CS 576 lecture of 3/14/17
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1. Syntactic and Semantic Theories and their Inclusions

Recall from early lectures that given an order-sorted first-order signature (Σ, Π) (equivalently an algebraic signature $\Sigma \cup \Pi$) we have a Galois connection with anti-monotonic functions:

$$\mathcal{P}(FO(\Sigma)) \underset{mod_\Sigma}{\overset{th_\Sigma}{\rightleftarrows}} \mathcal{P}(\underline{Mod}_{(\Sigma,\Pi)})$$

where for $\Gamma \subseteq FO(\Sigma)$, $mod_\Sigma(\Gamma) = \underline{Mod}_{((\Sigma,\Pi),\Gamma)} = \{\mathbb{M} \in \underline{Mod}_{(\Sigma,\Pi)} \mid \forall \varphi \in \Gamma \ \mathbb{M} \models \varphi\}$

and for $\mathbb{A} \subseteq \mathcal{P}(\underline{Mod}_{(\Sigma,\Pi)})$ $th_\Sigma(\mathbb{A}) = \{\varphi \in FO(\Sigma) \mid \forall \mathbb{M} \in \mathbb{A} \ \ \mathbb{M} \models \varphi\}$

Recall also the notations:

$$\Gamma^{\bullet} = th_\Sigma(mod_\Sigma(\Gamma))$$

$$\mathbb{A}^{\#} = mod_\Sigma(th_\Sigma(\mathbb{A}))$$

and the fact that, since this is a Galois connection, we have for each $\Gamma \subseteq FO(\Sigma)$, $\mathbb{A} \subseteq \underline{Mod}_{(\Sigma,\Pi)}$

(1) $\Gamma \subseteq th(\mathbb{A}) \iff mod(\Gamma) \supseteq \mathbb{A}$

(2) $\mathbb{A}^{\#\#} = \mathbb{A}^{\#}$

(3) $\Gamma^{\bullet\bullet} = \Gamma^{\bullet}$

The standard notion of theory (sometimes called theory presentation) is a pair (Σ, Γ), with $\Gamma \subseteq FO(\Sigma)$. When (Σ, Γ) is called a theory presentation, $(\Sigma, \Gamma^{\bullet})$ is then called the theory *presented* by $(\overset{(\Sigma,\Pi)}{\Sigma},\Gamma)$, making the point that $\Gamma^\bullet$ can be presented in many ways. /

A syntactically oriented notion of theory as a pair $(\overset{(\Sigma,\Pi)}{\Sigma},\Gamma)$ has *obvious advantages*, particularly when Γ is either finite or a *recursive* set of sentences, since in this case $(\overset{(\Sigma,\Pi)}{\Sigma},\Gamma)$ is called an *axiomatizable theory* (meaning effectively axiomatizable), so that we have an *effective* way of *defining syntactically* the semantic class of models $\underline{Mod}(\overset{(\Sigma,\Pi)}{\Sigma},\Gamma)$.

But this notion also has *obvious disadvantages*, associated with the intrinsic limitations of first-order logic to pin down a desired class of models by the exclusive means of logical axioms. Because of the upwards and downwards Löwenheim-Skolem theorems, except for theories having only finite models we *cannot pin down* the *cardinality* of the models of a theory. Therefore, a fortiori, we cannot hope to use (Σ,Γ) to pin down the class $\{M\}_{\cong}$ of models isomorphic to a given one, say, for example, $\{(\mathbb{N},+,\geq,>,0,1)\}_{\cong}$, the *desired* or *standard* model of Presburger arithmetic. In fact, the best we can hope for is a theory $(\overset{(\Sigma,\Pi)}{\Sigma},\Gamma)$ that is both *consistent*, i.e., $mod(\Gamma) \neq \emptyset$, and *complete* i.e., for each (Σ,Π)-sentence φ (i.e, rans$/\varphi) = \emptyset$) either $\Gamma \vdash \varphi$ or $\Gamma \nvdash \varphi$. In that case we can pin down the models of Γ in the best way available within the limits of first-order logic because we have:

<u>Lemma</u> If (Σ,Π) is complete (and consistent), then any two models $\mathcal{M}, \mathcal{M}' \in$ <u>Mod</u> are <u>elementarily equivalent</u>, denoted $\mathcal{M} \equiv \mathcal{M}'$, where $\mathcal{M} \equiv \mathcal{M}'$ is defined by the equivalence:

$$\mathcal{M} \equiv \mathcal{M}' \Leftrightarrow \forall \varphi \in sen(FO(\Sigma))\ \mathcal{M} \models \varphi \Leftrightarrow \mathcal{M}' \models \varphi .$$

<u>Proof</u>. Exercise. Hints: use the above Galois connection to observe that $\Gamma^{\circ} \subseteq th(\mathcal{M}), th(\mathcal{M}')$; then use notion of completeness to conclude $\Gamma^{\circ} = th(\mathcal{M}) = th(\mathcal{M}')$, and therefore $\mathcal{M} \equiv \mathcal{M}'$.

