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A Fully Syntactic AC-RPO

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We present the first fully syntactic (i.e., non-interpretation-based) AC-compatible *recursive path ordering* (RPO). It is simple, and hence easy to implement, and its behaviour is intuitive as in the standard RPO. The ordering is AC-total and defined uniformly for both ground and nonground terms, as well as for partial precedences. More important, it is the first one that can deal *incrementally* with partial precedences, an aspect that is essential, together with its intuitive behaviour, for interactive applications such as Knuth–Bendix completion.

Key Words: term rewriting; built-in associative and commutative theories; termination; term orderings.

1. INTRODUCTION

Rewrite-based methods with built-in associativity and commutativity (AC, for short) properties for some of the operators are well known to be crucial in theorem proving and programming. Therefore a lot of work has been done on the development of suitable *AC-compatible* reduction or simplification orderings, like [1–3, 6–9, 11–13]. An essential additional property of the ordering that is needed in order to preserve the completeness of most rewrite-based theorem proving techniques (modulo AC) is *AC-totality*, i.e., the totality on (AC-different) ground terms.

Since the initial attempts, it has always been an aim to obtain AC-compatible versions of Dershowitz' *recursive path ordering* (*RPO*) [5], as it is simple, easy to automate and use, and normally orients the rules in an adequate direction. In [13] we gave the first RPO-based AC-total and AC-compatible reduction ordering without any restriction on the number of AC-symbols or on the precedence over the signature. Unfortunately, although being defined in terms of RPO, it does not behave like RPO; e.g., it does not orient the distributivity rule in the "right" (i.e., distributing) way, since a transformation on the terms is applied before using RPO (this approach, with different transformations, is also used in [3] among others). Therefore, a better approach seems to be to directly apply an RPO-like scheme, treating as the only special case the *AC-equal-top case*, that is, when both terms to be compared are headed by the same AC-symbol. In this direction the first AC-compatible simplification ordering with an RPO scheme was defined in [11] and the first AC-total one on ground terms in [9]. Other simpler proposals for AC-orderings with RPO scheme were given in [14] and in [10].

However, all these AC-orderings need to interpret terms (apart from *flattening*) in some way, which makes their behaviour less intuitive, unlike what happens with the standard RPO, whose simple fully syntactic definition has been an important reason for its success.

In this paper we propose the first fully syntactic AC-RPO; i.e., no interpretation is needed apart from flattening. It is simple, and hence easy to implement, and its behaviour is intuitive as for the standard RPO. The ordering is AC-total and defined uniformly for both ground and nonground terms, as well as for partial precedences.

Moreover, precisely due to the fact that it is not interpretation-based, it is the first AC-RPO that can deal *incrementally* with partial precedences, i.e., if $s \succ t$, then $s \succ t$ under any extension of the precedence. This aspect is essential, together with its intuitive behaviour, for interactive applications such as Knuth–Bendix completion. Of course, previously existing orderings could work with partial precedences, but in a useless way, simply by considering an arbitrarily chosen total extension of the partial precedence, and hence losing incrementality.

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In order to introduce the concepts smoothly we give the ordering in three steps, first for ground terms and total precedences, then for terms with variables and total precedences, and finally for terms with variables and partial precedences, each definition strictly extending the previous one. For this reason we prove all properties only for the last one, showing that it is indeed an AC-compatible simplification ordering.

The paper is organized as follows. In the following section we give some basic notions and definitions. In Section 3 we introduce the ordering for ground terms and total precedences. Section 4 is devoted to terms with variables and total precedences. In Section 5 we generalize the previous ordering for dealing with partial precedences and in Section 6 we prove that it is an AC-compatible simplification ordering. Some improvements related to the implementation of the ordering are presented in Section 7. Conclusions are given in Section 8.

2. PRELIMINARIES

In the following we consider that $\mathcal{F}$ is a finite set of function symbols that is (partially) ordered by a precedence $\succ_{\mathcal{F}}$, where $\mathcal{F}_{AC}$ is the subset containing all AC-symbols of $\mathcal{F}$.

The arity of a function symbol f is a natural number that indicates the number of arguments that f may take. A function symbol may have several arities, and, in particular if $f \in \mathcal{F}_{AC}$ then it has all arities greater than or equal to 2. A function symbol has an *unbounded arity* if it has infinitely many arities. $\mathcal{T}(\mathcal{F})$ and $\mathcal{T}(\mathcal{F}, \mathcal{X})$ are defined as usual according to these arities, if $\mathcal{X}$ is a set of variables, whose elements will be denoted by $x, y, z, \ldots$, possibly with subscripts. The size of a term t, i.e., the number of symbols of t, is denoted by $|t|$.

We denote by $=_{AC}$ the congruence generated on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ by the associativity and commutativity axioms for the symbols in $\mathcal{F}_{AC}$. In what follows we will ambiguously use $=_{AC}$ to also denote the standard extension of AC-equality to multisets (and in fact to any other structure). In general a relation with subscript AC denotes that syntactic equality is replaced by AC-equality. For instance, we may write $M \supseteq_{AC} N$, $M \cap_{AC} N$, or $M \setminus_{AC} N$, for multisets M and N,

2.1. Rewriting Modulo AC

A term rewriting system (TRS) is a (possibly infinite) set of rules $l \to r$ where l and r are terms. Given a TRS R, s rewrites to t with R, denoted by $s \to_R t$, if there is some rule $l \to r$ in R, $s|_p = l\sigma$ for some position p and substitution σ and $t = s[r\sigma]_p$.

Given a TRS R, a terms s rewrites to t with R modulo AC, denoted by $s \to_{R/AC} t$, if $s =_{AC} s'$, $s'|_p = l\sigma$ for some term s', position p and substitution σ, and $t =_{AC} s'[r\sigma]_p$. A TRS R is terminating for rewriting modulo AC if there is some AC-compatible simplification ordering $\succ$ such that $l \succ r$ for all rules $l \to r$ in R.

2.2. Flattened Terms

In the following, terms are flattened wrt the AC-symbols. The flattening of t, denoted by $\bar{t}$, is the normal for of t wrt the infinite TRS containing the rules

$$f(x_1, \ldots, x_n, f(y_1, \ldots, y_m), z_1, \ldots, z_r) \to f(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_r)$$

for every $f \in \mathcal{F}_{AC}$ and $n, m, r \geq 0$. Due to flattening, the AC-symbols have an unbounded variable arity.

Let s and t be two terms such that $\bar{s} = f(s_1, \ldots, s_m)$ and $\bar{t} = g(t_1, \ldots, t_n)$. If $s =_{AC} t$ then $f = g$, $m = n$, and $\bar{s}$ is equal to $\bar{t}$ up to permutation of arguments for the AC-symbols. We will denote this equality up to permutation of arguments also by $=_{AC}$. The top-flattening of a term s wrt an AC-symbol f, denoted by $\mathit{tf}_f(s)$, is a string of terms defined as $\mathit{tf}_f(s) = s_1, \ldots, s_n$ if $f(s_1, \ldots, s_n)$ is the result of normalizing s by the flattening rules applied only at the top position of s, and $\mathit{tf}_f(s) = s$ if $\mathit{top}(s) \neq f$.

2.3. AC-Orderings

Let s, t, s', and t' be arbitrary terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$, let u be a nonempty context in $\mathcal{T}(\mathcal{F}, \mathcal{X})$, and let σ be a substitution. Then a (strict) ordering $\succ$ on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is a transitive irreflexive relation. In general we will consider that $\succeq$ is the union of a given ordering $\succ$ and $=_{AC}$. An ordering $\succ$ is *monotonic* if $s \succ t$ implies $u[s] \succ u[t]$ and *stable under substitution* if $s \succ t$ implies $s\sigma \succ t\sigma$. Monotonic orderings that are stable under substitution are called *rewrite orderings*. An ordering $\succ$ fulfills the *subterm property* if $u[t] \succ t$ and the *deletion property* if $f(\ldots s \ldots) \succ f(\ldots \;\; \ldots)$ for every symbol f with unbounded arity. A rewrite ordering that fulfills the subterm property and the deletion property is called a *simplification ordering* and is *well-founded*: there are no infinite sequences $t_1 \succ t_2 \succ \cdots$. An ordering is AC-total on ground terms if, when s and t are ground terms, either $s \succ t$ or $t \succ s$ or $s =_{AC} t$. Finally an ordering $\succ$ is *AC-compatible* if $s' =_{AC} s \succ t =_{AC} t'$ implies $s' \succ t'$.

THEOREM 1. *A TRS R is terminating for rewriting modulo AC iff there is an AC-compatible reduction ordering $\succ$ s.t. $l \succ r$ for all rules $l \to r \in R$.*

COROLLARY 1. *A TRS R is terminating for rewriting modulo AC if there is an AC-compatible simplification ordering $\succ$ s.t. $l \succ r$ for all rules $l \to r \in R$.*

2.4. Lexicographic and Multiset Extensions Modulo AC

Given a relation $\succ$, the (AC-)lexicographic extension of $\succ$ on sequences, denoted by $\succ_{lex}$, is defined by: $\langle s_1, \ldots, s_n\rangle \succ_{lex} \langle t_1, \ldots, t_m\rangle$ if there is some $j \in \{1 \ldots n\}$ s.t. $s_i =_{AC} t_i$ for all $i < j$ and either $s_j \succ t_j$ or $j = m + 1$.

Given a relation $\succ$, the (AC-)multiset extension of $\succ$ on finite multisets, denoted by $\succ\!\!\succ$, is defined as the smallest transitive relation containing

$$X \cup \{s\} \succ\!\!\succ_1 Y \cup \{t_1, \ldots, t_n\} \quad \text{if } X =_{AC} Y \quad \text{and} \quad s \succ t_i \quad \text{for all } i \in \{1 \ldots n\}.$$

The following properties show another way to define the multiset extension.

LEMMA 1. *Let $\succ$ be an AC-compatible ordering and let M and N be multisets of terms. Then $M \succ\!\!\succ N$ iff there are multisets X, Y and $N_1 \ldots N_n$ and terms $s_1, \ldots, s_n$ s.t. $M = X \cup \{s_1, \ldots, s_n\}$, $N = Y \cup N_1 \cup \cdots \cup N_n$, $n \geq 1$, $X =_{AC} Y$, $\{s_1, \ldots, s_n\} \cap_{AC} (N_1 \cup \cdots \cup N_n) = \emptyset$, and $s_i \succ t'$ for all $i \in \{1 \ldots n\}$ and $t' \in N_i$.*

Proof. For the right to left implication we proceed by induction on n, building a nonempty sequence from M to N using $\succ\!\!\succ_1$. Let M_1 be $X' \cup \{s_1, \ldots, s_{n-1}\}$, with $X' = Y \cup N_n$. By definition we have $M = X \cup \{s_1, \ldots, s_{n-1}\} \cup \{s_n\} \succ\!\!\succ_1 Y \cup \{s_1, \ldots, s_{n-1}\} \cup N_n = M_1$. If $n = 1$ then we are done, since $M_1 = N$. Otherwise, we have $N = X' \cup N_1 \cup \cdots \cup N_{n-1}$, with $n - 1 \geq 1$, $\{s_1, \ldots, s_{n-1}\} \cap_{AC} (N_1 \cup \cdots \cup N_{n-1}) = \emptyset$, and $s_i \succ t'$ for all $i \in \{1 \ldots n-1\}$ and $t' \in N_i$. Therefore, by induction hypothesis $M_1 = Y \cup N_n \cup \{s_1, \ldots, s_{n-1}\} \succ\!\!\succ_1^+ Y \cup N_n \cup N_1 \cup \cdots \cup N_{n-1} = N$, which implies $M \succ\!\!\succ_1^+ N$.

For the left to right implication we first prove the result without the restriction $\{s_1, \ldots, s_n\} \cap_{AC} (N_1 \cup \cdots \cup N_n) = \emptyset$ by induction on the length of the sequence $M \succ\!\!\succ_1^+ N$, and afterward we will show that we can also add the restriction.

Then we have $M \succ\!\!\succ_1 M_1 \succ\!\!\succ_1^* N$ for some M_1 s.t. M and M_1 are respectively $X' \cup \{s\}$ and $Y' \cup N''$, with $X' =_{AC} Y'$ and $s \succ t'$ for all $t' \in N''$. If $M_1 = N$, it holds. Otherwise, by induction hypothesis, there are multisets X_1, Y_1, and $N'_1 \ldots N'_m$ and terms $s'_1, \ldots, s'_m$ s.t. $M_1 = X_1 \cup \{s'_1, \ldots, s'_m\}$ and $N = Y_1 \cup N'_1 \cup \cdots \cup N'_m$, with $m \geq 1$, $X_1 =_{AC} Y_1$, and $s'_i \succ t'$ for all $i \in \{1 \ldots m\}$ and $t' \in N'_i$.

