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1. Conservative and Expansive Theory Extensions

<u>Notation</u>. Given a theory inclusion $T \subseteq T'$, we call T a <u>subtheory</u> of T', and T' an <u>extension</u> of T. We will often speak of a <u>subtheory</u> $T \subseteq T'$ (meaning T), and an <u>extension</u> $T' \supseteq T$ (meaning T').

<u>Def</u>. $T' \supseteq T$ is a <u>conservative extension</u> of T iff $th(T') \cap FO(sig(T)) = th(T)$.

For syntactic theories $((\Sigma,\Pi),\Gamma) \subseteq ((\Sigma',\Pi'),\Gamma')$ this boils down to what one would expect:

$$\forall \varphi \in FO(\Sigma,\Pi) \quad \Gamma \vdash \varphi \iff \Gamma' \vdash \varphi.$$

<u>Notation</u>. Given $(\Sigma,\Pi) \subseteq (\Sigma',\Pi')$ and $\mathcal{M}' \in \underline{Mod}(\Sigma',\Pi')$ we call $\mathcal{M}'|_{(\Sigma,\Pi)}$ the (Σ,Π)-<u>reduct</u> of $\mathcal{M}'$.

Also, given $\mathcal{M} \in \underline{Mod}(\Sigma,\Pi)$ we call $\mathcal{M}' \in Mod(\Sigma',\Pi')$ a (Σ',Π')-<u>expansion</u> of $\mathcal{M}$ iff $\mathcal{M}'|_{(\Sigma,\Pi)} = \mathcal{M}$.

This suggest yet another semantic theory operation associated to $(\Sigma,\Pi) \subseteq (\Sigma',\Pi')$ and the function

$$\_\uparrow^{(\Sigma',\Pi')} : \mathcal{P}(\underline{Mod}(\Sigma,\Pi)) \ni \mathbb{A} \longmapsto \mathbb{A}\uparrow^{(\Sigma',\Pi')} \in \mathcal{P}(\underline{Mod}(\Sigma',\Pi'))$$

where $\mathbb{A}\!\uparrow^{(\Sigma',\Pi')} = \{\mathcal{M}' \in Mod(\Sigma',\Pi') \mid \mathcal{M}'|_{(\Sigma,\Pi)} \in \mathbb{A}\}$

This induces a semantic theory operation $((\Sigma,\Pi),\mathbb{A}) \mapsto ((\Sigma,\Pi),\mathbb{A}\!\uparrow^{(\Sigma',\Pi')})$.

Exercise. Prove that if $((\Sigma,\Pi),\mathbb{A}) \cong ((\Sigma,\Pi),\Gamma)$, and $(\Sigma,\Pi) \subseteq (\Sigma',\Pi')$, $((\Sigma',\Pi'),\mathbb{A}\!\uparrow^{(\Sigma',\Pi')}) \cong ((\Sigma',\Pi'),\Gamma)$.

Definition. Call a theory extension $T' \supseteq T$ expansive iff $mod(T')|_{sig(T)} = mod(T)$. Call $((\Sigma',\Pi'),\mathbb{B}) \supseteq ((\Sigma,\Pi),\mathbb{A})$ literally expansive iff $(\mathbb{B}|_{(\Sigma,\Pi)})_{\cong} = \mathbb{A}_{\cong}$

Examples

1. $((\Sigma,\Pi),\Gamma) \subseteq ((\Sigma(Y),\Pi),\Gamma)$ where we have added to Σ a set Y of new fresh constants is expansive. Indeed, for $\mathcal{M} \in \underline{Mod}((\Sigma,\Pi),\Gamma)$ $\{\mathcal{M}\}\!\uparrow^{(\Sigma',\Pi')} = \{(\mathcal{M},a) \mid a \in [Y \to M]\}$ so that, $mod(T')|_{sig(T)} = mod(T)$. Note that, in fact, arguments about satisfiability take place in $Mod((\Sigma(Y),\Pi),\Gamma)$ for $Y = fvars(\varphi)$.

2. $(\{+\}, +_{AC}) \subseteq (\{+,*\}, +_{AC}, *_{AC})$ is expansive, because for any $\mathcal{A} = (A, +_{\mathcal{A}})$, the algebra $\mathcal{A}' = (A, +_{\mathcal{A}}, *_{\mathcal{A}'})$ where $*_{\mathcal{A}'} = +_{\mathcal{A}}$ is such that $\mathcal{A}'|_{\{+\}} = \mathcal{A}$.