The limitations of a syntactic notion of theory as a pair $((\Sigma,\Pi),\Gamma)$ are felt particularly strongly in the case of many <u>decision procedures for satisfiability</u>. In some cases the problem does <u>not</u> arise, as for theories like $((\Sigma,\Pi),\emptyset)$, or $((\Sigma,\Pi),\Delta_{AC})$ with $\Delta \subseteq \Sigma$ binary function symbols, where we are interested in <u>all the models</u>. They are however strongly felt when we want to obtain decision procedures for various <u>data structures</u> used in Computer Science or, more generally, any <u>models</u> used in mathematics. In such cases, a syntactic approach to theories as the one taken, for example, in the Bradley and Manna book seems unreal, since:

(i) it fails to explain what is the intended model
(ii) knowledge of such intended model either cannot be used or has to be smuggled with some hand-waving, and

(iii) the syntactic approach can easily <u>hide from view</u> the unsatisfactory fact that the syntactic theory $((\Sigma,\Pi),\Gamma)$ may after all <u>fail to capture</u> the intuitively intended model or models for which it was deployed in the first place. I will illustrate this fact with examples such as arrays and tables.

For all these reasons, at least from the paper by Ganzinger, "Shostak Light", LNCS (LNAI) 2392, 332-346, 2002, the notion of what I call a <u>semantic theory</u>, i.e., a pair $((\Sigma,\Pi), \mathcal{A})$ where $\mathcal{A} \subseteq \underline{Mod}_{(\Sigma,\Pi)}$ has become increasingly accepted. The point is that it is more general and flexible than what I call a <u>syntactic theory</u> $((\Sigma,\Pi),\Gamma)$ since, after all $((\Sigma,\Pi)\Gamma)$ can be expressed as $((\Sigma,\Pi), \underline{Mod}_{((\Sigma,\Pi),\Gamma)})$ and semantic theories in particular allows us to completely pin down an intended model $\mathcal{M}$ as $((\Sigma,\Pi), \{\mathcal{M}\})$. For example, $((\Sigma,\Pi),\{(\mathbb{N},+,0,1,\geq)\})$ as the intended model of Presburger arithmetic, and to <u>reason semantically</u> on such intended model $\mathcal{M}$ to obtain, for example, a decision procedure for satisfiability in $\mathcal{M}$.

Of course we can have our cake an eat it too, i.e., we can have the <u>combined advantages</u> of syntactic and semantic theories, and <u>move back and forth</u> between them using the Galois connection between sets of sentences and classes of models.

How about the notion of a <u>theory inclusion</u>? A general enough notion at the syntactic level is that $((\Sigma,\Pi),\Gamma) \subseteq ((\Sigma',\Pi'),\Gamma')$ iff $(\Sigma,\Pi) \subseteq (\Sigma',\Pi')$, i.e., poset of sorts, operations, and predicates of (Σ,Π) is a subposet, resp. subsets, of those of (Σ',Π'), <u>and</u> $\Gamma \subseteq \Gamma'^{\bullet}$. What is the corresponding notion for <u>semantic</u> theories?

<u>Definition</u>. A semantic theory $((\Sigma,\Pi), \mathbb{A})$ is a <u>subtheory</u> of, or is <u>included</u> into, another $((\Sigma',\Pi'), \mathbb{B})$ iff:

1. $(\Sigma,\Pi) \subseteq (\Sigma',\Pi')$
2. $\mathbb{B}|_{(\Sigma,\Pi)} \subseteq \mathbb{A}^{\#}$, where $\mathbb{B}|_{(\Sigma,\Pi)} = \{ M|_{(\Sigma,\Pi)} \mid M \in \mathbb{B} \}$.