Since M_1 is $Y' \cup N''$ and $X_1 \cup \{s'_1, \ldots, s'_m\}$, there are multisets Y'_1 and N''_1 s.t. $X_1 = Y'_1 \cup N''_1$ and $Y' = Y'_1 \cup \{s'_{i_1}, \ldots, s'_{i_{n-1}}\}$, $N'' = N''_1 \cup \{s'_{j_1}, \ldots, s'_{j_r}\}$, and $\{s'_1, \ldots, s'_m\} = \{s'_{i_1}, \ldots, s'_{i_{n-1}}, s'_{j_1}, \ldots, s'_{j_r}\}$ for indexes $\{i_1, \ldots, i_{n-1}, j_1, \ldots, j_r\} = \{1, \ldots, m\}$ and $n - 1 + r = m$.

Now we split $M = X \cup \{s_1, \ldots, s_{n-1}, s\}$, where $X =_{AC} Y'_1$ and $s_k =_{AC} s'_{i_k}$ for all $k \in \{1, \ldots, n-1\}$, and $N = Y \cup N_1 \cup \cdots \cup N_{n-1} \cup N_n$, where $Y \subseteq Y_1$ and $Y =_{AC} X$, and $N_k = N''_{i_k}$ for all $k \in \{1, \ldots, n-1\}$

and $N_n = (Y_1 \backslash Y) \cup N''_{j_1} \cup \cdots \cup N''_{j_r}$. Then, by AC-compatibility, $s_k =_{AC} s'_{i_k} \succ t'$ implies $s_k \succ t'$ for all $k \in \{1, \ldots, n-1\}$ and $t' \in N_k = N''_{i_k}$. Finally, we conclude by proving that $s \succ t'$ for all $t' \in N_n$: (i) since $(Y_1 \backslash Y) =_{AC} N''_1 \subseteq N''$ we have by AC-compatibility that $s \succ t'$ for all $t' \in (Y_1 \backslash Y)$; (ii) since $s \succ s'_{j_k}$ and $s'_{j_k} \succ t'$ for all $k \in \{1, \ldots, r\}$ and $t' \in N''_{j_k}$, by transitivity we have $s \succ t'$ for all $t' \in N''_{j_k}$.

The last part of the proof is devoted to show that we can add the restriction $\{s_1, \ldots, s_n\} \cap_{AC} (N_1 \cup \cdots \cup N_n) = \emptyset$. We proceed by induction on the number of terms in the intersection. Assume that the intersection is not empty, and suppose wlog that $s_n =_{AC} t$ and $N_j = \{t\} \cup N'$ for some $j \in \{1 \ldots n\}$. By irreflexivity, $j \neq n$, and hence $n > 1$, and, by transitivity, $s_j \succ t'$ for all $t' \in N' \cup N_n$. Therefore we take $M = X' \cup \{s_1, \ldots, s_{n-1}\}$, with $X' = X \cup \{s_n\}$ and $N = Y' \cup N'_1 \cup \cdots \cup N'_{n-1}$, with $Y' = Y \cup \{t\}$ and $N'_i = N_i$ for all $i \neq j$ and $i \in \{1 \ldots n-1\}$, and $N'_j = N' \cup N_n$. Finally we conclude by induction hypothesis, since we have one element less in the intersection. ∎

COROLLARY 2. *Let $\succ$ be an AC-compatible ordering, and let M, N, and X be multisets of terms. Then $M \succ\!\!\succ N$ iff $M \cup X \succ\!\!\succ N \cup X$.*

COROLLARY 3. *Let $\succ$ be an AC-compatible ordering, and let M and N be multisets of terms and t be a term s.t. $t \in N$ and $t \notin M$. Then $M \succ\!\!\succ N$ implies $M \succ\!\!\succ N \cup \{t_1, \ldots, t_n\}$ for all terms $t_1, \ldots, t_n$ s.t. $t \succeq t_i$ for all $i \in \{1 \ldots n\}$.*

COROLLARY 4. *Let $\succ$ be an AC-compatible, and let M and N be multisets of terms and t be a term s.t. $t \not\preceq t'$ for all $t' \in N$. Then $M \succeq\!\!\succeq N$ implies $M \backslash T \succeq\!\!\succeq N$ where T is the multiset containing all occurrences of t in M.*

If $\succ$ is an AC-compatible ordering on a set S then $\succ\!\!\succ$ and $\succ_{lex}$ are respectively an AC-compatible ordering on multisets of elements in S and an AC-compatible ordering on sequences of elements in S. Being more precise, in order to fulfil transitivity we need $\succ$ to be both transitive and AC-compatible. Additionally, if $\succ$ is stable under substitutions then $\succ\!\!\succ$ is stable under substitutions.

3. THE ORDERING FOR GROUND TERMS

In this section we consider only ground terms and assume that the precedence is total on the set of function symbols. First we introduce two different sets of terms obtained from a term headed by an AC-symbol.

DEFINITION 1. Let s be a term of the form $f(s_1, \ldots, s_n)$ with $f \in \mathcal{F}_{AC}$.

- The set of terms embedded in s through an argument headed by a small symbol, denoted by *EmbSmall*(s), is defined as $\{f(s_1, \ldots, \mathrm{tf}_f(v_j), \ldots, s_n) \mid s_i = h(v_1, \ldots, v_r) \wedge f \succ_{\mathcal{F}} h \wedge j \in \{1 \ldots r\}\}$.
- The multiset of arguments of s headed by a big function symbol, denoted by *BigHead*(s), is defined as $\{s_i \mid 1 \leq i \leq n \wedge top(s_i) \succ_{\mathcal{F}} f\}$.

We will now give the definition of the ordering $\succ$ on ground terms. The set of non-AC symbols is split into two sets *Lex* and *Mul* which denote respectively the set of symbols whose arguments are compared lexicographically and the set of symbols whose arguments are compared with the multiset extension of the ordering. All symbols with unbounded arity are in *Mul*.

DEFINITION 2. Let s and t be terms in $\mathcal{T}(\mathcal{F})$. Then $s = f(s_1, \ldots, s_n) \succ g(t_1, \ldots, t_m) = t$ if and only if

1. $s_i \succeq t$ for some $i \in \{1 \ldots n\}$, or
2. $f \succ_{\mathcal{F}} g$ and $s \succ t_i$ for all $i \in \{1 \ldots m\}$, or
3. $f = g \in Lex$ and $\langle s_1, \ldots, s_n \rangle \succ_{lex} \langle t_1, \ldots, t_m \rangle$ and $s \succ t_i$ for all $i \in \{1 \ldots m\}$, or
4. $f = g \in Mul$ and $\{s_1, \ldots, s_n\} \succ\!\!\succ \{t_1, \ldots, t_m\}$, or
5. $f = g \in \mathcal{F}_{AC}$ and $s' \succeq t$ for some $s' \in$ *EmbSmall*(s), or
6. $f = g \in \mathcal{F}_{AC}$ and $s \succ t'$ for all $t' \in$ *EmbSmall*(t) and *BigHead*$(s) \succeq\!\!\succeq$ *BigHead*(t) and either

(a) $BigHead(s) \gg BigHead(t)$ or
(b) $n > m$ or
(c) $n = m$ and $\{s_1, \ldots, s_n\} \gg \{t_1, \ldots, t_m\}$.

The first four cases of this definition correspond to the standard RPO with status, with the only difference that we allow the user to have in *Lex* symbols with variable arity provided it is not unbounded. The reason why we can do that is simple: the only property that cannot be proved for symbols in *Lex* is the deletion property, but, since it is only required for symbols with unbounded arity, *Lex* can contain any symbol with bounded arity. Cases 5 and 6 apply when both terms are headed by the same AC-symbol f. The intuition behind them is very simple. On the one hand, in order to obtain AC-compatibility, terms are considered in flattened form. On the other hand, the symbols that disappear under flattening must still be taken into account in order to obtain monotonicity. Let us consider an example.

Assume $f \succ_{\mathcal{F}} g$. Then, as in the standard RPO, we have of course $f(a, a) \succ g(a)$. By monotonicity, if we add the context $f(a, [\,])$ and flatten, we must have $f(a, a, a) \succ f(a, g(a))$, that is, the symbol f that has been removed under flattening is important in order to "take care" of the g. The number of such *implicit* f's depends of course on the number of arguments.

But, similarly, if $g \succ_{\mathcal{F}} f$, then $g(a) \succ f(a, a)$ and by monotonicity we should have $f(a, g(a)) \succ f(a, a, a)$. Clearly, in this kind of situation where the comparison of arguments headed by big symbols is conclusive, the number of such implicit f's is not important.

This motivates the three stage hierarchy in Case 6: (a) first consider the multisets of arguments headed by symbols bigger than f; (b) if these sets coincide, then compare the number of arguments (i.e., the number of implicit f's); (c) finally, if both terms are equal under the previous two measures, then we can safely compare the multisets of all (or only the small-headed ones) arguments in the usual (multiset) way.

Of course, since any simplification ordering must contain the *embedding* relation, we must have $s[g(\ldots t \ldots)]_p \succ s[t]_p$ for all s, t, g, and p. This indicates that the use of $EmbSmall(s)$ and $EmbSmall(t)$ in Cases 5 and 6 is no real restriction.

But the ideas of the three-stage approach of Case 6 can be safely applied precisely due to the precondition stating that $s \succ t'$ for all $t' \in EmbSmall(t)$, which prevents situations where t is a term like $f(a, h(\ldots u \ldots))$, and where by removing h (with $f \succ_{\mathcal{F}} h$) we get $f(a, u)$, where u can be headed by a big symbol, or, if u is headed by f, the number of arguments increases.

The following examples show the behaviour of the ordering when comparing terms headed by the same AC-symbol.

Example 1. Let $h \succ_{\mathcal{F}} f \succ_{\mathcal{F}} g \succ_{\mathcal{F}} a \succ_{\mathcal{F}} b$ be the precedence. Then we have

1. $f(g(f(h(a), a)), a) \succ f(h(a), a, a)$ by Case 5.

2. $s = f(h(a), g(a)) \succ f(g(h(a)), a) = t$ by Case 6, since $s \succ f(h(a), a) \in EmbSmall(t)$ by Case 5 and $BigHead(s) = \{h(a)\} \gg \emptyset = BigHead(t)$.

3. $s = f(g(h(a)), b, b, b) \succ f(g(f(h(a), a)), a) = t$ by Case 6b, since $n = 4 > 2 = m$ and $BigHead(s) = \emptyset = BigHead(t)$ and $s \succ f(h(a), a, a) = t' \in EmbSmall(t)$ by applying first Case 5 and then $s' = f(h(a), b, b, b) \succ t'$ by Case 6b, since $BigHead(s') = \{h(a)\} = BigHead(t')$, $EmbSmall(t') = \emptyset$, and $n = 4 > 3 = m$.

4. $s = f(h(a), a) \succ f(h(a), b) = t$, by Case 6c, since we have $EmbSmall(t) = \emptyset$, $BigHead(s) = \{h(a)\} = BigHead(t)$, $n = m = 2$ and $\{h(a), a\} \gg \{h(a), b\}$.

Lemma 2. *If $\succ_{\mathcal{F}}$ is total on the set of function symbols and Mul is empty then $\succ$ is AC-total on ground terms.*

Proof. Let $s = f(s_1, \ldots, s_n)$ and $t = g(t_1, \ldots, t_m)$ be ground terms. Then either $s \succ t$ or $t \succ s$ or $s =_{AC} t$. We proceed by induction on $|s| + |t|$.

By induction hypothesis for every s_i we have either $s_i \succeq t$ or $t \succ s_i$, and for every t_j we have either $s \succ t_j$ or $t_j \succeq s$. On the other hand by totality of the precedence, either $f \succ_{\mathcal{F}} g$ or $g \succ_{\mathcal{F}} f$ or $f = g$. Therefore, either we conclude $s \succ t$ or $t \succ s$ by Cases 1 or 2 or $s \succ t_j$ for all t_j and $t \succ s_i$ for all s_i and $f = g$.

