CS 576 Lecture 23, 4/4/17

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1. Signature Maps and Theory Interpretations (and III)

A theory interpretation $H : T \to T'$, called a view in Maude, allows us to "view" a model $\mathcal{M}' \in mod(T')$ as a model of T by interpreting (i.e., giving a meaning to) the theory T in the theory T'.

For example, T can be POSET, the theory of partially ordered sets, an T' can be the semantic theory NAT with $\Sigma = \{0, 1, +, *\}$, $\Pi = \{\geq, -|-\}$ and sorts $NzNat < Nat$, so that $NAT = ((\Sigma, \Pi), \{\mathbb{N}_{0,1,+,*,\geq,-|-}\})$, where the typing of all operators in Nat, except for the divisibility predicate $-|- : NzNat\ NzNat$, where $\mathbb{N}_{0,1,+,\geq,-|-}$ is the natural numbers with $+, *, \geq$ and $-|-$ defined in the standard way. We can the view NAT as a poset in various ways such as:

1. mapping the only sort Elt in POSET to Nat, and $\leq$ to $\leq$.
2. mapping the only sort Elt in POSET to NzNat and $\leq$ to $-|-$

This views NAT as a POSET in two completely different ways: (1) with its standard order, or (2) with it divisibility order.

<u>Definition</u>. Given OS-FO theories T and T' an <u>interpretation</u> of T in T', denoted $H : T \to T'$ is a signature map $H : sig(T) \to sig(T')$ such that one of the two equivalent conditions holds:

(1) $\quad H(th(T)) \subseteq th(T')$

(1') $\quad mod(T')|_H \subseteq mod(T)$. □

Conditions (1) and (1') are indeed equivalent, since we have

$$(1') \Leftrightarrow \forall m' \in mod(T') \;\; m'|_H \in mod(T) \Leftrightarrow \forall m' \in mod(T') \;\; m'|_H \models th(T)$$
$$\Leftrightarrow \forall m' \in mod(T') \;\; m' \models H(th(T)) \Leftrightarrow (1).$$

Note that theory interpretations <u>compose</u>, since if we have $T \xrightarrow{H} T' \xrightarrow{G} T''$, then $mod(T'')|_G \subseteq mod(T') \Rightarrow$ $mod(T'')|_{H;G} \subseteq mod(T'')|_H \subseteq mod(T)$. Note also that since any signature inclusion $\Sigma \subseteq \Sigma'$ is a very simple kind of signature map, the preorder category <u>ThIncl</u> of theory inclusions is a subcategory <u>ThIncl</u> $\subseteq$ <u>Th</u> of the category <u>Th</u> of theories and theory interpretations.

Of course, depending on the nature of T and T' we may find conditions (1) or (1') easier to check:

(a) if $H : (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ is an interpretation between syntactic theories, we will usually check the correctness of H by checking:

$$\forall \varphi \in \Gamma \quad \Gamma' \vdash H(\varphi)$$

which is the classical definition of theory interpretation. Instead, if $H : (\Sigma, \mathbb{A}) \longrightarrow (\Sigma', \mathbb{B})$ is an interpretation of semantic theories, we will prefer to check

$$\mathbb{B}|_H \subseteq \mathbb{A}^{\#}$$

1.1 New Theory Operations

By generalizing the relationships between theories from mere inclusion in Th Incl to theory interpretations in Th, some previous theory operations get generalized, and some new theory operations emerge. Specifically:

A signature map $H : \Sigma \to \Sigma'$ induces functions

$$-|_H : \mathcal{P}(Mod(\Sigma')) \ni \mathbb{B} \mapsto \mathbb{B}|_H \in \mathcal{P}(Mod(\Sigma))$$

$$-\!\uparrow^H : \mathcal{P}(Mod(\Sigma)) \ni \mathbb{A} \longmapsto \mathbb{A}\!\uparrow^H \in \mathcal{P}(Mod(\Sigma'))$$

where $\mathbb{A}\!\uparrow^H = \{ M' \in Mod(\Sigma') \mid M'|_H \in \mathbb{A} \}$