Note that this agrees with the syntactic notion of theory inclusion $((\Sigma,\Pi),\Gamma) \subseteq ((\Sigma',\Pi'),\Gamma')$, since (exercise!) it is easy to prove using the Galois connections at the (Σ,Π) and (Σ',Π') levels that we have, $Mod_{((\Sigma',\Pi'),\Gamma')}|_{(\Sigma,\Pi)} \subseteq \underline{Mod}_{((\Sigma,\Pi),\Gamma)} = \underline{Mod}_{((\Sigma,\Pi),\Gamma^{\bullet})}$.

But to be able to <u>move back and forth</u> between syntactic and semantic theories we can do better than that:

<u>Definition</u>. A <u>first-order theory</u> T is either a syntactic theory $T=((\Sigma,\Pi),\Gamma)$, or a semantic theory $T=((\Sigma,\Pi),\mathbb{A})$.

A <u>theory inclusion</u> $T \subseteq T'$ is defined as above when T and T' are both syntactic (resp. semantic) theories, and otherwise is defined as follows:

(i) $((\Sigma,\Pi),\Gamma) \subseteq ((\Sigma',\Pi'),\mathbb{B})$ iff

- $(\Sigma,\Pi) \subseteq (\Sigma',\Pi')$
- $\Gamma \subseteq th_{\Sigma'}(\mathbb{B})$

(ii) $((\Sigma,\Pi),\mathbb{A}) \subseteq ((\Sigma',\Pi'),\Gamma')$ iff

- $(\Sigma,\Pi) \subseteq (\Sigma',\Pi')$
- $mod_{(\Sigma',\Pi')}(\Gamma')\big|_{(\Sigma,\Pi)} \subseteq \mathbb{A}^{\#}$

Under this definition it is easy to prove (exercise!)

<u>Lemma</u>. If $T \subseteq T'$ and $T' \subseteq T''$, then $T \subseteq T''$. $\square$

<u>Remarks</u>.

1. This shows that theories and theory inclusions are a very simple kind of <u>category</u>, namely a preorder.
2. This can be generalized to an <u>actual category of theories</u> by considering not just signature inclusions $(\Sigma,\Pi) \subseteq (\Sigma',\Pi')$, but in fact <u>signature morphisms</u> $H : (\Sigma,\Pi) \to (\Sigma',\Pi')$. (in Maude terminology <u>"views"</u> $H : (\Sigma,\Pi) \to (\Sigma',\Pi')$). For example, syntactically $H : ((\Sigma,\Pi),\Gamma) \to ((\Sigma',\Pi'),\Gamma')$ holds iff $H : (\Sigma,\Pi) \to (\Sigma',\Pi')$ is a signature morphism <u>and</u> $H(\Gamma) \subseteq \Gamma'^{\bullet}$, and semantically $H : ((\Sigma,\Pi),\mathbb{A}) \to ((\Sigma',\Pi'),\mathbb{B})$ holds iff H is a signature morphism <u>and</u> $\mathbb{B}|_H \subseteq \mathbb{A}^{\#}$. The mixed case is then defined as in (i)-(ii) above changing $|_\Sigma$ by $|_H$.

2. Theory Transformations

How can we use some theories to define other theories of interest in a simple and modular way? For syntactic theories, theory transformations are used. For example, for (Σ, E) an equational theory the Knuth–Bendix completion process can be viewed (when it succeeds) as a theory transformation

$$(\Sigma, E) \mapsto (\Sigma, KB_{<}(E))$$

where $<$ is a chosen ordering on Σ-terms, and $KB_{<}(E)$ is a set of convergent and $>$-terminating Σ-equations such that $E^{\circ} = KB_{<}(E)^{\circ}$.

What can be done for semantic theories to increase the ability to define new theories from prior semantic or syntactic ones? We can use transformations on classes of models, which induce associated transformations in the corresponding theories.

Here are some examples:

1. Given $(\Sigma, \Pi) \subseteq (\Sigma', \Pi')$,

$$-|_{(\Sigma,\Pi)} : \mathcal{P}(\underline{Mod}_{(\Sigma',\Pi')}) \ni \mathbb{B} \mapsto \mathbb{B}|_{(\Sigma,\Pi)} \in \mathcal{P}(\underline{Mod}_{(\Sigma,\Pi)})$$

is a transformation on classes of models inducing a theory transformation $((\Sigma', \Pi'), \mathbb{B}) \longmapsto ((\Sigma, \Pi), \mathbb{B}|_{(\Sigma,\Pi)})$