If $f \notin \mathcal{F}_{AC}$ then, since *Mul* is empty, $f \in Lex$ and, by induction hypothesis, either $s =_{AC} t$ or we can conclude $s \succ t$ or $t \succ s$ by Case 3.

Finally if $f \in \mathcal{F}_{AC}$ then, by induction hypothesis, either $s \succ t'$ or $t' \succeq s$ for all $t' \in \mathit{EmbSmall}(t)$; and either $t \succ s'$ or $s' \succeq t$ for all $s' \in \mathit{EmbSmall}(s)$. Therefore either $s \succ t$ or $t \succ s$ by Case 5 or $s \succ t'$ for all $t' \in \mathit{EmbSmall}(t)$ and $t \succ s'$ for all $s' \in \mathit{EmbSmall}(s)$. By induction hypothesis, either $\mathit{BigHead}(s) \ggcurly_p \mathit{BigHead}(t)$ or $\mathit{BigHead}(t) \ggcurly_p \mathit{BigHead}(s)$ or $\mathit{BigHead}(s) =_{AC} \mathit{BigHead}(t)$. Therefore either $s \succ t$ or $t \succ s$ by Case 6a or $\mathit{BigHead}(s) =_{AC} \mathit{BigHead}(t)$ and then either $s \succ t$ or $t \succ s$ by Case 6b, or $m = n$, and by induction hypothesis, we have either $s \succ t$ or $t \succ s$ by Case 6c, or $s =_{AC} t$. ■

The following theorem follows from Corollary 5 and Theorem 4.

THEOREM 2. *$\succ$ is an AC-compatible simplification ordering on $\mathcal{T}(\mathcal{F})$ which is AC-total if Mul is empty.*

3.1. Totality with Multiset Comparison

As said, in order to achieve AC-totality we need *Mul* to be empty, otherwise we only obtain AC-totality up to permutation of arguments of symbols in *Mul*. On the other hand, it is necessary that the non-AC symbols with unbounded arity belong to *Mul* in order to fulfil the deletion property. Therefore, we cannot have non-AC symbols with unbounded arity.

An easy way to achieve AC-totality and deletion property for symbols with unbounded arity is to extend the multiset case in the following way.

4. $f = g \in \mathit{Mul}$ and $\{s_1, \ldots, s_n\} \ggcurly \{t_1, \ldots, t_m\}$, or $\{s_1, \ldots, s_n\} =_{AC} \{t_1, \ldots, t_m\}$ and $\langle s_1, \ldots, s_n\rangle \succ_{lex} \langle t_1, \ldots, t_m\rangle$.

4. TERMS WITH VARIABLES

In this section we consider terms with variables, but we still assume that the precedence is total on the set of function symbols. First, due to the presence of variables, the counting of arguments has to be adapted, since one cannot know how many arguments a variable will include when instantiated and flattened.

Therefore in Cases 6b and 6c, instead of n and m we will use the following notion of $\#(s)$ and $\#(t)$, and $n > m$ and $n = m$ become diophantine inequations over the positive integers.

DEFINITION 3. Let s be a term. Then $\#(s)$ is an expression with variables on the positive integers, defined as $\#(f(s_1, \ldots, s_n)) = \#_v(s_1) + \cdots + \#_v(s_n)$, where $\#_v(x) = x$ and $\#_v(t) = 1$ if t is not a variable.

For example, we have $\#(f(x, y, g(x))) = x + y + 1 > x + y = \#(f(x, y))$, which is necessary to achieve stability under substitution.

In addition we have to replace the set $\mathit{BigHead}(s)$ by $\mathit{NoSmallHead}(s)$, which may include variables, in one of its uses.

DEFINITION 4. Let s be of the form $f(s_1, \ldots, s_n)$ with $f \in \mathcal{F}_{AC}$. The multiset of arguments of s headed by a symbol not smaller than f, denoted by $\mathit{NoSmallHead}(s)$, is defined as $\{s_i \mid 1 \leq i \leq n \wedge f \not\succ_{\mathcal{F}} \mathit{top}(s_i)\}$.

DEFINITION 5. Let s and t be terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Then $s = f(s_1, \ldots, s_n) \succ_v g(t_1, \ldots, t_m) = t$ if and only if

1. $s_i \succeq_v t$ for some $i \in \{1 \ldots n\}$, or
2. $f \succ_{\mathcal{F}} g$ and $s \succ_v t_i$ for all $i \in \{1 \ldots m\}$, or
3. $f = g \in \mathit{Lex}$ and $\langle s_1, \ldots, s_n\rangle (\succ_v)_{lex} \langle t_1, \ldots, t_m\rangle$ and $s \succ_v t_i$ for all $i \in \{1 \ldots m\}$, or
4. $f = g \in \mathit{Mul}$ and $\{s_1, \ldots, s_n\} \ggcurly_v \{t_1, \ldots, t_m\}$, or
5. $f = g \in \mathcal{F}_{AC}$ and there is some $s' \in \mathit{EmbSmall}(s)$ s.t. $s' \succeq_v t$, or
6. $f = g \in \mathcal{F}_{AC}$ and $s \succ_v t'$ for all $t' \in \mathit{EmbSmall}(t)$ and $\mathit{NoSmallHead}(s) \succeq\!\!\succeq_v \mathit{NoSmallHead}(t)$ and either

(a) $BigHead(s) \gg_v BigHead(t)$ or
(b) $\#(s) > \#(t)$ or
(c) $\#(s) \geq \#(t)$ and $\{s_1, \ldots, s_n\} \gg_v \{t_1, \ldots, t_m\}$.

Note that the difference between $NoSmallHead(s)$ and $BigHead(s)$ is that the former includes the variables that are arguments of s and the latter does not. Then, on the one hand, the condition $NoSmallHead(s) \succeq\!\!\succeq_v NoSmallHead(t)$ ensures that every variable in t is taken care of by a variable in s or by an argument of s headed by a big symbol. Then if, by instantiation, some variable becomes a term headed by a big symbol, we know that some argument of the (instantiation) of s headed by a big symbol takes care of it. On the other hand, the condition $BigHead(s) \gg_v BigHead(t)$ prevents us from using variables that can become small terms by instantiation. The combination of both conditions is crucial to prove stability under substitutions.

EXAMPLE 2. Let $h \succ_{\mathcal{F}} f \succ_{\mathcal{F}} g$ be the precedence. Then we have

1. $f(g(f(h(x), x)), x) \succ_v f(h(x), x, x)$ by Case 5.

2. $s = f(h(x), g(x)) \succ_v f(g(h(x)), x) = t$ by Case 6a, since we have $s \succ_v f(h(x), x) \in EmbSmall(t)$ by Case 5, and $NoSmallHead(s) = \{h(x)\} \gg_v \{x\} = NoSmallHead(t)$ and $BigHead(s) = \{h(x)\} \gg_v \emptyset = BigHead(t)$.

3. $s = f(g(h(x)), x, x, y) \succ_v f(g(f(h(x), y)), x) = t$ by Case 6b, since we have $\#(s) = 2x + y + 1 > x + 1 = \#(t)$ and $NoSmallHead(s) = \{x, x, y\} \gg_v \{x\} = NoSmallHead(t)$ and $s \succ f(h(x), y, x) = t' \in EmbSmall(t)$ by applying first Case 5 and then $s' = f(h(x), x, x, y) \succ t'$ by Case 6b, since $NoSmallHead(s') = \{h(x), x, x, y\} \gg_v \{h(x), y, x\} = NoSmallHead(t')$, $EmbSmall(t') = \emptyset$ and, since x is a positive integer, $\#(s') = 2x + y + 1 > x + y + 1 = \#(t')$.

4. $s = f(g(g(x)), x) \succ f(g(x), g(x)) = t$, by Case 6c, since $s \succ_v f(g(x), x) \in EmbSmall(t)$ by Case 5 (note that the symmetric case follows in the same way), $NoSmallHead(s) = \{x\} \gg_v \emptyset = NoSmallHead(t)$ and $\#(s) = x + 1 \geq 2 = \#(t)$ and $\{g(g(x)), x\} \gg_v \{g(x), g(x)\}$.

LEMMA 3. *Let s and t be ground terms. Then $s \succ t$ if and only if $s \succ_v t$.*

Proof. The result is trivial since both definitions coincide when applied to ground terms. Note that if $s = f(s_1, \ldots, s_n)$ is ground then we have $\#(s) = n$ and $NoSmallHead(s) = BigHead(s)$. ■

The following theorem follows from Lemma 5 and Theorem 4.

THEOREM 3. *$\succ_v$ is an AC-compatible simplification ordering on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ that is AC-total on ground terms if Mul is empty.*

EXAMPLE 3. *Rings*. With $+, * \in \mathcal{F}_{AC}$, and $* \succ_{\mathcal{F}} I \succ_{\mathcal{F}} + \succ_{\mathcal{F}} 0$, the ordering orients (and hence proves termination of) the following TRS:

$$\begin{aligned} x + 0 &\to x \\ x + I(x) &\to 0 \\ I(0) &\to 0 \\ I(I(x)) &\to x \\ I(x + y) &\to I(x) + I(y) \\ x * (y + z) &\to (x * y) + (x * z) \\ x * 0 &\to 0 \\ x * I(y) &\to I(x * y). \end{aligned}$$

EXAMPLE 4. With $+, * \in \mathcal{F}_{AC}$, and $* \succ_{\mathcal{F}} + \succ_{\mathcal{F}} s \succ_{\mathcal{F}} 0$, the ordering orients (and hence proves termination of) the following TRS:

$$\begin{aligned} x + 0 &\to x \\ x + s(y) &\to s(x + y) \\ x * 0 &\to 0 \\ x * s(y) &\to x * y + x \\ x * (y + z) &\to (x * y) + (x * z). \end{aligned}$$

5. PARTIAL PRECEDENCES

First, in order to deal with partial precedences, we need to weaken the ordering when applied to compare the multisets *NoSmallHead*(s) and *NoSmallHead*(t), since otherwise we cannot ensure incrementality. We define in a general way the restriction of an ordering with respect to an AC-symbol.

DEFINITION 6. Let $\succ$ be an ordering on terms and let $\succ_{\mathcal{F}}$ be a (partial) precedence. The ordering restriction of $\succ$ wrt an AC-symbol f in the precedence $\succ_{\mathcal{F}}$, denoted by $\succ_f$, is defined as

$$s \succ_f t \quad \text{iff} \quad s \succ t \quad \text{and} \quad \text{if } top(s) \not\preceq_{\mathcal{F}} f \quad \text{then } top(s) \succeq_{\mathcal{F}} top(t).$$

The following property allows us to apply all desired properties on multisets to the multiset extension $\gg_f$ of $\succ_f$.

LEMMA 4. *Let $\succ$ be an AC-compatible ordering, let f be AC-symbol, and let $\succ_{\mathcal{F}}$ be a (partial) precedence. Then $\succ_f$ is an AC-compatible ordering.*

Proof. Irreflexivity trivially follows. For transitivity, let $s \succ_f t$ and $t \succ_f u$. By definition, we have (i) $s \succ t$ and if $top(s) \not\preceq_{\mathcal{F}} f$ then $top(s) \succeq_{\mathcal{F}} top(t)$, and (ii) $t \succ u$ and if $top(t) \not\preceq_{\mathcal{F}} f$ then $top(t) \succeq_{\mathcal{F}} top(u)$. Then by transitivity of $\succ$, we have $s \succ u$, and if $top(s) \not\preceq_{\mathcal{F}} f$, by transitivity of $\succeq_{\mathcal{F}}$, $top(t) \not\preceq_{\mathcal{F}} f$, and hence, by transitivity of $\succeq_{\mathcal{F}}$, we have $top(s) \succeq_{\mathcal{F}} top(u)$, which implies $s \succ_f u$.

Finally, for AC-compatibility, if $s' =_{AC} s \succ_f t =_{AC} t'$, by AC-compatibility of $\succ$, we have $s' \succ t'$, and, since $s' =_{AC} s$ implies $top(s) = top(s')$ and $t =_{AC} t'$ implies $top(t) = top(t')$, we have that $top(s') \not\preceq_{\mathcal{F}} f$ then $top(s') \succeq_{\mathcal{F}} top(t')$, which implies $s' \succ_f t'$. ∎

Now we adapt the set *EmbSmall*(s) to allow embedding through symbols not bigger than the head.

