CS 576 Lecture 22 3/30/17

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1. Signature Maps and Theory Interpretations (II)

1.1. Putting Signatures and Models Together

In any non-toy practical applications, theories are not presented in a monolithic way, as a single "kitchen sink" theory. Instead, they are structured, for very good modularity and reusability reasons, as combinations of simpler theories. For example, the Nelson-Oppen combination method exploits a structuring of a theory T as a disjoint union of simpler theories

$$T = T_1 \dot{\cup} \cdots \dot{\cup} T_n$$

each of

for which satisfiability is decidable to obtain a decision procedure for T-satisfiability under reasonable assumptions about the $T_1, \ldots, T_n$.

So an obvious question is: how do we put theories together? One, by now obvious answer is the Goguen-Burstall answer, supported by languages such as Maude:

Of course, by pushouts!

So, what is a pushout of theories? To answer this question we need to answer a more basic one:

<u>What is a pushout of signatures</u>?

Although I have shown in "Membership Algebra as a Logical Framework for Equational Specification," Proc. WRLA'97, 18-61, Springer LNCS 1376, that the category of order-sorted signatures has arbitrary pushouts, for our present purposes it will be enough to consider a special form of pushout appearing very often in practice and having very good properties, namely, pushouts for <u>nicely intersectable</u> signatures.

<u>Def.</u> Order-sorted FO signatures $\Sigma_1 = ((S_1, \leq_1), F_1, P_1)$ and $\Sigma_2 = ((S_2, \leq_2), F_2, P_2)$ are called <u>nicely intersectable</u> iff:

1. for each $s_0 \in S_1 \cap S_2$ either

   (i) $[s_0]_1 = [s_0]_2$, where

   $[s]_i$ abbreviates $[s]_{\equiv_{\leq_i}}$,

   and $\leq_1 \cap [s_0]_1^2 = \leq_2 \cap [s_0]_2^2$.

or (ii) $[s_0]_1 \cap [s_0]_2 = \{s_0\}$

2. for each $f \in fun(\Sigma_1) \cap fun(\Sigma_2)$

$F_1(f) = F_2(f)$ and, furthermore,

if $(s_1 .. s_n, s) \in F_1(f) = F_2(f)$, then

$[s_1]_1 = [s_1]_2, \ldots, [s_n]_1 = [s_n]_2$, and $[s]_1 = [s]_2$

3. for each $p \in pred(\Sigma_1) \cap pred(\Sigma_2)$, $P_1(p) = P_2(p)$,

and if $(s_1 .., s_n) \in P_1(p) = P_2(p)$, then

$[s_1]_1 = [s_1]_2, \ldots, [s_n]_1 = [s_n]_2$.

Under these conditions we define their <u>intersection</u>

$\Sigma_0 = \Sigma_1 \cap \Sigma_2 = ((S_0, \leqslant_0), F_0, P_0)$, where $S_0 = S_1 \cap S_2$

and $(S_0, \leqslant_0)$ is the disjoint union of posets

$([s_0]_1, \leqslant_1) = ([s_0]_2, \leqslant_2)$ when $s_0 \in S_1 \cap S_2$ and

$[s_0]_1 = [s_0]_2$, and otherwise the singleton poset

$(\{s_0\}, =_{\{s_0\}})$ when $\{s_0\} = [s_0]_1 \cap [s_0]_2$,

and where $fun(\Sigma_0) = fun(\Sigma_1) \cap fun(\Sigma_2)$, and

$pred(\Sigma_0) = pred(\Sigma_1) \cap pred(\Sigma_2)$, with

$F_0 = F_1|_{fun(\Sigma_0)} = F_2|_{fun(\Sigma_0)}$, $P_0 = P_1|_{pred(\Sigma_0)} = P_2|_{pred(\Sigma_0)}$.

Before defining the union of signatures $\Sigma_1 \cup \Sigma_2$ for nicely intersectable signatures and its pushout property, we need an elementary lemma about posets.

<u>Lemma</u>. Let $(A_1, <_1)$ and $(A_2, <_2)$ be posets such that $A_1 \cap A_2 = \{a\}$ for some a. Then $(A_1 \cup A_2, (<_1 \cup <_2)^+)$, where R^+ denotes the transitive closure of a relation R is a poset.