2. Given (Σ, Π) with set of sorts S and a cardinal α,

$$-_{<\alpha} : \mathcal{P}(\underline{Mod}_{(\Sigma,\Pi)}) \ni \mathbb{A} \mapsto \mathbb{A}_{<\alpha} \in \mathcal{P}(\underline{Mod}_{(\Sigma,\Pi)})$$

and $\_{\leq\alpha} : \mathcal{P}(\underline{\mathrm{Mod}}_{(\Sigma,\Pi)}) \ni \mathbb{A} \mapsto \mathbb{A}_{\leq\alpha} \in \mathcal{P}(\underline{\mathrm{Mod}}_{(\Sigma,\Pi)})$

are transformations on classes of models defined by $\mathbb{A}_{<\alpha} = \{\mathcal{M} \in \mathbb{A} \mid |\mathcal{M}|_{<\alpha}\}$, $\mathbb{A}_{<\alpha} = \{\mathcal{M} \in \mathbb{A} \mid |\mathcal{M}| \leq \alpha\}$, where if $\mathcal{M} = (M, \_^{\mathcal{M}})$, then $|\mathcal{M}|_{<\alpha}$ (resp. $|\mathcal{M}|_{\leq\alpha}$) iff $|\bigcup_{s\in S} M_s| < \alpha$ (resp. $|\bigcup_{s\in S} M_s| \leq \alpha$).

This induces theory transformations

$$((\Sigma,\Pi), \mathbb{A}) \mapsto ((\Sigma,\Pi), \mathbb{A}_{<\alpha})$$

and $$((\Sigma,\Pi), \mathbb{A}) \mapsto ((\Sigma,\Pi), \mathbb{A}_{\leq\alpha}).$$

3. Given (Σ,Π),

$$f.g. : \mathcal{P}(\underline{\mathrm{Mod}}_{(\Sigma,\Pi)}) \ni \mathbb{A} \mapsto f.g.(\mathbb{A}) \in \mathcal{P}(\underline{\mathrm{Mod}}_{(\Sigma,\Pi)})$$

is a transformation on classes of models extracting the subclass $f.g.(\mathbb{A}) \subseteq \mathbb{A}$ of finitely generated models, where $\mathcal{M} = (M, \_^{\mathcal{M}})$ is finitely generated iff there is a finite set of variables Y and an assignment $a \in [Y \to M]$ such that the unique Σ-homomorphism

$$\_a : T_\Sigma(Y) \longrightarrow \mathcal{M}|_\Sigma$$

is surjective (i.e., $\mathcal{M}$ is generated by the finite set of elements $a(Y)$).

Note that if the set S of sorts is countable,

$$\mathbb{A}_{<\omega} \subseteq f.g.(\mathbb{A}) \subseteq \mathbb{A}_{\leq\omega}$$

and the inclusions can be strict.

This transformation induces: $((\Sigma,\Pi), \mathbb{A}) \mapsto ((\Sigma,\Pi), f.g.(\mathbb{A}))$.

4. Given (Σ, Π),

$$\_{\cong} : \mathcal{P}(\underline{\mathbf{Mod}}_{(\Sigma,\Pi)}) \ni \mathbb{A} \longmapsto \mathbb{A}_{\cong} \in \mathcal{P}(\underline{\mathbf{Mod}}_{(\Sigma,\Pi)})$$

is the <u>closure under isomorphism</u> transformation, so that

$$\mathbb{A}_{\cong} = \{ \mathcal{M}' \in \underline{\mathbf{Mod}}_{(\Sigma,\Pi)} \mid \exists \mathcal{M} \in \mathbb{A} \text{ s.t. } \mathcal{M} \cong \mathcal{M}' \}.$$

This induces a theory transformation $((\Sigma,\Pi), \mathbb{A}) \mapsto ((\Sigma,\Pi), \mathbb{A}_{\cong})$.

5. Given $((\Sigma,\Pi), \Gamma)$ a <u>Horn theory with equality</u>, i.e., Γ is a set of formulas of the form $A_1 \wedge \ldots \wedge A_n \Rightarrow A$, with the $A, A_1, \ldots, A_n$ atoms, including possibly equality atoms, then

$$\underline{\mathit{Init}}((\Sigma,\Pi), \Gamma) = \mathcal{T}_{(\Sigma,\Pi)/\Gamma}$$

is the <u>initial model</u> in the category $\underline{\mathbf{Mod}}_{((\Sigma,\Pi),\Gamma)}$.

For example:

(i) when Π consists only of unary predicates, $\mathcal{T}_{(\Sigma,\Pi)/\Gamma}$ is the initial model of a theory in <u>membership equational logic</u>, so that, in Maude, <u>fmod</u> $((\Sigma,\Pi), \Gamma)$ endfm is a functional module whose <u>semantics</u> is precisely $\underline{\mathit{Init}}((\Sigma,\Pi), \Gamma)$.