DEFINITION 7. Let s be a term of the form $f(s_1, \ldots, s_n)$ with $f \in \mathcal{F}_{AC}$. The set of terms embedded in s through an argument headed by a non-big symbol, denoted by *EmbNoBig*(s), is defined as

$$\{f(s_1, \ldots, \mathrm{tf}_f(v_j), \ldots, s_n) \mid s_i = h(v_1, \ldots, v_r) \wedge h \not\succ_{\mathcal{F}} f \wedge j \in \{1 \ldots r\}\}.$$

DEFINITION 8. Let s and t be terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Then $s = f(s_1, \ldots, s_n) \succ_p g(t_1, \ldots, t_m) = t$ if and only if

1. $s_i \succeq_p t$ for some $i \in \{1 \ldots n\}$, or
2. $f \succ_{\mathcal{F}} g$ and $s \succ_p t_i$ for all $i \in \{1 \ldots m\}$, or
3. $f = g \in Lex$ and $\langle s_1, \ldots, s_n\rangle(\succ_p)_{lex}\langle t_1, \ldots, t_m\rangle$ and $s \succ_p t_i$ for all $i \in \{1 \ldots m\}$, or
4. $f = g \in Mul$ and $\{s_1, \ldots, s_n\} \gg_p \{t_1, \ldots, t_m\}$, or
5. $f = g \in \mathcal{F}_{AC}$ and there is some $s' \in$ *EmbNoBig*(s) s.t. $s' \succeq_p t$, or
6. $f = g \in \mathcal{F}_{AC}$ and $s \succ_p t'$ for all $t' \in$ *EmbNoBig*(t) and *NoSmallHead*$(s) \succeq\!\!\succeq_{pf}$ *NoSmallHead*(t) and either
   (a) *BigHead*$(s) \gg_p$ *BigHead*(t) or
   (b) $\#(s) > \#(t)$ or
   (c) $\#(s) \geq \#(t)$ and $\{s_1, \ldots, s_n\} \gg_p \{t_1, \ldots, t_m\}$.

Recall that by $\gg_{pf}$ we mean the multiset extension of $\succ_{pf}$, which is the restriction of $\succ_p$ wrt f in $\succ_{\mathcal{F}}$ (see Definition 6). The reason to ask, in case of having a partial precedence, for *NoSmallHead*$(s) \succeq\!\!\succeq_{pf}$ *NoSmallHead*(t), instead of using simply $\succeq\!\!\succeq_p$ is that if, by extending the precedence, an argument t' of t headed by a symbol incomparable with f becomes a term headed by a big symbol then the argument in s that takes care of t' becomes a term headed by a big symbol as well; and if, by extending the precedence, an argument s' of s headed by a symbol incomparable with f becomes a term headed by a small symbol, then all arguments in t taken care of by s' become terms headed by a small symbol as well.

LEMMA 5. *Let s and t be terms. If the precedence is total then $s \succ_v t$ if and only if $s \succ_p t$.*

Proof. The result is trivial since both definitions coincide when applied with a total precedence. Note that if the precedence is total then we have $EmbNoBig(s) = EmbSmall(s)$ and that for all $s' \in NoSmallHead(f(s_1, \ldots, s_n))$ either $top(s') \succ_{\mathcal{F}} f$ or s' is a variable, which implies that $\succ_{pf}$ and $\succ_p$ coincide. ■

COROLLARY 5. *Let s and t be ground terms. If the precedence is total then $s \succ t$ if and only if $s \succ_p t$.*

The proof of the following theorem is given in the next section.

THEOREM 4. *$\succ_p$ is an AC-compatible simplification ordering on $\mathcal{T}(\mathcal{F}, \mathcal{X})$, AC-total on ground terms if the precedence is total and Mul is empty, and incremental wrt the precedence.*

EXAMPLE 5. Let f be an AC-symbol.

1. With any precedence we have $s = f(g(g(x)), x) \succ_p f(g(x), g(x)) = t$ by Case 6c, since $s \succ_p t' = f(g(x), x) \in EmbNoBig(t)$, by Case 5, and $NoSmallHead(s) = \{g(g(x)), x\} \succeq_{pf} \{g(x), g(x)\} = NoSmallHead(t)$, and $\#(s) = 1 + x \geq 2 = \#(t)$ and $\{g(g(x)), x\} \not\succeq \{g(x), g(x)\}$.
2. With precedence $g \succ_{\mathcal{F}} h$ we have $s = f(x, x, g(x)) \succ_p f(x, h(x)) = t$ by Case 6b, since $s \succ_p t' = f(x, x) \in EmbNoBig(t)$, by Case 6b, and $NoSmallHead(s) = \{x, x, g(x)\} \succeq_{pf} \{x, h(x)\} = NoSmallHead(t)$, and $\#(s) = 2x + 1 > x + 1 = \#(t)$ (note that x is a positive integer).

EXAMPLE 6 (Milners's nondeterministic machines). With $+ \in \mathcal{F}_{AC}$ and $T \succ_{\mathcal{F}} +$ and $L \succ_{\mathcal{F}} +$, the ordering orients (and hence proves termination of) the following TRS. Note that the decision about the precedence relation between T and $+$ is not needed until the last rule.

$$
\begin{aligned}
0 + x &\to x \\
x + x &\to x \\
L(T(x)) &\to L(x) \\
L(T(y) + x) &\to L(x + y) + L(y) \\
T(T(x)) &\to T(x) \\
T(x) + x &\to T(x) \\
T(x + y) + x &\to T(x + y) \\
T(T(y) + x) &\to T(x + y) + T(y)
\end{aligned}
$$

6. THE PROPERTIES OF THE ORDERING

Here we prove all the properties of the ordering $\succ_p$. All proofs are quite simple, except the one for stability under substitutions, which involves some technical problems caused by the arguments of t that are variables instantiated by terms headed by a non-big symbol.

LEMMA 6. *If $s =_{AC} t$ and $top(s) \in \mathcal{F}_{AC}$ then $EmbNoBig(s) =_{AC} EmbNoBig(t)$, $NoSmallHead(s) =_{AC} NoSmallHead(t)$, $BigHead(s) =_{AC} BigHead(t)$, and $\#(s) = \#(t)$.*

Proof. Let $s = f(s_1, \ldots, s_n)$ with $f \in \mathcal{F}_{AC}$. Since $s =_{AC} t$, we have $t = f(t_1, \ldots, t_n)$ and $\{s_1, \ldots, s_n\} =_{AC} \{t_1, \ldots, t_n\}$. Therefore, since if $s_i =_{AC} t_j$ then $top(s_i) = top(t_j)$, we trivially have $NoSmallHead(s) =_{AC} NoSmallHead(t)$, $BigHead(s) =_{AC} BigHead(t)$ and $\#(s) = \#(t)$. Finally, for $EmbNoBig(s) =_{AC} EmbNoBig(t)$, if some $s_i = h(v_1, \ldots, v_p)$ with $h \not\succ_{\mathcal{F}} f$ then there is some $t_j =_{AC} s_i$ with $t_j = h(v'_1, \ldots, v'_p)$ and $\{v_1, \ldots, v_p\} =_{AC} \{v'_1, \ldots, v'_p\}$, and hence for all $k \in \{1 \ldots p\}$ there is some $l \in \{1 \ldots p\}$ satisfying $f(s_1, \ldots, tf_f(v_k), \ldots, s_n) =_{AC} f(t_1, \ldots, tf_f(v'_l), \ldots, t_n)$. ■

LEMMA 7. *$\succ_p$ is AC-compatible.*

Proof. Since after flattening AC-equal terms are equal up to permutation of arguments of AC-symbols, we have to prove the compatibility of $\succ_p$ wrt this permutative equality, which we also call $=_{AC}$. Then $s' =_{AC} s \succ_p t =_{AC} t'$ implies $s' \succ_p t'$. We proceed by induction on $|s| + |t|$ and case analysis on the proof of $s \succ_p t$.

1. If $s = f(s_1, \ldots, s_n) \succ_p t$ by Case 1 then there is some $s_i \succeq_p t$, and since $s' =_{AC} s$, there is some $s'_j =_{AC} s_i$, and by induction hypothesis $s'_j \succeq_p t'$, and hence $s' \succ_p t'$ by Case 1.

2. If $s \succ_p g(t_1, \ldots, t_m) = t$ by Case 2 then $top(s) \succ_{\mathcal{F}} g$ and $s \succ_p t_i$ for all $i \in \{1 \ldots m\}$, and, since $t =_{AC} t'$, we have $t' = g(t'_1, \ldots, t'_m)$ and for all $j \in \{1 \ldots m\}$ there is some $i \in \{1 \ldots m\}$ s.t. $t_i =_{AC} t'_j$. Therefore, by induction hypothesis $s' \succ_p t'_j$ for all $j \in \{1 \ldots m\}$ and $top(s') = top(s) \succ_{\mathcal{F}} g$, which implies $s' \succ_p g(t'_1, \ldots, t'_m) = t'$ by Case 2.

3. If $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by Case 3 then $f \notin \mathcal{F}_{AC}$ and $\langle s_1, \ldots, s_n \rangle (\succ_p)_{lex} \langle t_1, \ldots, t_m \rangle$ and $s \succ_p t_i$ for all $i \in \{1 \ldots m\}$. Since $s =_{AC} s'$ and $t =_{AC} t'$, we have $s' = f(s'_1, \ldots, s'_n)$ with $s_i =_{AC} s'_i$ for all $i \in \{1 \ldots n\}$ and $t' = f(t'_1, \ldots, t'_m)$ with $t_i =_{AC} t'_i$ for all $i \in \{1 \ldots m\}$. Therefore, by induction hypothesis, $\langle s'_1, \ldots, s'_n \rangle (\succ_p)_{lex} \langle t'_1, \ldots, t'_m \rangle$ and $s' \succ_p t'_i$ for all $i \in \{1 \ldots m\}$, which implies $s' \succ_p t'$ by Case 3.

4. If $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by Case 4 then $f \notin \mathcal{F}_{AC}$ and $\{s_1, \ldots, s_n\} \ggcurly_p \{t_1, \ldots, t_m\}$. Then, by induction hypothesis, $\{s_1, \ldots, s_n\} =_{AC} \{s'_1, \ldots, s'_n\} \ggcurly_p \{t'_1, \ldots, t'_m\} =_{AC} \{t_1, \ldots, t_m\}$, which implies, by Case 4, $s' = f(s'_1, \ldots, s'_n) \succ_p f(t'_1, \ldots, t'_m) = t'$.

5. If $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by Case 5 then $f \in \mathcal{F}_{AC}$ and there is some $u \in EmbNoBig(s)$ s.t. $u \succeq_p t$ then, by Lemma 6, there is some $u' \in EmbNoBig(s')$ s.t. $u =_{AC} u'$ and hence, by induction hypothesis $u' \succeq_p t'$, which implies $s' \succ_p t'$ by Case 5.

6. If $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by Case 6 then $f \in \mathcal{F}_{AC}$ and $s \succ_p u$ for all $u \in EmbNoBig(t)$ and $NoSmallHead(s) \succeq\!\!\succeq_{pf} NoSmallHead(t)$. Since $s =_{AC} s'$ and $t =_{AC} t'$, we have $s' = f(s'_1, \ldots, s'_n)$ with $\{s_1, \ldots, s_n\} =_{AC} \{s'_1, \ldots, s'_n\}$ and $t' = f(t'_1, \ldots, t'_m)$ with $\{t_1, \ldots, t_m\} =_{AC} \{t'_1, \ldots, t'_m\}$.

By Lemma 6, we have $EmbNoBig(t) =_{AC} EmbNoBig(t')$, $NoSmallHead(s) =_{AC} NoSmallHead(s')$ and $NoSmallHead(t) =_{AC} NoSmallHead(t')$, and by induction hypothesis, we have that $s' \succ_p u'$ for all $u' \in EmbNoBig(t')$ and $NoSmallHead(s') \succeq\!\!\succeq_{pf} NoSmallHead(t')$.