<u>Proof</u>. Since $(<_1 \cup <_2)$ is transitive by construction, we just need to show that it is irreflexive. But this follows trivially if we can prove that $(<_1 \cup <_2)^+ \cap A_i^2 = <_i$, $1 \leq i \leq 2$. This itself follows trivially by induction on the length n of a sequence

$$b_0 <_1 \cup <_2 b_1 \cdots b_{n-1} <_1 \cup <_2 b_n . \square$$

<u>Definition</u>. Let Σ_1, Σ_2 be nicely intersectable OS-FO signatures with $\Sigma_1 = ((S_1, <_1), F_1, P_1)$, $\Sigma_2 = ((S_2, <_2), F_2, P_2)$. Then their <u>union</u> $\Sigma_1 \cup \Sigma_2$ is defined as the signature

$$\Sigma_1 \cup \Sigma_2 = ((S_1 \cup S_2, (<_1 \cup <_2)^+), F_1 \cup F_2, P_1 \cup P_2).$$

Let us check that this definition makes sense. If $s_0 \in S_1 \cap S_2$, then either $[s_0]_1 = [s_0]_2$, and $\leqslant_1 \cap [s_0]_1^2 = \leqslant_2 \cap [s_0]_2^2$, so that, trivially,

$$\left(\leqslant_1 \cap [s_0]_1^2 \cup <_2 \cap [s_0]_2^2 \right)^+ = <_i \cap [s_0]_i^2, \; 1 \leq i \leq 2,$$

or $[s_0]_1 \cap [s_0]_2 = \{s_0\}$ and we are in the situation of the above lemma, so that

$$\left([s_0]_1 \cup [s_0]_2 , \left(<_1 \cap [s_0]_1^2 \cup <_2 \cap [s_0]_2^2 \right)^+ \right)$$

is a poset. Of course, if $s_1 \in S_1 - S_2$ we have a connected component $[s_1]_1$ with $[s_1]_1 \cap S_2 = \emptyset$ and likewise if $s_2 \in S_2 - S_1$. Note that $(S_1 \cup S_2, (<_1 \cup <_2)^+)$ is just the union of such connected components.

Likewise, since F_1 and F_2 agree on $fun(\Sigma_1) \cap fun(\Sigma_2)$ and P_1 and P_2 on $pred(\Sigma_1) \cap pred(\Sigma_2)$, $F_1 \cup F_2$ and $P_1 \cup P_2$ are well-defined functions.

This gives us the following diagram of signature inclusions:

$$\begin{tikzcd}
\Sigma_1 \arrow[r, hook, "J_2'"] & \Sigma_1 \cup \Sigma_2 \\
\Sigma_0 \arrow[u, hook, "J_1"] \arrow[r, hook, "J_2"'] & \Sigma_2 \arrow[u, hook, "J_1'"']
\end{tikzcd}$$

which has the following <u>pushout property</u>:

<u>Theorem</u>. Let Σ_1, Σ_2 be nicely intersectable signatures with $\Sigma_0 = \Sigma_1 \cap \Sigma_2$. Let $H_1 : \Sigma_1 \longrightarrow \Sigma$, $H_2 : \Sigma_2 \to \Sigma$ be signature maps such that $J_1 ; H_1 = J_2 ; H_2$.
Then there is a unique signature map, denoted $H_1 \cup H_2 : \Sigma_1 \cup \Sigma_2 \longrightarrow \Sigma$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & \Sigma \\
\Sigma_1 \arrow[r, hook, "J_2'"] \arrow[urr, bend left, "H_1"] & \Sigma_1 \cup \Sigma_2 \arrow[ur, dashed, "H_1 \cup H_2"] & \\
\Sigma_0 \arrow[u, hook, "J_1"] \arrow[r, hook, "J_2"'] & \Sigma_2 \arrow[u, hook, "J_1'"'] \arrow[uur, "H_2"'] &
\end{tikzcd}$$