(ii) The special case when $\Pi = \emptyset$ yields the case of <u>order-sorted conditional equational theories</u>, so that, in Maude, fmod(Σ, E) endfm is a functional module whose semantics is precisely $\underline{\mathit{Init}}((\Sigma,\emptyset), E)$.

This induces a theory transformation

$$((\Sigma,\Pi),\Gamma) \mapsto ((\Sigma,\Pi),\{\mathcal{T}_{(\Sigma,\Pi)/\Gamma}\})$$

in this case from a syntactic theory to a semantic one.

Note that this transformation is very useful and concise, since it implicitly has the power of adding <u>second-order axioms</u> to $((\Sigma,\Pi),\Gamma)$ including all <u>inductive axioms</u>.

For example, for $\Sigma = \{0, 1, +\}$ and $\Gamma = \{AC_{+}, x+0=x,$ $x+y \geq y,\ x+y+1 > y\}$, where $\Pi = \{\geq, >\}$, $((\Sigma,\Pi),\ Init((\Sigma,\Pi),\Gamma))$ is the theory of <u>Peano arithmetic</u>, since we have theory inclusions $((\Sigma,\Pi),\{\mathcal{T}_{(\Sigma,\Pi)/\Gamma}\}) \subseteq ((\Sigma,\Pi), th(\mathcal{T}_{(\Sigma,\Pi)/\Gamma}))$ and $((\Sigma,\Pi), th(\mathcal{T}_{(\Sigma,\Pi)/\Gamma})) \subseteq ((\Sigma,\Pi),\{\mathcal{T}_{(\Sigma,\Pi)/\Gamma}\})$ making both theories isomorphic.

A more interesting case is $\Sigma = \{0, 1, +, *\}$, and $E = \{AC_{+}, AC_{*}, x+0=x, x*1=x, x*(y+z)=(x*y)+(x*z)\}$ where $\mathcal{T}_{\Sigma/E} = (\mathbb{N}, +_{\mathbb{N}}, *_{\mathbb{N}}, 0_{\mathbb{N}}, 1_{\mathbb{N}})$, so that $(\Sigma,\{\mathcal{T}_{\Sigma/E}\})$ is the theory of the natural numbers with addition and multiplication. The implicit inductive axioms hidden in $(\Sigma,\{\mathcal{T}_{\Sigma/E}\})$ are evidenced by the fact that we have <u>strict</u> inclusions

$$(\Sigma,E) \subsetneq PA \subsetneq (\Sigma,\{\mathcal{T}_{\Sigma/E}\})$$

where PA is Peano arithmetic, and by Gödel's incompleteness theorem $(\Sigma, \{J_{\Sigma/E}\})$ is <u>not</u> axiomatizable, so that the inclusion $PA \subseteq (\Sigma, \{J_{\Sigma/E}\})$ must be strict.

In short, the <u>Init</u> transformation gives us the power and convenience of <u>pinpointing the model of interest</u> by <u>purely syntactic methods</u> (namely specifying $((\Sigma,\Pi),\Gamma)$.

6. Given a theory inclusion $J : ((\Sigma,\Pi),\Gamma) \subseteq ((\Sigma',\Pi'),\Gamma')$ between Horn theories with equality,

$$\mathbb{F}_J : \mathcal{P}(\underline{Mod}_{((\Sigma,\Pi),\Gamma)}) \ni \mathbb{A} \mapsto \mathbb{F}_J(\mathbb{A}) \in \mathcal{P}(\underline{Mod}_{((\Sigma',\Pi'),\Gamma')})$$

defined by $\mathbb{F}_J(\mathbb{A}) = \{\mathbb{F}_J(\mathcal{M}) \mid \mathcal{M} \in \mathbb{A}\}$, where

$$\mathbb{F}_J : \underline{Mod}_{((\Sigma,\Pi),\Gamma)} \longrightarrow \underline{Mod}_{((\Sigma',\Pi'),\Gamma')}$$ is the <u>left</u>

<u>adjoint</u> to the functor $-|_{(\Sigma,\Pi)} : \underline{Mod}_{((\Sigma',\Pi'),\Gamma')} \longrightarrow \underline{Mod}_{((\Sigma,\Pi),\Gamma)}$.

The detailed construction of $\mathbb{F}_J$ is given in J. A. Goguen and J. Meseguer, "Models and Equality for Logic Programming", Springer LNCS 250, 1-22, 1987.