3. $(\{*,1\}, MON) \subseteq (\{*,1\}, MON \cup \{(\forall x)\exists y\,.\,x*y=1\})$ with MON the theory of monoids is not an expansive extension. For example, $\mathcal{2} = (2, OR, 0)$ with $2 = \{0,1\}$, $*_{\mathcal{2}} = OR$, $1_{\mathcal{2}} = 0$ has no expansion to a group.

3'. Suppose (Σ, Π) and $(\Sigma \overset{+}{\cup} \Delta, \Pi \overset{+}{\cup} \Pi_\Delta)$ are order-sorted signatures with same poset $(S, \leq)$ of sorts and Δ and Π_Δ declare new function and predicate symbols not appearing in Σ, resp. Π. Assume, further, for simplicity, that each function symbol f in Δ has a top typing $f: s_1 \ldots s_n \to s$, so that for any other typing $f: s'_1 \ldots s'_n \to s'$ we have $s_i \geq s'_i$, $s \geq s'$, and each predicate symbol in Π_Δ has likewise a top typing $p: s_1 \ldots s_n$ so that any other typing $p: s'_1 \ldots s'_n$ is such that $s'_i \leq s_i$, $1 \leq i \leq n$. Call T' in the inclusion $T' =$

$T = ((\Sigma, \Pi), \Gamma) \subseteq ((\Sigma \overset{+}{\cup} \Delta, \Pi \overset{+}{\cup} \Pi_\Delta), \Gamma \overset{+}{\cup} def(\Delta) \overset{+}{\cup} def(\Pi_\Delta))$

a definitional extension of T if:

(i) $def(\Delta)$ has an equation $f(x_1 : s_1, \ldots, x_n : s_n) = t_f(x_1 : s_1, \ldots, x_n : s_n)$ for each f in Δ, where $f: s_1 \ldots s_n \to s$ is the top typing, and for any typing $f: s'_1 \ldots s'_n \to s'$, $t_f(x_1 : s'_1, \ldots, x_n : s'_n) \in T_\Sigma(X)_{s'}$.

(ii) $def(\Pi_\Delta)$ has equivalences $p(x_1 : s_1, \ldots, x_n : s_n) \Leftrightarrow \varphi_p$ where $vars(\varphi_p) \subseteq \{x_1 : s_1, \ldots, x_n : s_n\}$ and for each $p \in \Pi_\Delta$; where $p: s_1 \ldots s_n$ is the top typing.

Then $T \subseteq T'$ is expansive. Indeed, for each $\mathcal{M} \in mod(T')$ there is a unique expansion $\mathcal{M}\uparrow^{T'}$, where for each $f: s_1 \ldots s_n \to s$ in Δ (resp. $p: s_1 \ldots s_n$ in Π_Δ) and $a \in [\{x_1^{:s_1}, \ldots, x_n^{:s_n}\} \to M]$ $f_{\mathcal{M}\uparrow T}(a(x_1), \ldots, a(x_n)) = t_f\, a$ and $(a_1(x_1), \ldots, a_n(x_n)) \in p_{\mathcal{M}\uparrow T} \Leftrightarrow \mathcal{M}, a \models \varphi$. Thus $T \subseteq T'$ is expansive.

4. For $(\Sigma', E') \supseteq (\Sigma, E)$ a <u>protecting</u> extension of equational theories, $\underline{Init}(\Sigma', E') \supseteq Init(\Sigma, E)$ is literally expansive.
In the <u>VarSat</u> paper we have systematically exploited such literally expansive extensions

$$(\Sigma, \{\mathcal{T}_{\Sigma/E \cup B}\}) \supseteq (\Omega, \{\mathcal{T}_{\Omega/E_\Omega \cup B_\Omega}\})$$

5. For $(\{X\}, \emptyset) \overset{J}{\subseteq} (\Sigma, \Gamma)$, where (Σ, Γ) are, for example, the theories of lists, compact lists, multisets, sets, and HF-sets parameterized by the sort X, the extensions

$$(\Sigma, \mathbb{F}_J(\underline{NeSet})) \supseteq (\{X\}, \underline{NeSet})$$

are literally expansive extensions, since for each non-empty set A we have the <u>parameter-protection property</u>:

$$\mathbb{F}_J(A)\big|_{\{X\}} \cong A$$

For example, $\underline{List}(\mathbb{N})|_{\{X\}} \cong \mathbb{N}$.