<u>Proof</u>. The notation $H_1 \cup H_2$ is for the moment a <u>formal notation</u>; we need to show that it actually makes sense. Uniqueness is obvious, since H_1 and H_2 are themselves families of functions (for sorts, function symbols and predicate symbols) which <u>agree</u> on Σ_0, if a corresponding (family of) functions is supposed

to define a signature map $\Sigma_1 \cup \Sigma_2 \to \Sigma$, it
must be the (component-wise) set-theoretic union
of H_1 and H_2. By the nice intersectability
property and the equality $J_1 ; H_1 = J_2 ; H_2$
That
$F_1 \cup F_2$ and $P_1 \cup P_2$ are indeed functions.
$H_1 \cup H_2 : (S_1 \cup S_2, (\leqslant_1 \cup \leqslant_2)^+) \to (S, \leqslant)$
they and
satisfy the requirements of a signature map boils
down to showing $H_1 \cup H_2$ monotonic. But this
is easy to show by focusing on each connected
component of $(S_1 \cup S_2, (\leqslant_1 \cup \leqslant_2)^+)$. The
only possibly problematic case is when $[s_0]_1 \cap [s_0]_2 = \{s_0\}$.
But in that case $H_1 \cup H_2|_{[s_0]_1 \cup [s_0]_2} : ([s_0]_1 \cup [s_0]_2, (\leqslant_1 \cup \leqslant_2)^+|_{[s_0]_1 \cup [s_0]_2}^2)$
$\to (S, \leq_S)$ is indeed monotonic. For $s \leqslant s'$ with
$s, s' \in [s_0]_i$, $1 \leq i \leq 2$, this follows from the proof
of the above lemma, since then $H_1 \cup H_2(s) = H_i(s)$,
$H_1 \cup H_2(s') = H_i(s)$, and $s (\leqslant_1 \cup \leqslant_2)^+ s'$ iff
$s \leq_i s'$, so that $H_i(s) \leqslant_S H_i(s')$.
For $s < s'$ with, say, $s \in S_1$ and $s' \in S_2$
we must have $s \leqslant_1 s_0 \leqslant_2 s'$. But this gives us
$H_1(s) \leqslant_S H_1(s_0) = H_2(s_0) \leqslant_S H_2(s')$, as desired. $\square$

This is all very well. But how can we now put models together?
That is, suppose $\mathcal{M}_1 \in \underline{Mod}(\Sigma_1)$ and $\mathcal{M}_2 \in Mod(\Sigma_2)$.
Can we the combine them into a model, say
$[\mathcal{M}_1, \mathcal{M}_2] \in Mod(\Sigma_1 \cup \Sigma_2)$ which we may call their
amalgamation? Existence of such amalgamations is, for example, a crucial issue for Nelson-Oppen combinations to get off the ground.

Let us think categorically for a moment. Consider the functor $\underline{Mod}: \underline{\text{OS-FO Sign}}^{op} \longrightarrow \underline{Cat}$

since $\underline{\text{OS-FO-Sign}}^{op}$ is the dual of $\underline{\text{OSFO-Sign}}$
what does a pushout in $\underline{\text{OS-FO-Sign}}$ looks like in $\underline{\text{OS-FO-Sign}}^{op}$? Of course as it dual,

namely a pullback.

Viewed in this, enlightened, way, what is the existence amalgamations for a pushout of theories all about?

It is all about $\underline{Mod}$ mapping that pullback in $\underline{\text{OS-FO-Sign}}^{op}$ to a pullback in $\underline{Cat}$.

I call this in my General Logics paper an exactness property about $\underline{Mod}$.

It turns out that <u>Mod</u> is <u>not</u> <u>exact</u> for arbitrary pushouts of OS-signatures (see J. Meseguer, "Order-Sorted Parameterization and Induction", Springer LNCS 5700, 43–80, 2009 for a counterexample). I will, however show that it <u>is</u> exact for the $\Sigma_1 \cup \Sigma_2$ pushout when Σ_1 and Σ_2 are nicely intersectable.

So, <u>what is amalgamation</u>? Or, put in a more enlightened way, what is a pullback in <u>Cat</u> like? If we disregard morphisms, which we can for satisfyability purposes, the answer is easy:

<u>Exactly like a pullback in Set</u>

So, what is a pullback in the category of sets? That is extremely easy to answer:

Given function $f_1 : A_1 \to A_0$, $f_2 : A_2 \to A_0$ their pullback $A_1 \times_{A_0} A_2$ is the set

$$A_1 \times_{A_0} A_2 = \{ (a_1, a_2) \in A_1 \times A_2 \mid f_1(a_1) = f_2(a_2) \}$$

together with the projection functions $p_i : A_1 \times_{A_0} A_2 \to A_i$, $1 \leq i \leq 2$, and it has the following universal property: for any two function $g_i : A \to A_i$ $1 \leq i \leq 2$ such that $g_1 ; f_1 = g_2 ; f_2$

there is a unique function $(g_1, g_2) : A \to A_1 \times_{A_0} A_2$ such that the diagram

$$\begin{tikzcd}
 & & A \arrow[dll, bend right, "g_1"'] \arrow[dl, dashed, "{(g_1,g_2)}"'] \arrow[ddl, bend left, "g_2"] \\
A_1 \arrow[d, "f_1"'] & A_1 \times_{A_0} A_2 \arrow[l, "p_1"] \arrow[d, "p_2"] & \\
A_0 & A_2 \arrow[l, "f_2"] &
\end{tikzcd}$$

commutes. (g_1, g_2) does the expected thing:

$(g_1, g_2) : a \mapsto (g_1(a), g_2(a)) \in A_1 \times_{A_0} A_2$.

So, what is the <u>amalgamation property</u> for our pushout $\Sigma_1 \cup \Sigma_2$?

Disregarding homomorphisms and sticking only to models the claim that the commutative diagram

$$(\star) \quad \begin{tikzcd}
\mathrm{Mod}(\Sigma_1) \arrow[d, "-|_{\Sigma_0}"'] & \mathrm{Mod}(\Sigma_1 \cup \Sigma_2) \arrow[l, "-|_{\Sigma_1}"'] \arrow[d, "-|_{\Sigma_2}"] \\
\mathrm{Mod}(\Sigma_0) & \mathrm{Mod}(\Sigma_2) \arrow[l, "-|_{\Sigma_0}"]
\end{tikzcd}$$

is, up to a bijective change of representation, a <u>pullback in Set</u>. That is, the claim that the models of $\mathrm{Mod}(\Sigma_1 \cup \Sigma_2)$ are exactly <u>pairs</u> $[M_1, M_2]$ with $M_1 \in \mathrm{Mod}(\Sigma_1)$, $M_2 \in \mathrm{Mod}(\Sigma_2)$, and $M_1|_{\Sigma_0} = M_2|_{\Sigma_0}$.

(Amalgamation Property).

Theorem. Let Σ_1 and Σ_2 be nicely intersectable OS-FO signatures. Then, the diagram ($\star$) is a pullback in Set.

Proof. We just need to show that given $\mathcal{M}_1 \in \underline{Mod}(\Sigma_1)$, $\mathcal{M}_2 \in \underline{Mod}(\Sigma_2)$ such that $\mathcal{M}_1|_{\Sigma_0} = \mathcal{M}_2|_{\Sigma_0}$, there is a unique expansion to a model, denoted $[\mathcal{M}_1, \mathcal{M}_2]$ in $\underline{Mod}(\Sigma_1 \cup \Sigma_2)$, and that any model $\mathcal{M} \in Mod(\Sigma_1 \cup \Sigma_2)$ is exactly of the form $\mathcal{M} = [\mathcal{M}|_{\Sigma_1}, \mathcal{M}|_{\Sigma_2}]$.

Note, first of all, that, for any $\mathcal{M} \in Mod(\Sigma_1 \cup \Sigma_2)$ we have $(\mathcal{M}|_{\Sigma_1})|_{\Sigma_0} = (\mathcal{M}|_{\Sigma_2})|_{\Sigma_0}$ because of ($\star$).

Therefore, all we need to prove is:

1. Uniqueness. If $\mathcal{M}, \mathcal{M}' \in Mod(\Sigma_1 \cup \Sigma_2)$ with $\mathcal{M}_i = \mathcal{M}|_{\Sigma_i} = \mathcal{M}'|_{\Sigma_i}$, $1 \le i \le 2$, then $\mathcal{M} = \mathcal{M}'$.

   Let $\mathcal{M} = (M, \_^F_{\mathcal{M}}, \_^P_{\mathcal{M}})$, $\mathcal{M}' = (M', \_^F_{\mathcal{M}'}, \_^P_{\mathcal{M}'})$.

   $M = M'$ follows easily from
   $M = \{M_s\}_{s \in S_1 \cup S_2} = \{M'_s\}_{s \in S_1 \cup S_2} = M'$.