This construction is enormously useful, since it gives a declarative programming language like Maude the power of <u>parametric polymorphism</u>, and the ability to define <u>parameterized data types</u>.

This then defines a theory $((\Sigma', \Pi'), \mathbb{F}_J(\underline{Mod}_{((\Sigma,\Pi),\Gamma)}))$.
In Maude such a theory, associated to $J : T \to T'$ is defined by the syntax:

$\underline{fth}$ T $\underline{endfth}$

$\underline{fmod}$ $T'\{X : T\}$ $\underline{endfm}$

whose $\underline{semantics}$ is precisely $((\Sigma', \Pi'), \mathbb{F}_J(\underline{Mod}_{((\Sigma,\Pi),\Gamma)}))$ when $T = ((\Sigma,\Pi), \Gamma)$ and $T' = ((\Sigma', \Pi'), \Gamma')$.

We have seen already many examples of this construction such as those for parameterized lists, compact lists, multisets, sets, and hereditarily finite sets in the Variant Satisfiability paper. Additional examples involving parameterized data types for finite relations, finite partial functions, arrays, and tables and illustrating the expressive power of Maude's membership equational logic will be given soon.

For specific examples, the abstract $\mathbb{F}_J$ notation can be made more readable an intuitive by indicating the parameterized data type in question, for example:

$\underline{List}(\underline{NeSet}) = \{\underline{List}(X) \mid X \in \underline{NeSet}\}$

denotes the semantics of the parameterized data type of lists whose actual parameters are non-empty sets in $\underline{NeSet} = \underline{Mod}_{(\{Elt\}, (\emptyset,\emptyset), \emptyset)}$, where $(\{Elt\}, (\emptyset, \emptyset), \emptyset)$, denoted TRIV in Maude, has single sort Elt, $(\Sigma, \Pi) = \emptyset$, and no axioms.

7. Given a signature $((\Sigma,\Pi))$,

$$\underline{Sub} : \mathcal{P}(\underline{Mod}_{(\Sigma,\Pi)}) \ni \mathbb{A} \mapsto \underline{Sub}(\mathbb{A}) \in \mathcal{P}(\underline{Mod}_{(\Sigma,\Pi)})$$

is the <u>closure under submodels</u> transformation,

so that $\underline{Sub}(\mathbb{A}) = \{\mathcal{M}' \in \underline{Mod}_{(\Sigma,\Pi)} \mid \exists \mathcal{M} \in \mathbb{A} \text{ s.t. } \mathcal{M}' \subseteq \mathcal{M}\}$,

where $\mathcal{M}'$ is a <u>submodel</u> of $\mathcal{M}$, denoted $\mathcal{M}' \subseteq \mathcal{M}$

iff : (i) $\mathcal{M}'|_\Sigma$ is a subalgebra of $\mathcal{M}$, i.e.,

if $\mathcal{M} = (M, -_{\mathcal{M}})$ and $\mathcal{M}' = (M', -_{\mathcal{M}'})$,

then $M' \subseteq M$ and the Σ-operations $f_{\mathcal{M}'}$

on $\mathcal{M}'$ are the <u>restrictions</u> of $f_{\mathcal{M}}$ to M'.

(ii) $\forall p \in \Pi_{s_1 \dots s_n}$ $\forall (a_1 \dots, a_n) \in M'_{s_1} \times \dots \times M'_{s_n}$

$$(a_1, \dots, a_n) \in p_{\mathcal{M}'} \Leftrightarrow (a_1 \dots, a_n) \in p_{\mathcal{M}},$$

i.e., $p_{\mathcal{M}'} = p_{\mathcal{M}} \cap (M'_{s_1} \times \dots \times M'_{s_n})$;

again $p_{\mathcal{M}'}$ is the <u>restriction</u> of $p_{\mathcal{M}}$ to M'.

This induces a theory transformation

$$((\Sigma,\Pi), \mathbb{A}) \mapsto ((\Sigma,\Pi), \underline{Sub}(\mathbb{A}))$$

<u>Exercise</u>. Prove that if $\Gamma \subseteq \forall QF(\Sigma)$ (universal quantification of quantifier-free sentences), we have

$$\underline{Sub}(\underline{Mod}_{((\Sigma,\Pi),\Gamma)}) = \underline{Mod}_{((\Sigma,\Pi),\Gamma)}$$

but that this fails for arbitrary $\Gamma \subseteq FO(\Sigma)$.