Finally, if $s \succ_p t$ by Case 6a then, since, by Lemma 6, $BigHead(s') =_{AC} BigHead(s) \ggcurly_p BigHead(t) =_{AC} BigHead(t')$, by induction hypothesis, we have $BigHead(s') \ggcurly_p BigHead(t')$, and hence $s' \succ_p t'$ by Case 6a. If $s \succ_p t$ by Case 6b, then, since, by Lemma 6, $\#(s') = \#(s) > \#(t) = \#(t')$, we have $s' \succ_p t'$ by Case 6b. Otherwise, $s \succ_p t$ by Case 6c, and $\#(s') = \#(s) \geq \#(t) = \#(t')$ and $\{s'_1, \ldots, s'_n\} =_{AC} \{s_1, \ldots, s_n\} \ggcurly_p \{t_1, \ldots, t_m\} =_{AC} \{t'_1, \ldots, t'_m\}$, which implies by induction hypothesis that $\{s'_1, \ldots, s'_n\} \ggcurly_p \{t'_1, \ldots, t'_n\}$, and hence $s' \succ_p t'$ by Case 6c. ■

LEMMA 8. *Let f be an AC-symbol. If $n > m$ and $1 \leq i_1 < \cdots < i_m \leq m$ then $f(s_1, \ldots, s_n) \succ_p f(s_{i_1}, \ldots, s_{i_m})$.*

Proof. Let s be $f(s_1, \ldots, s_n)$ and let t be $f(s_{i_1}, \ldots, s_{i_m})$. We proceed by induction on $|s| + |t|$.

First we have that for all $t' \in EmbNoBig(f(s_{i_1}, \ldots, s_{i_m}))$ of the form $f(s_{i_1}, \ldots, tf_f(v_j), \ldots, s_{i_m})$ there is some $s' \in EmbNoBig(f(s_1, \ldots, s_n))$ of the form $f(s_1, \ldots, tf_f(v_j), \ldots, s_n)$, s.t. by induction hypothesis, $s' \succ_p t'$ and, hence, by Case 5, we have $s \succ_p t'$. Then, since $NoSmallHead(s) \supseteq NoSmallHead(t)$ and $\#(s) > \#(t)$, by Case 6b, $s \succ_p t$ holds. ■

LEMMA 9. *$\succ_p$ fulfils the deletion property for symbols with unbounded arity.*

Proof. All symbols f with unbounded arity are either in $\mathcal{F}_{AC}$ or in *Mul*. If $f \in \mathcal{F}_{AC}$ we have $f(\ldots s \ldots) \succ_p f(\ldots \ldots)$ by Lemma 8. Otherwise, since $f \in Mul$ and $\{\ldots s \ldots\} \ggcurly_p \{\ldots \ldots\}$ we have $f(\ldots s \ldots) \succ_p f(\ldots \ldots)$ by Case 4. ■

LEMMA 10. *Let f be an AC-symbol. If $n > m$ and $1 \leq i_1 < \cdots < i_m \leq m$ then $s \succeq_p f(t_1, \ldots, t_n)$ implies $s \succ_p f(t_{i_1}, \ldots, t_{i_m})$.*

Proof. We proceed by induction on $|s|$. If $s =_{AC} f(t_1, \ldots, t_n)$ then it holds by Lemma 8 and AC-compatibility. Otherwise there are several cases according to the case applied in the proof of $s \succ_p f(t_1, \ldots, t_n)$.

1. If Case 1 applies then $s_i \succeq_p f(t_1, \ldots, t_n)$ and by induction hypothesis $s_i \succ_p f(t_{i_1}, \ldots, t_{i_n})$, which implies $s \succ_p f(t_{i_1}, \ldots, t_{i_n})$ by Case 1.

2. If Case 2 applies then trivially $s \succ_p f(t_{i_1}, \ldots, t_{i_n})$ by Case 2.

3. If Case 5 applies then $s = f(s_1, \ldots, s_k)$ and there is some $s' \in \mathit{EmbNoBig}(s)$ s.t. $s' \succeq_p f(t_1, \ldots, t_n)$. Since, by induction hypothesis, $s' \succ_p f(t_{i_1}, \ldots, t_{i_m})$, we have $s \succ_p f(t_{i_1}, \ldots, t_{i_m})$ by Case 5.

4. If Case 6 applies, we have that for all $t'' \in \mathit{EmbNoBig}(f(t_{i_1}, \ldots, t_{i_m}))$ of the form $f(t_{i_1}, \ldots, \mathit{tf}_f(v_j), \ldots, t_{i_m})$ there is some $t' \in \mathit{EmbNoBig}(f(t_1, \ldots, t_n))$ of the form $f(t_1, \ldots, \mathit{tf}_f(v_j), \ldots, t_n)$, and therefore, since $s \succ_p t'$ for all $t' \in \mathit{EmbNoBig}(f(t_1, \ldots, t_n))$, by induction hypothesis, we have $s \succ_p t''$ for every $t'' \in \mathit{EmbNoBig}(f(t_{i_1}, \ldots, t_{i_m}))$.

On the other hand, $\mathit{NoSmallHead}(s) \succeq\!\succeq_{pf} \mathit{NoSmallHead}(f(t_1, \ldots, t_n))$ implies $\mathit{NoSmallHead}(s) \succeq\!\succeq_{pf} \mathit{NoSmallHead}(f(t_{i_1}, \ldots, t_{i_m}))$.

Now if Case 6a applies then we have $\mathit{BigHead}(s) \succ\!\succ_p \mathit{BigHead}(f(t_{i_1}, \ldots, t_{i_m}))$, and hence $s \succ_p f(t_{i_1}, \ldots, t_{i_n})$ by Case 6a. Otherwise, if cases 6b or 6c apply then, since $n > m$ we have $\#(s) \geq \#(t) > \#(f(t_{i_1}, \ldots, t_{i_n}))$ and we conclude by Case 6b. ■

LEMMA 11. *If $s \succeq_p t$ then $s \succ_p t_i$, for all t_i arguments of t.*

Proof. By induction on the $|s| + |t|$. Let s be $f(s_1, \ldots, s_n)$ and t be $g(t_1, \ldots, t_m)$. If $s =_{AC} t$ then for every t_i there is some s_j s.t. $s_j =_{AC} t_i$ and therefore by Case 1 we have $s \succ_p t_i$. Otherwise $s \succ_p t$ and we distinguish several cases according to the definition.

1. If $s \succ_p t$ by Case 1 then $s_j \succeq_p t$ for some $j \in \{1 \ldots n\}$, and by induction hypothesis $s_j \succ_p t_i$ which implies by Case 1 $s \succ_p t_i$.

2. If $s \succ_p t$ by Case 2 or Case 3 then $s \succ_p t_i$ by definition.

3. If $s \succ_p t$ by Case 4 then, since $\{s_1, \ldots, s_n\} \succ\!\succ_p \{t_1, \ldots, t_m\}$, there is some $s_j \succeq_p t_i$ and hence $s \succ_p t_i$ by Case 1.

4. If $s \succ_p t$ by Case 5 then there is some $s' \in \mathit{EmbNoBig}(s)$ s.t. $s' \succeq_p t$. By induction hypothesis $s' \succ_p t_i$. Now we prove, by a second induction on $|t_i|$, that if $s' \succ_p t_i$ then $s \succ_p t_i$. Let t_i be $h(v_1, \ldots, v_r)$. If $h = f$ then it holds by Case 5. If $f > h$, then since $s' \succ_p t_i$ by the outer induction hypothesis $s' \succ_p v_j$ for all $j \in \{1 \ldots r\}$, and hence by the inner induction hypothesis $s \succ_p v_j$ and hence $s \succ_p t_i$ by Case 2. Otherwise, $s' \succ_p t_i$, by Case 1, and hence by Case 1 (maybe applied more than once), $s \succ_p t_i$.

5. If $s \succ_p t$ by Case 6 then there are three cases

   (a) If $f \succ_{\mathcal{F}} \mathit{top}(t_i)$ and t_i is a constant, it holds trivially, by Case 2.

   (b) If $f \succ_{\mathcal{F}} \mathit{top}(t_i)$ and $t_i = h(v_1, \ldots, v_r)$ with $r > 0$, then for all $j \in \{1 \ldots r\}$ there is some $t' \in \mathit{EmbNoBig}(t)$ s.t. $t' = f(t_1, \ldots, \mathit{tf}_f(v_j), \ldots, t_m)$ and $s \succ_p t'$. Now, for all v_j either $\mathit{top}(v_j) \neq f$ and $\mathit{tf}_f(v_j) = v_j$ and hence by induction hypothesis $s \succ_p v_j$, or $\mathit{top}(v_j) = f$, and then by Lemma 10 we have $s \succ_p f(\mathit{tf}_f(v_j)) = v_j$. Therefore, since $f \succ_{\mathcal{F}} h$, by Case 2, we have $s \succ_p t_i$.

   (c) If $f \not\succ_{\mathcal{F}} \mathit{top}(t_i)$ then, since $\mathit{NoSmallHead}(s) \succeq\!\succeq_{pf} \mathit{NoSmallHead}(t)$, there is some s_j (with $f \not\succ_{\mathcal{F}} \mathit{top}(s_j)$) s.t. $s_j \succeq_p t_i$ and hence $s \succ_p t_i$ by Case 1. ■

COROLLARY 6. *$\succ_p$ fulfils the subterm property.*

LEMMA 12. *Let s and t be terms. Then $\mathit{NoSmallHead}(s) \succeq\!\succeq_{pf} \mathit{NoSmallHead}(t)$ implies $\mathit{BigHead}(s) \succeq\!\succeq_p \mathit{BigHead}(t)$.*

Proof. It follows directly from Lemma 1. ■

LEMMA 13. *$\succ_p$ is transitive.*

Proof. We prove $s \succ_p t$ and $t \succ_p u$ implies $s \succ_p u$ by induction on $|s| + |t| + |u|$ and case analysis on the definition. Let s be $f_1(s_1, \ldots, s_n)$, t be $f_2(t_1, \ldots, t_m)$, and u be $f_3(u_1, \ldots, u_p)$.

1. If $s \succ_p t$ by Case 1 then there is some $s_i \succeq_p t$ and by induction hypothesis and AC-compatibility $s_i \succ_p u$, which implies $s \succ_p u$ by Case 1.

2. If $t \succ_p u$ by Case 1 then there is some $t_i \succeq_p u$. By Lemma 11 we have $s \succ_p t_i$ and by induction hypothesis and AC-compatibility $s \succ_p u$.

3. If $s \succ_p t$ by Case 2 and $t \succ_p u$ by Cases 2–6 then $f_1 \succ_{\mathcal{F}} f_3$. By Lemma 11, $t \succ_p u_i$ for all i, which by induction hypothesis implies $s \succ_p u_i$ for all i, and therefore $s \succ_p u$ by Case 2.

4. If $s \succ_p t$ by Case 3–6 and $t \succ u$ by Case 2, then $f_1 \succ_{\mathcal{F}} f_3$, and we conclude as in the previous case.

5. If both $s \succ_p t$ and $t \succ_p u$ by Case 3 then $\langle s_1, \ldots, s_n\rangle \succ_{lex} \langle t_1, \ldots, t_m\rangle$ and $\langle t_1, \ldots, t_m\rangle \succ_{lex} \langle u_1, \ldots, u_p\rangle$ and $t \succ_p u_i$ for all i, and by induction hypothesis $\langle s_1, \ldots, s_n\rangle \succ_{lex} \langle u_1, \ldots, u_r\rangle$ and $s \succ_p u_i$ for all i, which implies $s \succ_p t$ by Case 3.

6. If both $s \succ_p t$ and $t \succ_p u$ by Case 4 then we have $\{s_1, \ldots, s_n\} \ggcurly_p \{t_1, \ldots, t_m\}$ and $\{t_1, \ldots, t_m\} \ggcurly_p \{u_1, \ldots, u_p\}$, which implies, by induction hypothesis, that $\{s_1, \ldots, s_n\} \ggcurly_p \{u_1, \ldots, u_p\}$, and hence $s \succ_p t$ by Case 4.

7. If $s \succ_p t$ by Case 5 and $t \succ_p u$ by Case 5 or 6, then there is some $s' \in \mathit{EmbNoBig}(s)$ s.t. $s' \succeq_p t$, and by induction hypothesis and AC-compatibility $s' \succ_p u$, which implies $s \succ_p u$ by Case 5.

8. If $s \succ_p t$ by Case 6 and $t \succ_p u$ by Case 5 then there is some $t' \in \mathit{EmbNoBig}(t)$ s.t. $t' \succeq_p u$, and since $s \succ_p t'$ for all $t' \in \mathit{EmbNoBig}(t)$, by induction hypothesis and AC-compatibility we have $s \succ_p u$.