and these functions induce theory transformations

$$(\Sigma', \mathbb{B}) \mapsto (\Sigma, \mathbb{B}|_H)$$

$$(\Sigma, \mathbb{A}) \longmapsto (\Sigma', \mathbb{A}\!\uparrow^H)$$

such that we have theory interpretations

$$H : (\Sigma, \mathbb{B}|_H) \longrightarrow (\Sigma', \mathbb{B})$$

$$H : (\Sigma, \mathbb{A}) \longrightarrow (\Sigma', \mathbb{A}\!\uparrow^H)$$

Note that, in fact, any theory interpretation $H : (\Sigma, \mathbb{A}) \to (\Sigma', \mathbb{B})$ factors as a composition:

$$\begin{tikzcd}
(\Sigma, \mathbb{A}) \arrow[r, "H"] \arrow[d, hook] & (\Sigma', \mathbb{B}) \\
(\Sigma, \mathbb{B}|_H) \arrow[r, "H"] & (\Sigma', \mathbb{A}\!\uparrow^H) \arrow[u, hook]
\end{tikzcd}$$

Nicely intersectable signatures Σ_1 and Σ_2 with $\Sigma_1 \cap \Sigma_2 = \Sigma_0$ induce a funtion

$$- \times_{\Sigma_0} - \; : \mathcal{P}(Mod(\Sigma_1)) \times \mathcal{P}(Mod(\Sigma_2)) \ni (\mathbb{A}, \mathbb{B}) \mapsto \mathbb{A}_1 \times_{\Sigma_0} \mathbb{A}_2 \in \mathcal{P}(Mod(\Sigma_1 \cup \Sigma_2))$$

where $\mathbb{A}_1 \times_{\Sigma_0} \mathbb{A}_2 = \{ [\mathcal{M}_1, \mathcal{M}_2] \in Mod(\Sigma_1 \cup \Sigma_2) \mid \mathcal{M}_1 \in \mathbb{A}_1 \wedge \mathcal{M}_2 \in \mathbb{A}_2 \}$

and this induces a theory transformation, called <u>theory union</u>

$$((\Sigma_1, \mathbb{A}_1), (\Sigma_2, \mathbb{A}_2)) \mapsto (\Sigma_1 \cup \Sigma_2, \mathbb{A}_1 \times_{\Sigma_0} \mathbb{A}_2)$$

so that, if $T_1 = (\Sigma_1, \mathbb{A}_1)$ and $T_2 = (\Sigma_2, \mathbb{A}_2)$, then $(\Sigma_1 \cup \Sigma_2, \mathbb{A}_1 \times_{\Sigma_0} \mathbb{A}_2)$ is denoted $T_1 \cup T_2$.

Note that this notation makes sense and extends naturally an operation of theory union for syntactic theories

$$((\Sigma_1, \Gamma_1), (\Sigma_2, \Gamma_2)) \mapsto (\Sigma_1 \cup \Sigma_2, \Gamma_1 \cup \Gamma_2)$$

since we have $mod(\Sigma_1 \cup \Sigma_2, \Gamma_1 \cup \Gamma_2) = mod(\Sigma_1, \Gamma_1) \times_{\Sigma_0} mod(\Sigma_2, \Gamma_2)$.

The notions of conservative, resp. expansive, theory extension generalize naturally to theory interpretations:

<u>Definition</u> $H : T \to T'$ is called:

1. <u>conservative</u> iff $H(th(T)) = th(T') \cap H(FO(\Sigma))$

2. <u>expansive</u> iff $mod(T')|_H = mod(T)$

3. when $T = (\Sigma, \mathbb{A})$ and $T' = (\Sigma', \mathbb{B})$, <u>literally expansive</u> iff $(\mathbb{B}|_H)_{\cong} = \mathbb{A}_{\cong}$.

And the previous results again generalize easily:

a) H expansive $\Rightarrow$ H conservative

b) H literally expansive $\Rightarrow$ H conservative

But in general, H conservative $\not\Rightarrow$ H expansive,
H conservative $\not\Rightarrow$ H literally expansive,
H expansive $\not\Rightarrow$ H literally expansive, and
H literally expansive $\not\Rightarrow$ H expansive.

1.2 <u>More General Notions of Theory Interpretation</u>

As already mentioned, in Maude a view $H : (\Sigma, \Gamma) \to (\Sigma', \Gamma')$ need not map an operation $f : s_1 .. s_n \to s$ in Σ to $H(f) \in fun(\Sigma')$ such that $H(f)(x_1 : H(s_1), \dots, x_n : H(s_n)) \in T_{\Sigma'}(X_{S'})_{H(s)}$.
It can, more generally, map $f : s_1 .. s_n \to s$ to a <u>term</u> $t_f(x_1 : H(s_1), .., x_n : H(s_n)) \in T_{\Sigma'}(X_{S})_{H(s)}$.