<u>Lemma</u>. If $T' \supseteq T$ is expansive or literally expansive, then T is a conservative extension of T.

<u>Proof</u>. In both cases we have $th_{(\Sigma,\Pi)}(mod(T')|_{sig(T)}) = th(T)$, but $th_{(\Sigma,\Pi)}(mod(T')|_{sig(T)}) = th(T') \cap FO(sig(T))$. $\square$

The converse of this lemma is not true: there are conservative extensions that are not expansive. Furthermore, there are literally expansive extensions that are not expansive, and expansive extensions that are not literally expansive.
Most of these points can be illustrated by the following exercise:

<u>Exercise</u>. Let $\mathbb{R}$ denote the <u>set</u> of real numbers (no operations). Let $\Sigma_{\mathbb{R}}$ be the signature with a single sort Elt, $\Sigma_{\mathbb{R},0} = \mathbb{R}$, (constants), and $\Sigma_{\mathbb{R},n} = \emptyset$, $n > 0$.
Consider the literally expansive extension

$$T' = (\Sigma_{\mathbb{R}}, \{(\mathbb{R}, 1_{\mathbb{R}})\}) \supset (\{Elt\}, \{\mathbb{R}\}) = T$$

where $1_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is the interpretation of each constant $a \in \Sigma_{\mathbb{R},0}$ <u>by itself</u>. Prove that the, obviously conservative, extension (by above Lemma)

$$(\Sigma_{\mathbb{R}}, th(T')) \supseteq (\{Elt\}, th(T))$$

is <u>not</u> expansive.

<u>Hint</u>. Use the upward and downward Löwenheim-Skolem theorems in their <u>sharp</u> formulation in, e.g., Wilfrid Hodges, "A Shorter Model Theory", Cambridge UP, (as Corollary 3.1.4, and Corollary 5.1.4 there).

Signature Maps and Theory Interpretations

In SMT solving *extensibility* is a crucial goal: we want the methods, tools, and algorithms to apply *as widely as possible* to as many applications as possible. For this:

1. theory-generic satisfiability decision procedures such as those in the *VarSat* paper, and
2. ways of *shifting our ground*, so that a satisfiability procedure for a theory T can be *used* to decide the satisfiability of a *different* theory T'

are both crucial. As an example of (2), in the *VarSat* paper we have already encountered the method of *descent maps*. But, more generally, how can we "shift our ground"? One possible answer is: by *theory interpretations* $H : T \to T'$, which can often *translate* (as opposed to just *include*, as in the case of a theory inclusion $T \subseteq T'$) the *language* of T into that of T'.

But what is a theory interpretation? There are increasingly more general notions of "theory interpretation", so a single answer cannot cover them all. For Maude users a good, quite general notion is the notion of a *view* H from a theory T to a theory T',

which indeed defines a theory interpretation $H: T \to T'$.
But underlying the notion of theory interpretation we have the simpler, more basic notion of a *signature map*. So, not to begin the house by the roof, I will first explain signature maps. I will focus on a notion somewhat less general of signature map than that used in Maude views, and will leave the extension to that more general notion as a non-trivial exercise.

I first need some detailed notation. To simplify life (and notation) I will replace the notation (Σ, Π) for an order-sorted first-order theory with function symbols Σ and predicate symbols Π by the simpler notation

$$\Sigma = ((S, \leq), F, P)$$

where:

1. $(S, \leq)$ is the poset of sorts

2. F is a mapping

$$F : fun(\Sigma) \to \mathcal{P}(S^* \times S)$$

from a *set of function symbols* (here denoted $fun(\Sigma)$) to a *set of ranks* for such symbols such that for each $f \in fun(\Sigma)$,

2.1 $\quad F(f) \neq \emptyset$

2.1 $\quad$ (subsort *polymorphism*) if $(w, s), (w', s') \in F(f)$

then, (i) $|w| = |w'|$, and (ii) $w \equiv_{\leq} w'$ and $s \equiv_{\leq} s'$

Ranks such as, say, $(s_1 \dots s_n, s) \in F(f)$ are usually denoted: $f : s_1 \dots s_n \to s$. Subsort poly-morphism then means that if $f : s_1 \dots s_n \to s$ and $f : s'_1 \dots s'_n \to s'$ are typings for f, then $[s_i] = [s'_i]$, and $[s] = [s']$, where $[s]$ denotes the connected component of s in the poset $(S, \leq)$, that is, the equivalence class of s under the equivalence relation $\equiv_{\leq} = (\leq \cup \geq)^+$.

