order is a generic model for satisfaction of QF formulas for the TOSET theory of totally ordered sets with unsorted signature $\{\leq\}$, and axioms:

1. reflexivity $\quad x \leq x$
2. transitivity $\quad x \leq y \wedge y \leq z \Rightarrow x \leq z$
3. anti-symmetry $\quad x \leq y \wedge y \leq x \Rightarrow x = y$
4. total order $\quad x \leq y \vee y \leq x$

<u>Proof</u>. This follows by composition of the following reduction maps:

$$((\{\leq\}, \mathrm{TOSET}), QF(\leq)) \xrightarrow{1_{QF(\leq)}} ((\{\leq\}, \underline{\mathrm{Mod}}(\mathrm{TOSET})_{<\omega}), QF(\leq)) \xrightarrow{1_{QF(\leq)}} ((\{\leq\}, \{\mathbb{N}_{\leq}\}), QF(\leq))$$

where the first is a reduction map by Proposition 1 and the fact that TOSET is universal and $f.g.(\mathrm{Mod}(\mathrm{TOSET})) = \mathrm{Mod}(\mathrm{TOSET})_{<\omega}$, so that, by Proposition 3-(1) it has the finite model property. The fact that the second map is also a reduction map easily follows from Lemma 1, since any finite total order is isomorphic to a submodel of $\mathbb{N}$. Specifically, any such model is of the form $(\{a_0, .., a_n\}, \leq)$ with $a_0 < a_1 < a_2 < .. < a_n$ and we have:

$$\begin{tikzcd}
 & & n+1 \arrow[d, dash] \\
a_n \arrow[r, leftrightarrow, "\cong"] \arrow[d, dash] & n \arrow[r, leftrightarrow, "="] \arrow[d, dash] & n \arrow[d, dash] \\
a_{n-1} \arrow[r, leftrightarrow, "\cong"] \arrow[d, dotted, dash] & n-1 \arrow[r, leftrightarrow, "="] \arrow[d, dotted, dash] & n-1 \arrow[d, dotted, dash] \\
a_2 \arrow[r, leftrightarrow, "\cong"] \arrow[d, dash] & 2 \arrow[r, leftrightarrow, "="] \arrow[d, dash] & 2 \arrow[d, dash] \\
a_1 \arrow[r, leftrightarrow, "\cong"] \arrow[d, dash] & 1 \arrow[r, leftrightarrow, "="] \arrow[d, dash] & 1 \arrow[d, dash] \\
a_0 \arrow[r, leftrightarrow, "\cong"] & 0 \arrow[r, leftrightarrow, "="] & 0
\end{tikzcd}$$

Indeed, note that $(n+1, \leq)$, with $n+1 = \{0, 1, .., n\}$ is obviously a submodel of $\mathbb{N}_{\leq}$. In fact it is a <u>retract</u> submodel with $r : \mathbb{N}_{\leq} \to (n+1, \leq)$ defined as: $r(x) = \underline{\text{if}}\ x \leq n\ \underline{\text{then}}\ x\ \underline{\text{else}}\ n\ \underline{\text{fi}}$. Therefore,

φ satisfiable in $(\{a_1, .., a_n\}, \leq)$ iff φ satisfiable in $(n+1, \leq)$

$\Rightarrow$ φ satisfiable in $\mathbb{N}$ by Lemma 1.

Conversely, let $\varphi \in QF(\leq)$ be satisfiable in $\mathbb{N}_{\leq}$, $Y = vars(\varphi)$ and $a \in [Y \to \mathbb{N}]$ such that $\mathbb{N}_{\leq}, a \models \varphi$. If $Y = \emptyset$, then φ must be a Boolean combination of $\bot$ and $\top$ equivalent to $\top$, and $(1, \leq), a \models \varphi$. Otherwise let $n = \max\{a(y) \mid y \in Y\}$, then $a(Y) \subseteq n+1$ for some $n \in \mathbb{N}$, and $(n+1, \leq), a \models \varphi$. This finishes the proof. $\square$

<u>Corollary</u>. Satisfiability of QF formulas in TOSET is decidable.

<u>Proof</u>. Just compose with the simple ascent map

$$((\{\leq\}, \{\mathbb{N}_{\leq}\}), QF(\leq)) \underset{J}{\overset{J}{\rightrightarrows}} ((\{+, 0, 1, \leq\}, \{\mathbb{N}_{+,\leq}\}), QF(+, 0, 1, \leq)).$$