In this way we can, for example view NAT as a poset in descending order by:

1. mapping Elt to Nat, and $\leq$ to the term $x_2 : Nat \leq x_1 : Nat$
2. mapping Elt to NzNat and $\leq$ to the term $x_2 : NzNat \mid x_1 : NzNat$,

Likewise, rather than mapping $p \in pred(\Sigma)$ to $H(p) \in pred(\Sigma')$ we can map $p : s_1 \ldots s_n$ to a formula $\varphi_p \in FO(\Sigma')$ such that $vars(\varphi_p) \subseteq \{x_1 : H(s_1), \ldots, x_n : H(s_n)\}$ where such mapping for $f \in fun(\Sigma)$ and $p \in pred(\Sigma)$ are required to respect subsort polymorphism in the obvious way.

In this way, we obtain a more general category of signatures and signature maps $H : \Sigma \to \Sigma'$. The crucial property is that we again obtain an equivalence of truth under translation

$$\mathcal{M}'|_H, a \models \varphi \Leftrightarrow \mathcal{M}', \tilde{a} \models H(\varphi)$$

The extension of the results in Lecture 21 to this more general notion of signature map, so that we can likewise have a more general notion of theory interpretation $H : T \to T'$ is not hard and is left as a (non-trivial) exercise.

Note, however, that we can make do with the more restrictive notion of signature map by the trick of using definitional extensions (recall what they are from pg. 2' of Lecture 21).

For example, we can add to NAT a new predicate $\geq$ with the definitional extension $x \geq y \Leftrightarrow y \leq x$, to get a definitional extension $NAT \subseteq NAT_{\geq}$. Then we can obtain the more general theory interpretation $\tilde{H}$ described in (1), pg. 6 above "virtually" as follows:

$$\begin{tikzcd} & NAT_{\geq} \\ POSET \arrow[ur, "H"] \arrow[r, dashed, "\tilde{H}"'] & NAT \arrow[u, hook] \end{tikzcd}$$

where we get the effect of defining $\tilde{H}$ in the more general setting by defining it as the above, more restricted, H on a definitional extension of NAT. In a similar way, we can get the effect of composing these more general theory interpretations "virtually", as follows:

$$\begin{tikzcd} & & (T''_{ext})_{ext} \\ & T'_{ext} \arrow[ur, "G'"] & T''_{ext} \arrow[u, hook] \\ T \arrow[ur, "H"] \arrow[r, dashed, "\tilde{H}"'] & T' \arrow[u, hook] \arrow[ur, "G"] \arrow[r, dashed, "\tilde{G}"'] & T'' \arrow[u, hook] \end{tikzcd}$$

Even more general notions are possible. For example:

(a) we can consider monotonic maps
$$H : (S, \leq) \longrightarrow (S'^{*}, \leq')$$
so that a sort $s \in S$ is now mapped to $H(s) = s'_1 \ldots s'_n$ interpreted as the product sort $s'_1 \times \cdots \times s'_n$

(see N. Martí-Oliet, J. Meseguer, M. Palomino, "Theoroidal Maps and Algebraic Simulations", Proc. WADT'04, 126–143, Springer LNCS 3423, 2005 for a useful application of this idea to the verification of LTL properties)

(b) we can map a sort s not to a sort $H(s)$, but to a formula $\varphi_s(x : H(s)) \in FO(\Sigma')$ such that $fvars(\varphi_s) \subseteq \{x : H(s)\}$

In this way, given a model $\mathcal{M}' \in Mod(\Sigma')$, the sort s in $\mathcal{M}'|H$ is interpreted as the set

$$M'|_{H,s} = \{ a \in M'_{H(s)} \mid \mathcal{M}', \{(x : H(s), a)\} \models \varphi_s \}$$

In this case the translation $H : FO(\Sigma) \to FO(\Sigma')$ must relativize quantification:

$$H(\forall x{:}s\ \psi) = \forall x{:}H(s)\ \ \varphi_s \Rightarrow H(\psi)$$

$$H(\exists x{:}s\ \psi) = \exists x{:}H(s)\ \ \varphi_s \wedge H(\psi)$$

(c) We can combine (a) and (b) and map, more generally, a sort s not to $H(s) = s'_1 \dots s'_n$ but to a formula $\varphi_s(x_1 : H(s_1), \dots, x_n : H(s_n)) \in FO(\Sigma')$

In cases (b) and (c), additional conditions on $H : T \to T'$ must be given to make sure that the sets defined by φ_s are closed under the operations in Σ.