For example, we may have Nat < Int < Rat, and $F(+) = \{(\text{Nat Nat}, \text{Nat}), (\text{Int Int}, \text{Int}), (\text{Rat Rat}, \text{Rat})\}$.

This notion is slightly restrictive, but not much. It excludes:

(a) ad-hoc polymorphism, as in $\_+\_ : \text{Nat Nat} \to \text{Nat}$ for addition, and $\_+\_ : \text{Bool Bool} \to \text{Bool}$ for exclusive or, but such ad-hoc polymorphism can always be renamed away.

(b) using the same symbol f with different numbers of arguments, as $f : s\, s' \to s''$, and $f : s\, s'\, s'' \to s''$, but we can always distinguish them either by renaming, or by marking argument positions, as symbols: $f(\_,\_)$ and $f(\_,\_,\_)$.

3. P is a mapping

$$P : pred(\Sigma) \rightarrow \mathcal{P}(S^*)$$

from a <u>set of predicate symbols</u> (here denoted $pred(\Sigma)$) to a <u>set of arities</u> for such symbols such that for each $p \in pred(\Sigma)$

3.1 $P(p) \neq \emptyset$

3.2 (<u>subsort polymorphism</u>) if $w, w' \in P(p)$, then $|w| = |w'|$ and $w \equiv_{\leq} w'$. If $w \in P(p)$ we write $p : w$.

Recall from <u>VarSat</u> the notion of a Σ- model $\mathcal{M}$. It is: (a) an <u>order-sorted algebra</u> $(M, \_^F_{\mathcal{M}})$ for the signature $((S, \leq), F)$, and (b) together with for each $p : w$ a subset $p^w_{\mathcal{M}} \subseteq M^w$, where $M^\varepsilon = 1 = \{0\}$, and $M^{s_1 \ldots s_n} = M_{s_1} \times \ldots \times M_{s_n}$, such that if $p : w$ and $p : w'$, and $(a_1, \ldots, a_n) \in M^w \cap M^{w'}$, then

$$(a_1, \ldots, a_n) \in p^w_{\mathcal{M}} \iff (a_1, \ldots, a_n) \in p^{w'}_{\mathcal{M}}.$$

So, altogether $\mathcal{M} = (M, \_^F_{\mathcal{M}}, \_^P_{\mathcal{M}})$, where

$$(f : s_1, \ldots, s_n \rightarrow s)_{\mathcal{M}} = f_{\mathcal{M}} : M_{s_1} \times \ldots \times M_{s_n} \rightarrow M_s, \text{ and}$$

$$(p : w)_{\mathcal{M}} = p^w_{\mathcal{M}} \subseteq M^w$$

(given the subsort polymorphism we can omit the w in $p^w_{\mathcal{M}}$).

<u>Definition</u>. Given order-sorted signatures $\Sigma = ((S, \leq), F, P)$ and $\Sigma' = ((S', \leq'), F', P')$, a <u>signature map</u> $H : \Sigma \to \Sigma'$ is specified by:

1. a monotonic function $H : (S, \leq) \to (S', \leq')$
2. a function $H : fun(\Sigma) \to fun(\Sigma')$ such that:
   (i) if $f : s_1 \ldots s_n \to s$ in Σ, and $(w', s') \in F'(H(f))$ then $H(s_1) \ldots H(s_n) \equiv_{\leq'} w'$ and $H(s) \equiv_{\leq'} s'$.
   (ii) For each $f : s_1 \ldots s_n \to s$ in Σ,
   $$H(f)(x_1 : H(s_1), \ldots, x_n : H(s_n)) \in T_{\Sigma'}(X)_{H(s)}$$

Note that condition (ii) is <u>more flexible</u> than just requiring that $H(f) : H(s_1) \ldots H(s_n) \to H(s)$ is rank for $H(f)$ in F'!