   Likewise we have,

$\overset{F}{\twoheadrightarrow}_{\mathcal{M}} = \overset{F_1}{\twoheadrightarrow}_{\mathcal{M}_1} \cup \overset{F_2}{\twoheadrightarrow}_{\mathcal{M}_2} = \overset{F}{\twoheadrightarrow}_{\mathcal{M}'}$, and

$\overset{P}{\twoheadrightarrow}_{\mathcal{M}'} = \overset{P}{\twoheadrightarrow}_{\mathcal{M}_1} \cup \overset{P}{\twoheadrightarrow}_{\mathcal{M}_2} = \overset{P}{\twoheadrightarrow}_{\mathcal{M}'}$

<u>2. Existence</u>. Given $\mathcal{M}_1 \in \underline{Mod}(\Sigma_1)$, $\mathcal{M}_2 \in \underline{Mod}(\Sigma_2)$ with, say, $\mathcal{M}_1 = (M_1, \overset{F_1}{\twoheadrightarrow}_{\mathcal{M}_1}, \overset{P_1}{\twoheadrightarrow}_{\mathcal{M}_1})$, $\mathcal{M}_2 = (M_2, \overset{F_2}{\twoheadrightarrow}_{\mathcal{M}_2}, \overset{P_2}{\twoheadrightarrow}_{\mathcal{M}_2})$ define $[\mathcal{M}_1, \mathcal{M}_2] = ([M_1, M_2], \overset{F_1}{\twoheadrightarrow}_{\mathcal{M}_1} \cup \overset{F_2}{\twoheadrightarrow}_{\mathcal{M}_2}, \overset{P_1}{\twoheadrightarrow}_{\mathcal{M}_1} \cup \overset{P_2}{\twoheadrightarrow}_{\mathcal{M}_2})$, as follows: $[M_1, M_2] = \{[M_1, M_2]_s\}_{s \in S_1 \cup S_2}$, where if $s \in S_i$, then $[M_1, M_2]_s = M_{i,s}$, $1 \leq i \leq 2$, which is well-defined, since when $s_0 \in S_1 \cap S_2$ since $\mathcal{M}_1|_{\Sigma_0} = \mathcal{M}_2|_{\Sigma_0}$ we must have $M_{1,s_0} = M_{2,s_0}$. Furthermore, if $s, s' \in S_1 \cup S_2$ and $s \leq s'$ we do have $[M_1, M_2]_s \subseteq [M_1, M_2]_{s'}$. This follows trivially when $s, s' \in S_i$, $1 \leq i \leq 2$. Otherwise, if $s < s'$ with, say, $s \in S_1$; $s' \in S_2$, and $[s]_1 \cap [s']_2 = \{s_0\}$, we must have $s \leq s_0 \leq s'$, which forces

$$M_{1,s} \subseteq M_{1,s_0} = M_{2,s_0} \subseteq M_{2,s'}$$

as desired.

We just need to check that $-^{F_1}_{M_1} \cup -^{F_2}_{M_2}$ defines an order-sorted algebra structure on $[M_1, M_2]$ that yields the one on M_i as its Σ_i-reduct, $1 \leq i \leq 2$, and likewise for $-^{P_1}_{M_1} \cup -^{P_2}_{M_2}$ with respect to predicates. Let us show that this is the case for $-^{F_1}_{M_1} \cup -^{F_2}_{M_2}$, and leave the predicate case as an exercise. There are three cases:

1. $f \in fun(\Sigma_1) \cap fun(\Sigma_2) = fun(\Sigma_0)$

   then, by $M_1|_{\Sigma_0} = M_2|_{\Sigma_0}$ we must have

   $f_{M_1} = f_{M_2}$ for all the <u>shared</u>! types,

   since $F_1(f) = F_2(f)$.

2. $f \in fun(\Sigma_1) - fun(\Sigma_2)$. Then

   $(-^{F_1}_{M_1} \cup -^{F_2}_{M_2})(f)$ f_{M_1} for <u>all</u> the types

   of $F_1(f)$

3. Likewise, if $f \in fun(\Sigma_2) - fun(\Sigma_1)$,

   $(-^{F}_{M_2} \cup -^{F}_{M_2})(f) = f_{M_2}$

Therefore, $-^{F_1}_{M_1} \cup -^{F_2}_{M_2}$ (and likewise $-^{P_1}_{M_1} \cup -^{P_2}_{M_2}$) are well-defined, and yield the desired amalgamation $[M_1, M_2]$ with $[M_1, M_2]|_{\Sigma_i} = M_i$, $1 \leq i \leq 2$. $\square$