2. Ascent Maps

The goal of an <u>ascent map</u> $(H,\alpha) : (T,\mathcal{F}) \to (T',\mathcal{G})$ is to <u>reduce</u> the problem of satisfiability of $\mathcal{F}$-formulas in T to that of satisfiability of $\mathcal{G}$-formulas in T' by means of a theory interpretation $H : T \to T'$ and a formula translation $\alpha : \mathcal{F} \to \mathcal{G}$. Here is the general definition:

<u>Definition</u>. An <u>ascent map</u> from a theory T and a recursive class $\mathcal{F} \subseteq FO(sig(T))$ of formulas to another theory T and recursive class $\mathcal{F} \subseteq FO(sig(T'))$ of formulas, denoted $(H,\alpha) : (T,\mathcal{F}) \to (T',\mathcal{G})$ is a pair (H,α) with $H : T \to T'$ a theory interpretation and $\alpha : \mathcal{F} \to \mathcal{G}$ a recursive function such that for each $\varphi \in \mathcal{F}$ is T-satisfiable iff $\alpha(\varphi)$ is T'-satisfiable. (H,α) is called <u>effective</u> if given any $\varphi \in \mathcal{F}$, $\mathcal{M}' \in mod(T')$ and $\tilde{a} \in [fvars(\alpha(\varphi)) \to M']$ such that $\mathcal{M}', \tilde{a} \models \alpha(\varphi)$ there is an effective construction of a model $\mathcal{M} \in mod(T)$ and $a \in [fvars(\varphi) \to M]$ such that $\mathcal{M}, a \models \varphi$.

<u>Notation</u>. When H is a theory inclusion $J : T \subseteq T'$, we denote $(J,\alpha) : (T,\mathcal{F}) \to (T',\mathcal{G})$ as $(T,\mathcal{F}) \overset{\alpha}{\subseteq} (T',\mathcal{G})$.

Note that, indeed, if T'-satisfiability of $\mathcal{G}$-formulas is decidable, we can use an ascent map $(H,\alpha):(T,\mathcal{F}) \to (T',\mathcal{G})$ to obtain a decision procedure for T-satisfiability of $\mathcal{F}$-formulas.

The simplest class of ascent maps, called trivial ascent maps are of the form $(H,H):(T,\mathcal{F}) \longrightarrow (T',\mathcal{G})$ with $H: T \to T'$ (or literally expansive) expansive and $H(\mathcal{F}) \subseteq \mathcal{G}$. Such maps are indeed ascent maps, since any $\mathcal{M} \in mod(T)$ is of the form $\mathcal{M} = \mathcal{M}'|_H$ for some $\mathcal{M}' \in mod(T')$. Therefore, given $\varphi \in \mathcal{F}$ ~~$\mathcal{M}$~~ $\mathcal{M} \in mod(T)$, and $a \in [fvars(\varphi) \to M]$, with $\mathcal{M} = \mathcal{M}'|_H$, we have

$$\mathcal{M}, a \models \varphi \Leftrightarrow \mathcal{M}', \tilde{a} \models H(\varphi)$$

where if $Y = fvars(\varphi)$, then $\tilde{a} \in [H(Y) \to M']$, with $\tilde{a}(x{:}H(s)) = a(x{:}s)$.

Conversely, given $\mathcal{M}' \in mod(T')$ and $\tilde{a} \in [H(Y) \to M']$ we have

$$\mathcal{M}', \tilde{a} \models H(\varphi) \Longleftrightarrow \mathcal{M}'|_H, a \models \varphi .$$

Note also that trivial ascent maps are effective, since $\mathcal{M}', \tilde{a}$ and φ determine $\mathcal{M}'|_H$, and a.