9. If $s \succ_p t$ by Case 6 and $t \succ_p u$ by Case 6 then $f_1 = f_2 = f_3 = f$ and by induction hypothesis and AC-compatibility we have $s \succ_p u'$ for all $u' \in \mathit{EmbNoBig}(u)$ and $\mathit{NoSmallHead}(s) \succeq\!\!\succ_{pf} \mathit{NoSmallHead}(u)$.

Now if either $s \succ_p t$ or $t \succ_p u$ by Case 6a then, by Lemma 12, induction hypothesis, and AC-compatibility we have $s \succ_p u$ by Case 6a. Otherwise, if either $s \succ_p t$ or $t \succ_p u$ by Case 6b then we have $s \succ_p u$ by Case 6b. Otherwise, $s \succ_p t$ and $t \succ_p u$ by Case 6c, and then $\{s_1, \ldots, s_n\} \ggcurly_p \{t_1, \ldots, t_m\}$ and $\{t_1, \ldots, t_m\} \ggcurly_p \{u_1, \ldots, u_p\}$ implies, by induction hypothesis and AC-compatibility, $\{s_1, \ldots, s_n\} \ggcurly_p \{u_1, \ldots, u_p\}$, and hence $s \succ_p u$ by Case 6c. ■

Lemma 14. *$\succ_p$ is irreflexive.*

Proof. We proceed by induction on $|s|$. We show that we cannot apply any case to prove $s \succ_p s$.

1. If we apply Case 1 then we have $s_i \succeq_p s$ then since by Case 1 we have $s \succ_p s_i$, which by transitivity (and AC-compatibility) implies $s_i \succ_p s_i$, which cannot be by induction hypothesis.

2. Cases 2 and 6 cannot trivially apply.

3. If we apply Case 5 then there is some $s' \in \mathit{EmbNoBig}(s)$ s.t. $s' \succeq_p s$, but by Case 5 we have $s \succ_p s'$, which, by transitivity (and AC-compatibility), implies $s' \succ_p s'$, which cannot be by induction hypothesis.

4. Finally, by using the induction hypothesis on the multiset and lexicographic extension of $\succ_p$ applied on subterms, we have that neither Case 3 nor 4 nor 6a nor 6c can be applied. ■

For the proof of monotonicity and stability under substitutions we restrict the set $\mathit{EmbNoBig}(t)$ to have embedding only in a subset of the arguments.

Definition 9. Let t be a term of the form $f(t_1, \ldots, t_m)$ and S be a set of terms. The set $\mathit{EmbNoBig}_S(t)$ is defined as

$$\{f(t_1, \ldots, \mathrm{tf}_f(v_j), \ldots, t_n) \mid t_i = h(v_1, \ldots, v_r) \notin S \wedge h \not\succ_{\mathcal{F}} f \wedge j \in \{1 \ldots r\}\}.$$

Note that $u \in S$ in the previous definition means that there is a term in S which is equal to u up to permutation of the arguments. Note also that we only need to keep in $\mathit{EmbNoBig}_S(t)$ terms which are not equal up to permutation of arguments.

The following lemma shows that in Case 6a some terms in $\mathit{EmbNoBig}(t)$ do not need to be considered. We need to prove first an additional property.

Lemma 15. *Let s and t be terms headed by $f \in \mathcal{F}_{AC}$ and let S be a multiset of terms s.t. $top(u) \not\succeq_{\mathcal{F}} f$ for all $u \in S$. If $\mathit{NoSmallHead}(s) \succeq\!\!\succ_{pf} \mathit{NoSmallHead}(t)$ and $\mathit{BigHead}(s) \ggcurly_p \mathit{BigHead}(t) \cup S$ then $\mathit{NoSmallHead}(s) \succeq\!\!\succ_{pf} \mathit{NoSmallHead}(t) \cup S$.*

Proof. By Lemma 1, we have $BigHead(s) = X \cup \{w_1, \ldots, w_p\}$ and $BigHead(t) \cup S = Y \cup N_1 \cup \cdots \cup N_p$, s.t. $p \geq 1$, $X =_{AC} Y$, $\{w_1, \ldots, w_p\} \cap (N_1 \cup \cdots \cup N_p) = \emptyset$, and $w_j \succ_p w'$ for all $j \in \{1 \ldots p\}$ and $w' \in N_j$. Moreover, since $top(u) \not\succeq_{\mathcal{F}} f$ for all $u \in S$, we have $Y \cap S = \emptyset$.

On the other hand, we have that $NoSmallHead(s) = BigHead(s) \cup T_1$ and $NoSmallHead(t) = BigHead(t) \cup T_2$ where T_1 and T_2 are multisets of terms incomparable with f. Then, $BigHead(s) \ggcurly_p BigHead(t)$ implies $NoSmallHead(s) \neq_{AC} NoSmallHead(t)$ and hence $NoSmallHead(s) \ggcurly_{pf} NoSmallHead(t)$. Therefore, by Lemma 1 we have $NoSmallHead(s) = X' \cup \{w'_1, \ldots, w'_q\}$ and $NoSmallHead(t) = Y' \cup N'_1 \cup \cdots \cup N'_q$, s.t. $q \geq 1$, $X' =_{AC} Y'$, $\{w'_1, \ldots, w'_q\} \cap (N'_1 \cup \cdots \cup N'_q) = \emptyset$, and $w'_j \succ_{pf} w'$ for all $j \in \{1 \ldots q\}$ and $w' \in N'_j$. Moreover, since $T_1 \cap BigHead(t) = \emptyset$ and $T_2 \cap BigHead(s) = \emptyset$, we have $X' = X \cup X''$ with $X'' \subseteq T_1$ and $Y' = Y \cup Y''$ with $Y'' \subseteq T_2$, which implies that there are indexes $1 \leq i_1 < \cdots i_p \leq q$ s.t. $w_k = w'_{i_k}$ for all $k \in \{1 \ldots p\}$. Now, since all $u \in S$ are in some N_j, we have $w_j \succ_p u$ for some w_j, and therefore taking $N''_{i_j} = N'_{i_j} \cup (N_j \cap S)$ for all $j \in \{1 \ldots p\}$ and $N''_j = N'_j$ for all $j \in \{1 \ldots q\} \backslash \{i_1 \ldots i_p\}$, we have $w'_j \succ_{pf} w'$ for all $j \in \{1 \ldots q\}$ and $w' \in N''_j$. Since now $Y' \cup N''_1 \cup \cdots \cup N''_q = NoSmallHead(t) \cup S$, by Lemma 1, we have $NoSmallHead(s) \ggcurly_{pf} NoSmallHead(t) \cup S$. ■

LEMMA 16. *Let s be $f(s_1, \ldots, s_n)$ and t be $f(t_1, \ldots, t_m)$ with $f \in \mathcal{F}_{AC}$. If $BigHead(s) \ggcurly_p BigHead(t) \cup S$ for some set of terms S then $s \succ_p t'$ for all $t' \in EmbNoBig_S(t)$ and $NoSmallHead(s) \ggcurly_{pf} NoSmallHead(t)$ implies $s \succ_p t$.*

Proof. We proceed by induction on $|t|$. To conclude that $s \succ_p t$ by Case 6a, we only need to show that $s \succ_p t' = f(t_1, \ldots, \mathrm{tf}_f(v_j), \ldots, t_m)$ for all $t_i \in S$ with $t_i = h(v_1, \ldots, v_r)$, $h \not\succ_{\mathcal{F}} f$ and $j \in \{1 \ldots r\}$. Let T be the set $\{t_i \mid i \in \{1 \ldots n\} \wedge t_i \in S \wedge t_i = h(v_1, \ldots, v_r) \wedge h \not\succ_{\mathcal{F}} f\}$. Then $EmbNoBig_S(t) = EmbNoBig_T(t)$ and hence we have to consider only all $t_i \in T$.

Let $\mathrm{tf}_f(v_j) = B \cup N \cup E$ where B are the terms headed by a symbol greater than f, N are the terms headed by a symbol incomparable with f, and E are the terms headed by a symbol smaller than f.

Then, since $BigHead(s) \ggcurly_p BigHead(t) \cup T$ and $BigHead(s) \cap T = \emptyset$, by the subterm property and Corollary 3, we have that $BigHead(s) \ggcurly_p BigHead(t) \cup T \cup B \cup N \cup E = BigHead(t') \cup T \cup N \cup E$.

Now, $BigHead(s) \ggcurly_p BigHead(t) \cup T$ and $NoSmallHead(s) \ggcurly_{pf} NoSmallHead(t)$ implies, by Lemma 15, $NoSmallHead(s) \ggcurly_{pf} NoSmallHead(t) \cup T$. Therefore, we have $NoSmallHead(s) \ggcurly_{pf} NoSmallHead(t) \cup \{t_i\}$ and, by the subterm property, $NoSmallHead(s) \ggcurly_{pf} NoSmallHead(t) \cup B \cup N \cup E = NoSmallHead(t') \cup E$, and hence $NoSmallHead(s) \ggcurly_{pf} NoSmallHead(t')$.

Therefore, since $BigHead(s) \ggcurly_p BigHead(t') \cup T \cup N \cup E$, by induction hypothesis $s \succ_p t'$ if $s \succ t''$ for all $t'' \in EmbNoBig_{S'}(t')$ with $S' = T \cup N \cup E$. Now let $t'' \in EmbNoBig_{S'}(t')$ be $f(t_1, \ldots, \mathrm{tf}_f(u_k), \ldots, \mathrm{tf}_f(v_j), \ldots, t_n)$ for some $t_q \notin S'$ with $t_q = g(u_1, \ldots, u_p)$ and $g \not\succ_{\mathcal{F}} f$. Then, since $t_q \notin T$ as well, we have $s \succ_p t''' = f(t_1, \ldots, \mathrm{tf}_f(u_k), \ldots, t_i, \ldots, t_m) \in EmbNoBig_T(t)$, and $t''' \succ_p t''$ by Case 5, which implies $s \succ_p t''$ by transitivity. ■

LEMMA 17. *$\succ_p$ is monotonic.*

Proof. If $s \succ_p t$ then $\overline{f(\ldots s \ldots)} \succ_p \overline{f(\ldots t \ldots)}$ for every flattened context $f(\ldots[\,]\ldots)$, by induction on $|\overline{f(\ldots s \ldots)}| + |\overline{f(\ldots t \ldots)}|$. Note that if the context is not flattened, we can flatten it and then apply the result.

If f is not in $\mathcal{F}_{AC}$ then $\overline{f(\ldots s \ldots)} = f(\ldots s \ldots)$ and $\overline{f(\ldots t \ldots)} = f(\ldots t \ldots)$ and either by Case 3 or by Case 4 it holds.

If f is in $\mathcal{F}_{AC}$ then $cs = \overline{f(\ldots s \ldots)} = f(\ldots \mathit{tf}_f(s) \ldots)$ and $ct = \overline{f(\ldots t \ldots)} = f(\ldots \mathit{tf}_f(t) \ldots)$. Now, let $s = g_1(s_1, \ldots, s_n)$ and $t = g_2(t_1, \ldots, t_m)$; there are several cases to be considered according to the proof of $s \succ_p t$ and the head symbols of s and t.

If $s \succ_p t$ by Case 1 and $g_1 = f$ then $\mathit{tf}_f(s) = s_1 \ldots s_n$ and $s_i \succeq_p t$ for some s_i, and by induction hypothesis $f(\ldots \mathit{tf}_f(s_i) \ldots) \succeq_p f(\ldots \mathit{tf}_f(t) \ldots)$. By Lemma 8 we have $f(\ldots s_1 \ldots s_n \ldots) \succ_p f(\ldots s_i \ldots)$ and, since $\mathit{tf}_f(s_i) = s_i$, by transitivity and AC-compatibility, we obtain $cs \succ_p ct$.

If $s \succ_p t$ by Case 1 and $g_1 \not\succeq_{\mathcal{F}} f$ then $\mathit{tf}_f(s) = s$ and $s_i \succeq_p t$ for some s_i, and by induction hypothesis $s' = f(\ldots \mathit{tf}_f(s_i) \ldots) \succeq_p f(\ldots \mathit{tf}_f(t) \ldots)$. Since $g_1 \not\succeq_{\mathcal{F}} f$, we have $s' \in EmbNoBig(cs)$, and hence $cs \succ_p ct$ by Case 5.

Otherwise, either $g_1 \succ_{\mathcal{F}} f$ or $s \succ_p t$ by Cases 2–6. Let $u_1, \ldots, u_p$ be the arguments of the context.