3. a function $H : pred(\Sigma) \to pred(\Sigma')$ such that:
   (i) if $p : w$ in Σ and $H(p) : w'$ in Σ', then $|w| = |w'|$ and $H(w) \equiv_{\leq} w'$, where $H(\varepsilon) = H(\varepsilon)$, and $H(s w) = H(s) H(w)$.
   (ii) for each $p : w$ in Σ there is $H(p) : w'$ in Σ' such that $H(w) \leq' w'$.

Again, this is <u>more flexible</u> than requiring $H(w) \in P'(H(p))$.

<u>Exercise</u>. Prove that <u>signature maps compose</u>, i.e., if we have $\Sigma \xrightarrow{H} \Sigma' \xrightarrow{G} \Sigma''$, then $H;G : \Sigma \to \Sigma''$ is also a signature map.

Therefore, order-sorted FO-signatures and signature maps define a category <u>OS-FOSign</u>.

The key model-theoretic property of signature maps is that the mapping $\underline{\mathrm{Mod}} : \Sigma \mapsto \underline{\mathrm{Mod}}(\Sigma)$ extends to a <u>contravariant functor</u> $\underline{\mathrm{Mod}} : \underline{\text{OS-FOSign}}^{op} \to \underline{\mathrm{Cat}}$, where we use the notation:

$$\underline{\mathrm{Mod}}(\Sigma \xrightarrow{H} \Sigma') = \underline{\mathrm{Mod}}(\Sigma') \xrightarrow{-|_H} \underline{\mathrm{Mod}}(\Sigma)$$

and call $\mathrm{Mod}(H) = -|_H$ the H- <u>reduct functor</u>.

In the case when H is a theory inclusion $\Sigma \subseteq \Sigma'$, $-|_H$ becomes the usual reduct functor $-|_\Sigma$.

Let us define $-|_H : \mathrm{Mod}(\Sigma') \longrightarrow \mathrm{Mod}(\Sigma)$ in detail for the objects. Given $\mathcal{M}' \in \mathrm{Mod}(\Sigma')$, say $\mathcal{M}' = (M', \_^{F'}_{\mathcal{M}}, \_^{P'}_{\mathcal{M}})$, we define $\mathcal{M}'|_H$ as follows:

1. using the monotonic function $H : (S, \leq) \to (S', \leq')$ we define $\mathcal{M}'|_H = (M'|_H, \_^F_{\mathcal{M}'|_H}, \_^P_{\mathcal{M}'|_H})$ with

S-sorted family of sets $M'|_H = \{M'|_{H,s}\}_{s \in S}$,

where $M'|_{H,s} = M'_{H(s)}$. Note that, by monotonicity of H, if $s \leq s'$, then we have $M'|_{H,s} \subseteq M'|_{H,s'}$.

2. $\_^F_{\mathcal{M}'|_H}$ maps each $f : s_1 \dots s_n \to s$ in Σ to the function $M'_{H(s_1)} \times \dots \times M'_{H(s_n)} \ni (a_1, \dots, a_n) \mapsto H(f)_{\mathcal{M}'}(a_1, \dots, a_n) \in M'_{H(s)}$

which is well-defined since $H(f)(x_1 : H(s_1), \dots, x_n : H(s_n)) \in T_{\Sigma'}(X)_{H(s)}$

3. $\_^P_{\mathcal{M}'|_H}$ maps each $p : w$ in Σ to the subset

$$M'^{H(w)} \cap H(p)^{w'}_{\mathcal{M}'}$$

which is well-defined by choosing an $H(p) : w'$ in Σ' such that $H(w) \leq' w'$, and does not depend of the choice of w' because of subsort polymorphism.

In what sense does a signature map $H : \Sigma \to \Sigma'$ induce a <u>translation</u> $H : FO(\Sigma) \to FO(\Sigma')$ of the language of Σ into that of Σ'?

In the following sense. Let X be an infinite set of <u>names</u>. Define $X_S = \{X_s\}_{s \in S}$, where $X_s = X \times \{s\}$, and we write the pair (x, s) as the <u>typed variable</u> $x : s$.