For a trivial example of a trivial ascent map, consider

$$\big((\{\geq\},\{\mathbb{N}_{\geq}\}), QF(\{\geq\})\big) \overset{J}{\subseteq} \big((\{+,*,0,1,\geq\},\{\mathbb{N}_{+,*,0,1,\geq}\}), QF(\{+,*,0,1,\geq\})\big)$$

where we "borrow" Presburger arithmetic's decision procedure to decide satisfiability of formulas in $QF(\{\geq\})$ for the simpler semantic theory $(\{\geq\},\{\mathbb{N}_{\geq}\})$.

Note the ascent maps compose in an obvious way, since given $(T, \mathcal{F}) \xrightarrow{(H,\alpha)} (T', \mathcal{G}) \xrightarrow{(G,\beta)} (T'', K)$ we get $(T, \mathcal{F}) \xrightarrow{(H;G,\,\alpha;\beta)} (T'', K)$. Therefore, the form a category $\underline{\text{Ascent}}$.

For a not-so-trivial example of a trivial ascent map, consider one mapping QF formulas to equational QF-formulas, i.e., QF-formulas whose only atoms are equations. Recall that any OS-FO signature $\Sigma = ((S, \leq), F, P)$ can be transformed into an equational signature $\widetilde{\Sigma} = ((S \overset{+}{\cup} \{Pred\}, \leq \overset{+}{\cup} =_{\{Pred\}}), F \cup \widetilde{P}, \emptyset)$, where Pred is a fresh new sort unrelated to any sort in $(S, \leq)$, $fun(\widetilde{\Sigma}) = fun(\Sigma) \overset{+}{\cup} pred(\Sigma) \cup \{tt\}$, and for each $p \in pred(\Sigma)$, $\widetilde{P}(p) = \{(w, Pred) \mid w \in P(p)\}$, and, furthermore, $\widetilde{P}(tt) = \{(\varepsilon, Pred)\}$. That is, we just turn each $p : s_1 \ldots s_n$ into a function symbol $p : s_1 \ldots s_n \to Pred$, and add a "true" constant tt of sort Pred.

We can then define a generalized map of signatures in the sense of Section 1.2 $FUN : \Sigma \to \widetilde{\Sigma}$ that is the inclusion on sorts, the identity on function symbols F, and maps each $p : s_1 \ldots s_n$ to the equational formula $p(x_1 : s_1, \ldots, x_n : s_n) = tt$.

Note that FUN, as a formula translation $FUN : FO(\Sigma) \to FO(\tilde{\Sigma})$ maps any $\varphi \in FO(\Sigma)$ to an equational formula $FUN(\varphi) \in FO(\tilde{\Sigma})$.

For example,

$$FUN(\forall x\, \exists y\, .(\, y \geq x \wedge prime(y))) =$$

$$= \forall x\, \exists y\, .(\, y \geq x = tt \wedge prime(y) = tt\,).$$

I claim that for any $\Gamma \subseteq FO(\Sigma)$,

$$(FUN, FUN) : (\Sigma, \Gamma) \longrightarrow (\tilde{\Sigma}, FUN(\Gamma))$$

is a trivial ascent map. This is so because any $\mathcal{M} = (M, \_^{F}_{\mathcal{M}}, \_^{P}_{\mathcal{M}}) \in Mod(\Sigma)$ can be extended to a model $\mathcal{M}\uparrow^{\tilde{\Sigma}} = (M\uparrow, \_^{F \cup \tilde{P}}_{\mathcal{M}\uparrow\tilde{\Sigma}}, \emptyset) \in Mod(\tilde{\Sigma})$ as follows. For $s \in S$, $M\uparrow^{\tilde{\Sigma}}_{s} = M_s$. For the sort Pred we define

$$M\uparrow^{\tilde{\Sigma}} = \{tt\} \cup \{ (\{p\}, a_1, \dots, a_n) \mid p \in Pred(\Sigma) \wedge s_1 \cdots s_n \in P(p) \wedge (a_1, \dots, a_n) \in M_{s_1} \times \dots \times M_{s_n} \setminus p^{s_1 \dots s_n}_{\mathcal{M}} \}.$$

Then for each $f : s_1 \dots s_n \to s$ in F, $f_{\mathcal{M}\uparrow\tilde{\Sigma}} = f_{\mathcal{M}} : M_{s_1} \times \dots \times M_{s_n} \to M_s$ and for each $p : s_1 \dots s_n \to Pred$

$$p_{\mathcal{M}\uparrow\Sigma} : M_{s_1} \times \dots \times M_{s_n} \ni (a_1, \dots, a_n) \mapsto \underline{if}\ (a_1, \dots, a_n) \in p_{\mathcal{M}}\ \underline{then}\ tt\ else\ (\{p\}, a_1, \dots, a_n)\ \underline{fi}$$

and, of course $tt_{\mathcal{M}\uparrow\Sigma} = tt$.