First we prove that $cs \succ_p t'$ for all terms $t' \in EmbNoBig_S(ct)$ where S is the set $\{v \mid v \in tf_f(t)\}$. Then, for every $t' \in EmbNoBig_S(ct)$ there is some $i \in \{1, \ldots, p\}$ s.t. $u_i = h(v_1, \ldots, v_r)$ with $h \not\succ_{\mathcal{F}} f$ and some $j \in \{1, \ldots, r\}$ s.t. $t' = f(\ldots tf_f(v_j) \ldots tf_f(t) \ldots)$, and hence there is some $s' = f(\ldots tf_f(v_j) \ldots tf_f(s) \ldots) \in EmbNoBig(cs)$ s.t., by induction hypothesis, $s' \succ_p t'$, which implies $cs \succ_p t'$ by Case 5.

Now, since either $g_1 \succ_{\mathcal{F}} f$ or $s \succ_p t$ by Cases 2–6, we have the following four cases:

1. $g_1 \succ_{\mathcal{F}} f$. Then since $s \succ_p t$ (by Lemma 11 if $g_2 = f$) we have $s \succ_p v$ for all $v \in tf_f(t)$ and hence $\{s\} \ggg_p \{tf_f(t), tf_f(t)\}$. Therefore, if X is the multiset of the terms headed by a symbol greater than f in $u_1, \ldots, u_p$ then $BigHead(cs) = X \cup \{s\} \ggg_p X \cup \{tf_f(t), tf_f(t)\} \supseteq BigHead(ct) \cup \{tf_f(t)\}$, and, by Lemma 16, we only need $NoSmallHead(cs) \succeq_{pf} NoSmallHead(ct)$ and $cs \succ_p t'$ for all $t' \in EmbNoBig_S(t)$, where S is the set $\{v \mid v \in tf_f(t)\}$ to conclude $cs \succ_p ct$. The latter has already been proved. For the former, since $\{s\} \ggg_{pf} \{tf_f(t)\}$, if Y is the multiset of the terms headed by a symbol not smaller than f in $u_1, \ldots, u_p$ then we have $NoSmallHead(cs) = Y \cup \{s\} \ggg_{pf} Y \cup \{tf_f(t)\} \supseteq NoSmallHead(ct)$.

2. $g_1 = g_2 = f$ and $s \succ_p t$ by Case 5. Then there is some $s' \in EmbNoBig(s)$ with $s' = f(s_1, \ldots, tf_f(v_j), \ldots, s_n)$ for some $s_i = h(v_1, \ldots, v_r)$ and $h \not\succ_{\mathcal{F}} f$ s.t. $s' \succeq_p t$. Then, by induction hypothesis we have $f(\ldots tf_f(s') \ldots) \succeq_p ct$, and since $f(\ldots tf_f(s') \ldots) = f(\ldots, s_1, \ldots, tf_f(v_j), \ldots, s_n, \ldots) \in EmbNoBig(cs)$, we conclude $cs \succ_p ct$ by Case 5.

3. $g_1 = g_2 = f$ and $s \succ_p t$ by Case 6. Then $tf_f(t) = t_1 \ldots t_m$. As we have proved, $cs \succ_p t'$ for all $t' \in EmbNoBig_S(ct)$, with $S = \{t_1, \ldots, t_m\}$, in this case. For the other t' in $EmbNoBig(ct)$, we have $t' = f(\ldots, t_1, \ldots, tf_f(v_j), \ldots, t_m, \ldots)$ for some $t_i = h(v_1, \ldots, v_r)$ with $h \not\succ_{\mathcal{F}} f$. Since, $f(t_1, \ldots, tf_f(v_j), \ldots, t_m) \in EmbNoBig(t)$, and $s \succ_p w$ for every $w \in EmbNoBig(t)$, we have that $s \succ_p f(t_1, \ldots, tf_f(v_j), \ldots, t_m)$ which, by induction hypothesis, implies that $cs = f(\ldots, tf_f(s), \ldots) \succ_p f(\ldots, tf_f(w), \ldots) = t'$.

   On the other hand, if X is the multiset with the terms headed by a symbol not smaller than f in $u_1, \ldots, u_p$ then we have $NoSmallHead(cs) = X \cup NoSmallHead(s) \succeq_{pf} X \cup NoSmallHead(t) = NoSmallHead(ct)$.

   Finally, if $s \succ_p t$ by Case 6, we have $BigHead(cs) = Y \cup BigHead(s) \ggg_p Y \cup BigHead(t) = BigHead(ct)$, where Y is the multiset with the terms headed by a symbol greater than f in $u_1, \ldots, u_p$, and hence $cs \succ_p ct$ by Case 6a. Otherwise, if $s \succ_p t$ by Case 6b, we have $\#(cs) = e + \#(s) > e + \#(t) = \#(ct)$, where e is the expression coming from the $u_1, \ldots, u_p$, and hence $cs \succ_p ct$ by Case 6b. Otherwise, $s \succ_p t$ by Case 6c, and we have $\#(cs) = e + \#(s) \geq e + \#(t) = \#(ct)$, where e is as before, and $\{s_1, \ldots, s_n\} \ggg_p \{t_1, \ldots, t_m\}$, which implies $\{\ldots, s_1, \ldots, s_n, \ldots\} \ggg_p \{\ldots, t_1, \ldots, t_m, \ldots\}$, and therefore $cs \succ_p ct$ by Case 6c.

4. $g_1 \not\preceq_{\mathcal{F}} f$, $g_1 \succeq_{\mathcal{F}} g_2$. Then we have $g_2 \not\preceq_{\mathcal{F}} f$ as well. Again, since we have proved $cs \succ_p t'$ for all $t' \in EmbNoBig_S(ct)$, with $S = \{t\}$ in this case, for the rest of terms t' in $EmbNoBig(ct)$, we have $t' = f(\ldots, tf_f(t_i), \ldots)$ for some $i \in \{1 \ldots m\}$. Since $s \succ_p t$, by Lemma 11, we have $s \succ_p t_i$, and by induction hypothesis $cs = f(\ldots, tf_f(s), \ldots) \succ_p f(\ldots, tf_f(t_i), \ldots) = t'$. Now, let X be the multiset of the terms headed by a symbol not smaller than f in $u_1, \ldots, u_p$. Since $g_1 \succeq_{\mathcal{F}} g_2$, if $f \succ_{\mathcal{F}} g_1$ then $NoSmallHead(cs) = X = NoSmallHead(ct)$, otherwise $NoSmallHead(cs) = X \cup \{s\} \succeq_{pf} X \cup \{t\} \supseteq NoSmallHead(ct)$. Therefore, $NoSmallHead(cs) = \succeq_{pf} NoSmallHead(ct)$, and, since $s \succ_p t$ implies $\{\ldots, s, \ldots\} \ggg_p \{\ldots, t, \ldots\}$, we have $cs \succ_p ct$ by Case 6c. ■

The following properties and lemmas are used in the proof of stability under substitution.

LEMMA 18. *Let $u, s_1, \ldots, s_n$, and $t_1, \ldots, t_m$ be terms and let f be an AC-symbol. If $f(s_1, \ldots, u, \ldots, s_n) \succ_p f(t_1, \ldots, u, \ldots, t_m)$ then we have that $f(s_1, \ldots, v_1, \ldots, v_p, \ldots, s_n) \succ_p f(t_1, \ldots, v_1, \ldots, v_p, \ldots, t_m)$, for all terms $v_1, \ldots, v_p$.*

Proof. We proceed by induction on $|u| + |f(t_1, \ldots, v_1, \ldots, v_p, \ldots, t_m)|$. Then, let s be $f(s_1, \ldots, u, \ldots, s_n)$, t be $f(t_1, \ldots, u, \ldots, t_m)$, s_v be $f(s_1, \ldots, v_1, \ldots, v_p, \ldots, s_n)$, and t_v be $f(t_1, \ldots, v_1, \ldots, v_p, \ldots, t_m)$. If $s \succ_p t$ by Case 1 then using the deletion property, the subterm property, transitivity and monotonicity, it holds. If $s \succ_p t$ by Case 5 then by induction hypothesis (using the subterm property and monotonicity) we can conclude. Otherwise, $s \succ_p t$ by Case 6. We first prove that $s_v \succ_p t'_v$ for every $t'_v \in EmbNoBig(t_v)$. If $t'_v = f(t_1, \ldots, tf_f(u_j), \ldots, v_1, \ldots, v_p, \ldots, t_m)$ for some $t_i = h(u_1, \ldots, u_r)$, $h \not\succ_{\mathcal{F}} f$, and $j \in \{1 \ldots r\}$ then, since by assumption, we have $s \succ_p f(t_1, \ldots, tf_f(u_j), \ldots, u, \ldots, t_m) \in EmbNoBig(t)$, by induction hypothesis, $s_v \succ_p t'_v$. Otherwise, we have $t'_v = f(t_1, \ldots, v_1, \ldots, tf_f(u_j),$

$\ldots, v_p, \ldots, t_m)$, for some $v_i = h(u_1, \ldots, u_r)$, $h \not\succ_{\mathcal{F}} f$, and $j \in \{1 \ldots r\}$, and, by induction hypothesis, $f(s_1, \ldots, v_1, \ldots, tf_f(u_j), \ldots, v_p, \ldots, s_n) \succ_p t'_v$, which implies $s_v \succ_p t'_v$ by Case 5.

Second, we show that $NoSmallHead(s_v) \succeq\!\!\succeq_{pf} NoSmallHead(t_v)$. Since we have $NoSmallHead(s) \succeq\!\!\succeq_{pf} NoSmallHead(t)$, by Corollary 2, we have $NoSmallHead(s_v) = (NoSmallHead(s)\backslash\{u\}) \cup V_N \succeq\!\!\succeq_{pf} (NoSmallHead(t)\backslash\{u\}) \cup V_N = NoSmallHead(t_v)$, where $V_N = \{v_i \mid i \in \{1 \ldots r\} \wedge f \not\succ_{\mathcal{F}} top(v_i)\}$.

Now if $s \succ_p t$ by Case 6a then, by Corollary 22, $BigHead(s) \gg_p BigHead(t)$ implies $BigHead(s_v) = (BigHead(s)\backslash\{u\}) \cup V_B \gg_p (BigHead(t)\backslash\{u\}) \cup V_B = BigHead(t_v)$, where $V_B = \{v_i \mid i \in \{1 \ldots r\} \wedge top(v_i) \succ_{\mathcal{F}} f\}$, and hence, $s_v \succ_p t_v$ by Case 6a. Otherwise, if $s \succ_p t$ by Case 6b then $\#(s) > \#(t)$ implies $\#(s_v) > \#(t_v)$ and hence $s_v \succ_p t_v$ by Case 6b. Finally, if $s \succ_p t$ by Case 6c, then $\#(s) \geq \#(t)$ implies $\#(s_v) \geq \#(t_v)$, and, by Corollary 2, $\{s_1, \ldots, u, \ldots, s_n\} \gg_p \{t_1, \ldots, u, \ldots, t_m\}$ implies $\{s_1, \ldots, v_1, \ldots, v_p, \ldots, s_n\} \gg_p \{t_1, \ldots, v_1, \ldots, v_p, \ldots, t_m\}$, and hence $s_v \succ_p t_v$ by Case 6c. ■

LEMMA 19. *Let s and t be terms and let x be a variable s.t. $x \notin NoSmallHead(s)$ and $x \in NoSmallHead(t)$. Then $NoSmallHead(s) \succeq\!\!\succeq_{pf} NoSmallHead(t)$ implies $BigHead(s) \gg_p BigHead(t) \cup \{x, \ldots, x\}$.*

Proof. It follows directly from Lemma 1. ■

LEMMA 20. *Let σ be $\{x \mapsto q(y_1, \ldots, y_k)\}$ for some symbol q. If $s \succ_p t$ then $\overline{s\sigma} \succ_p \overline{t\sigma}$.*

Proof. The proof is done by induction on $|\overline{s\sigma} + \overline{t\sigma}|$ and case analysis on the proof of $s \succ_p t$. Let s be $f(s_1, \ldots, s_n)$.