Now note that $T_{\Sigma'}(X_{S'})|_H$ is an F-algebra and we have an S-sorted function $H : X_S \to X_{S'}|_H$,

$$H = \{ H_s : X_s \ni x : s \longmapsto x : H(s) \in X_{H(s)} = X_{S'}|_{H,s} \}_{s \in S}.$$

Therefore there is a unique F-homomorphism

$$H : T_\Sigma(X_S) \longrightarrow T_{\Sigma'}(X_{S'})|_H \quad \text{extending } H.$$

This also extends to <u>predicate atoms</u> in the obvious sense:

$$H : p(t_1, .., t_n) \longmapsto H(p)(H(t_1), .., H(t_n)).$$

$H : FO(\Sigma) \to FO(\Sigma')$ is then defined in the obvious recursive way:

(i) atoms $t = t'$ and $p(t_1 .. t_n)$ are mapped to $H(t) = H(t')$ and $H(p(t_1, .., t_n))$

(ii) $H(\neg \varphi) = \neg H(\varphi)$, $H(\varphi \boxtimes \psi) = H(\varphi) \boxtimes H(\psi)$
$\boxtimes \in \{\wedge, \vee, \Rightarrow\}$

(iii) $H(\forall x : s\ \varphi) = \forall x : H(s)\ H(\varphi)$, and
$H(\exists x : s\ \varphi) = \exists x : H(s)\ H(\varphi)$.

<u>Theorem</u>. Let $H : \Sigma \to \Sigma'$ be an OS-FO-signature map. Then for each $\mathcal{M}' \in \underline{\mathrm{Mod}}(\Sigma')$, $\varphi \in FO(\Sigma)$, and $a \in [fvars(\varphi) \to M'|_H]$ we have

$$\mathcal{M}'|_H , a \models \varphi \iff \mathcal{M}', \tilde{a} \models H(\varphi)$$

where $\tilde{a} \in [H(fvars(\varphi)) \to M']$

maps: $x : H(s) \longmapsto a(x : s) \in M'_{H(s)}$.

<u>Proof</u> (Sketch) [Filling in the details is a highly recomended exercise!].
The crucial step is to prove this for atoms. That is, for $t = t'$ a Σ-equation with $Y = vars(t = t')$ we need to show (1) $\mathcal{M}'|_H , a \models t = t' \iff \mathcal{M}', \tilde{a} \models H(t) = H(t')$
and for $p(t_1, \ldots, t_n)$ a predicate atom with $Y = vars(p(t_1 .. t_n))$ need to show (2) $\mathcal{M}'|_H, a \models p(t_1 .. t_n) \iff \mathcal{M}', \tilde{a} \models H(p)(H(t_1), .., H(t_n))$.
Let us show (1) and leave (2) as an exercise.
The key observations to prove (1) are the following:

(a) for each $a \in [Y \to M'|_H]$, $\tilde{a} \in [H(Y) \to M']$ extends uniquely to an F'-homomorphism

$$-\tilde{a} : \mathcal{T}_{\Sigma'}(H(Y)) \longrightarrow \mathcal{M}'$$

(b) since $\tilde{a}(x : H(s)) = a(x : s)$, a itself factors as:

$$\begin{tikzcd} Y \arrow[r, hook, "\eta_Y"] \arrow[ddr, "a"'] & T_\Sigma(Y) \arrow[d, "H"] \\ & T_\Sigma(H(Y))|_H \arrow[d, "\_\tilde{a}|_H"] \\ & M|_H \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}_\Sigma(H(Y)) \arrow[d, "\_\tilde{a}"] \\ \mathcal{M} \end{tikzcd}$$

which by the Freeness Corollary (Cf. CS 476 Lecture notes) gives us the identity of F-homomorphisms:

$$\begin{tikzcd} \mathcal{T}_\Sigma(Y) \arrow[rr, bend left, "\_a"] \arrow[r, "H"'] & \mathcal{T}_\Sigma(H(Y))|_H \arrow[r, "\_\tilde{a}|_H"'] & \mathcal{M}|_H \end{tikzcd} \quad =$$

Therefore, for any F-equation $t = t'$ we have

$ta = ta' \Leftrightarrow H(t)\tilde{a} = H(t')\tilde{a}$, and

therefore

$$\mathcal{M}'|_H, a \models t = t' \Leftrightarrow \mathcal{M}', \tilde{a} \models H(t) = H(t')$$

as desired. $\square$

Remark. The above theorem expresses the "invariance of truth under translation" and is crucial for many applications, including applications to SMT solving.