It is then easy to check that $\mathcal{M}\!\uparrow^{\tilde{\Sigma}}\big|_{FUN} = \mathcal{M}$, and of course $\mathcal{M}, a \models \varphi \iff \mathcal{M}\!\uparrow^{\tilde{\Sigma}}, a \models FUN(\varphi)$, $a \in [fvars(\varphi) \to M]$.

Therefore, $\mathcal{M} \models \Gamma \iff \mathcal{M}\!\uparrow^{\tilde{\Sigma}} \models \Gamma$, making $FUN : (\Sigma, \Gamma) \longrightarrow (\tilde{\Sigma}, FUN(\Gamma))$ expansive, and therefore $(FUN, FUN) : (\Sigma, \Gamma) \to (\tilde{\Sigma}, FUN(\Gamma))$ a (not-so-trivial), trivial ascent map.

Let us now consider two examples of <u>non-trivial</u> ascent maps.

1. Let Σ be an algebraic OS-signature, i.e., $\Sigma = ((S, \leq), F, \emptyset)$. Its <u>unsorted</u> version $\Sigma^u = ((\{U\}, =_{\{U\}}), F^u, \emptyset)$ has a single "universe" sort U, $fun(\Sigma) = fun(\Sigma^u)$, and F^u is defined as follows

$$F^u(f) = \{(U \overset{n}{\cdots} U, U) \mid (s_1 \cdots s_n, s) \in F(f)\}$$

Then, the unique mapping (assuming $S \neq \emptyset$!) $u : S \to \{U\}$ induces an obvious signature map $u : \Sigma \to \Sigma^u$ that maps each $f : s_1 \dots s_n \to s$ in Σ to $f : U \overset{n}{\cdots} U \to U$.

The map $(u, u) : ((\Sigma, \emptyset) QF(\Sigma)) \longrightarrow ((\Sigma^u, \emptyset), QF(\Sigma^u))$ is an ascent map. But proving this is non-trivial, because as soon as $(S, \leq)$ has more than one sort we obviously have $Mod(\Sigma^u)|_u \subsetneqq Mod(\Sigma)$. The proof that it is an ascent map is given in J. Meseguer, "Order-Sorted Rewriting and Congruence Closure", Proc. FOSSACS 2016, 493-509, Springer LNCS 9634, 2016. It is furthermore effective: we just transform a congruence closure Σ^u-model of a QF-formula φ^u into a Σ-model by restoring the sort information of the variables in $vars(\varphi)$. This reduces satisfiability of "the" theory of order-sorted uninterpreted function symbols to unsorted satisfiability by congruence closure

2. Likewise, if $\Delta \subseteq \Sigma$ is a family of binary symbols and AC_Δ are associativity-commutativity axioms for each $f \in \Delta$, the above-cited paper proves that

$$(u, u) : (\Sigma, \Delta_{AC}) \to (\Sigma^u, \Delta^u_{AC})$$

is an ascent map, reducing satisfiability of the order-sorted theory with AC function symbols Δ to that of unsorted symbols with AC Δ^u by AC congruence closure.

Finally, the compositions of ascent maps

$$((\Sigma, \emptyset), QF(\Sigma)) \xrightarrow{(FUN, FUN)} ((\tilde{\Sigma}, \emptyset), QF(\tilde{\Sigma})) \xrightarrow{(u,u)} ((\tilde{\Sigma}^u, \emptyset), QF(\Sigma^u))$$

$$((\Sigma, AC_\Delta), QF(\Sigma)) \xrightarrow{(FUN, FUN)} ((\tilde{\Sigma}, AC_\Delta), QF(\tilde{\Sigma})) \xrightarrow{(u,u)} ((\tilde{\Sigma}^u, AC^u_\Delta), QF(\Sigma^u))$$

make QF-satisfiability in $(\Sigma, \emptyset)$, resp. (Σ, AC_Δ) decidable by unsorted congruence closure (resp. AC-congruence closure).