1. $s \succ_p t$ by Case 1. Then $s_i \succeq_p t$ for some $i \in \{1 \ldots n\}$. If $f \in \mathcal{F}_{AC}$, $s_i = x$ and $q = f$ then $t = x$ and, since the arity of f is greater than 1, by Lemma 8, $\overline{s\sigma} = f(\ldots, y_1, \ldots, y_k, \ldots) \succ_p f(y_1, \ldots, y_k) = \overline{t\sigma}$. Otherwise, $\overline{s\sigma} = f(\ldots, \overline{s_i\sigma}, \ldots) \succ_p \overline{t\sigma}$, by induction hypothesis and Case 1.

2. $s \succ_p t = g(t_1, \ldots, t_m)$ by Case 2. Then $f \succ_{\mathcal{F}} g$ and $s \succ_p t_i$ for all $i \in \{1 \ldots m\}$. By induction hypothesis, we have $\overline{s\sigma} \succ_p \overline{t_i\sigma}$, and if $g \in \mathcal{F}_{AC}$, $t_i = x$ and $g = q$, then, by Lemma 11, $\overline{s\sigma} \succ_p y_j$ for all $j \in \{1 \ldots k\}$. Therefore, since $\overline{t\sigma}$ is of the form $g(t'_1, \ldots, t'_p)$ and $f \succ_{\mathcal{F}} g$ and $\overline{s\sigma} \succ_p t'_i$ for all $i \in \{1 \ldots p\}$, we have $\overline{s\sigma} \succ_p \overline{t\sigma}$, by Case 2.

3. $s \succ_p t = f(t_1, \ldots, t_m)$ by Case 3. Then $\langle s_1, \ldots, s_n\rangle(\succ_p)_{lex}\langle t_1, \ldots, t_m\rangle$ and $s \succ_p t_i$ for all $i \in \{1 \ldots m\}$ (and also $f \notin \mathcal{F}_{AC}$). By induction hypothesis, we have $\overline{s\sigma} \succ_p \overline{t_i\sigma}$ for all $i \in \{1 \ldots m\}$ and $\langle\overline{s_1\sigma}, \ldots, \overline{s_n\sigma}\rangle(\succ_p)_{lex}\langle\overline{t_1\sigma}, \ldots, \overline{t_m\sigma}\rangle$, and hence $\overline{s\sigma} = f(\overline{s_1\sigma}, \ldots, \overline{s_n\sigma}) \succ_p f(\overline{t_1\sigma}, \ldots, \overline{t_m\sigma}) = \overline{t\sigma}$ by Case 3.

4. $s \succ_p t = f(t_1, \ldots, t_m)$ by Case 4. Then $\{s_1, \ldots, s_n\} \gg_p \{t_1, \ldots, t_m\}$ (and $f \notin \mathcal{F}_{AC}$). By induction hypothesis, we have $\{\overline{s_1\sigma}, \ldots, \overline{s_n\sigma}\} \gg_p \{\overline{t_1\sigma}, \ldots, \overline{t_m\sigma}\}$, and hence $\overline{s\sigma} = f(\overline{s_1\sigma}, \ldots, \overline{s_n\sigma}) \succ_p f(\overline{t_1\sigma}, \ldots, \overline{t_m\sigma}) = \overline{t\sigma}$ by Case 4.

5. $s \succ_p t = f(t_1, \ldots, t_m)$ by Case 5. Then there is some $s' \in EmbNoBig(s)$ with $s' = f(s_1, \ldots, tf_f(v_j), \ldots, s_n)$ for some $s_i = h(v_1, \ldots, v_r)$ and $h \not\succ_{\mathcal{F}} f$ s.t. $s' \succeq_p t$. By induction hypothesis, we have $\overline{s'\sigma} \succeq_p \overline{t\sigma}$. If $h \in \mathcal{F}_{AC}$, $v_j = x$ and $h = q$ then, by Lemma 8, $\overline{s_i\sigma} = h(\ldots, y_1, \ldots, y_k, \ldots) \succ_p h(y_1, \ldots, y_k) = x\sigma$, and by monotonicity $\overline{s\sigma} \succ_p \overline{s'\sigma}$ and hence by transitivity $\overline{s\sigma} \succ_p \overline{t\sigma}$. Otherwise, $\overline{s'\sigma} \in EmbNoBig(\overline{s\sigma})$, and hence $\overline{s\sigma} \succ_p \overline{t\sigma}$ by Case 5.

6. $s \succ_p t = f(t_1, \ldots, t_m)$ by Case 6. For this case we proceed by a second induction on the number of arguments of s that are x. There are two new cases.

If there is some $i \in \{1 \ldots n\}$ and some $j \in \{1 \ldots m\}$ s.t. $s_i = t_j = x$, then by Lemma 18 we have that $s \succ_p t$ implies $s' = f(s_1, \ldots, x\sigma, \ldots, s_n) \succ_p f(t_1, \ldots, x\sigma, \ldots, t_m) = t'$, and hence since $|\overline{s\sigma} + \overline{t\sigma}| = |\overline{s'\sigma} + \overline{t'\sigma}|$ and s' has less arguments that are x than s, by the second induction we have $\overline{s\sigma} = \overline{s'\sigma} \succ_p \overline{t'\sigma} = \overline{t\sigma}$.

Otherwise, we first prove that $\overline{s\sigma} \succ_p w$ for all $w \in EmbNoBig(\overline{t\sigma})$ if no $t_j = x$ and for all $w \in EmbNoBig_S(\overline{t\sigma})$ where $S = \{x\sigma\}$, otherwise. We have that $\overline{t\sigma} = f(tf_f(\overline{t_1\sigma}), \ldots, tf_f(\overline{t_n\sigma}))$ and $w = f(tf_f(\overline{t_1\sigma}), \ldots, tf_f(v'_l), \ldots, tf_f(\overline{t_n\sigma}))$ for some $t_j = h(v_1, \ldots, v_r)$ and $h \not\succ_{\mathcal{F}} f$ s.t. $tf_f(\overline{t_j\sigma}) = \overline{t_j\sigma} = h(v'_1, \ldots, v'_p)$ and either $v'_l = \overline{v_k\sigma}$ for some $k \in \{1 \ldots r\}$ or $v'_l \in tf_h(\overline{v_k\sigma})$ for some $k \in \{1 \ldots r\}$ s.t. $v_k = x$ and $h = q \in \mathcal{F}_{AC}$. Then, by assumption, we have $s \succ_p t' = f(t_1, \ldots, tf_f(v_k), \ldots, t_m) \in EmbNoBig(t)$, and by induction hypothesis $\overline{s\sigma} \succ_p \overline{t'\sigma} = f(tf_f(\overline{t_1\sigma}), \ldots, tf_f(\overline{v_k\sigma}), \ldots, tf_f(\overline{t_n\sigma}))$. If $v'_l = \overline{v_k\sigma}$ then we are done since $\overline{t'\sigma} = w$, and otherwise, we have $\overline{t'\sigma} \succ_p w$ by Case 5 since $w \in EmbNoBig(\overline{t'\sigma})$, and hence $\overline{s\sigma} \succ_p w$ holds by transitivity.

Finally there are two cases.

(a) If no $s_i = x$ and some $t_j = x$ then we have $x \in NoSmallHead(t)$ and $x \notin NoSmallHead(s)$ and, since $NoSmallHead(s) \succeq_{pf} NoSmallHead(t)$, by Lemma 19, $BigHead(s) \gg_p BigHead(t) \cup X \cup \{x\}$ where X is the multiset containing all x in $NoSmallHead(t)$, and, by induction hypothesis, we have $BigHead(\overline{s\sigma}) = \overline{BigHead(s)\sigma} \gg_p \overline{BigHead(t)\sigma} \cup X\sigma \cup \{x\sigma\} \supseteq BigHead(\overline{t\sigma}) \cup \{x\sigma\}$. Therefore, by Lemma 16, we need to show that $\overline{s\sigma} \succ_p w$ for all $w \in EmbNoBig_S(\overline{t\sigma})$ where $S = \{x\sigma\}$, which has already been proved, and $NoSmallHead(\overline{s\sigma}) \succeq_{pf} NoSmallHead(\overline{t\sigma})$. By induction hypothesis, it holds that $NoSmallHead(\overline{s\sigma}) = \overline{NoSmallHead(s)\sigma} \succeq_{pf} \overline{NoSmallHead(t)\sigma}$. If $q \neq f$ then $\overline{NoSmallHead(t)\sigma} \supseteq NoSmallHead(\overline{t\sigma})$, and we are done. Otherwise, by the subterm property (and $top(x\sigma) = f$), we have $x\sigma \succ_{pf} y$ for all $y \in \{y_1, \ldots, y_k\}$, and since $NoSmallHead(\overline{t\sigma})$ is obtained from $\overline{NoSmallHead(t)\sigma}$ replacing each element $x\sigma$ by $y_1, \ldots, y_k$, we have, by Lemma 1, that $\overline{NoSmallHead(t)\sigma} \gg_{pf} NoSmallHead(\overline{t\sigma})$, and therefore, by transitivity, $NoSmallHead(\overline{s\sigma}) \gg_{pf} NoSmallHead(\overline{t\sigma})$.

(b) If no $t_j = x$ then $\overline{s\sigma} \succ_p w$ for all $w \in EmbNoBig(\overline{t\sigma})$ has already been proved. Now we prove $NoSmallHead(\overline{s\sigma}) \succeq_{pf} NoSmallHead(\overline{t\sigma})$. Since $NoSmallHead(s) \succeq_{pf} NoSmallHead(t)$ and $x \notin NoSmallHead(t)$, by Corollary 4, $NoSmallHead(s) \backslash X \succeq_{pf} NoSmallHead(t)$, where X is the multiset with all elements x in $NoSmallHead(s)$. Then, by induction hypothesis, we have that $\overline{(NoSmallHead(s) \backslash X)\sigma} \succeq_{pf} \overline{NoSmallHead(t)\sigma} = NoSmallHead(\overline{t\sigma})$, and hence $NoSmallHead(\overline{s\sigma}) \succeq_{pf} NoSmallHead(\overline{t\sigma})$, since $NoSmallHead(\overline{s\sigma}) \supseteq \overline{(NoSmallHead(s) \backslash X)\sigma}$.

If $s \succ_p t$ by Case 6a then, since $BigHead(s) \gg_p BigHead(t)$, by induction hypothesis, $BigHead(\overline{s\sigma}) \supseteq \overline{BigHead(s)\sigma} \gg_p \overline{BigHead(t)\sigma} = BigHead(\overline{t\sigma})$, which implies $\overline{s\sigma} \succ_p \overline{t\sigma}$ by Case 6a. If $s \succ_p t$ by Case 6b then, since $\#(s) > \#(t)$ implies $\#(\overline{s\sigma}) > \#(\overline{t\sigma})$, we have $\overline{s\sigma} \succ_p \overline{t\sigma}$ by Case 6b. Finally, if $s \succ_p t$ by Case 6c then $\#(s) \geq \#(t)$. If some $s_j = x$ and $q = f$ then $\#(\overline{s\sigma}) > \#(s) \geq \#(t) = \#(\overline{t\sigma})$ (note that no $t_j = x$), and we conclude by Case 6b. Otherwise, either no $s_j = x$ or $q \neq f$, and then $\#(\overline{s\sigma}) = \#(s) \geq \#(t) = \#(\overline{t\sigma})$ and $\overline{s\sigma} = f(\overline{s_1\sigma}, \ldots, \overline{s_n\sigma})$ and $\overline{t\sigma} = f(\overline{t_1\sigma}, \ldots, \overline{t_m\sigma})$. Then, since by induction hypothesis, $\{\overline{s_1\sigma}, \ldots, \overline{s_n\sigma}\} \gg_p \{\overline{t_1\sigma}, \ldots, \overline{t_m\sigma}\}$, we have $\overline{s\sigma} \succ_p \overline{t\sigma}$ by Case 6c. ■

LEMMA 21. *$\succ_p$ is stable under substitution.*

Proof. We have to prove that if $s \succ_p t$ then $\overline{s\sigma} \succ_p \overline{t\sigma}$ for every substitution σ. We proceed by induction on the $|\sigma|$ defined as the multiset $\{|w| \mid (x \mapsto w) \in \sigma\}$ and compared by the multiset extension of $>$.

If σ is empty it holds trivially. Otherwise $\sigma = \{x \mapsto q(w_1, \ldots, w_k)\} \cup \sigma'$. Then, by Lemma 20, we have $\overline{s\{x \mapsto q(y_1, \ldots, y_k)\}} \succ_p \overline{t\{x \mapsto q(y_1, \ldots, y_k)\}}$ and taking $\gamma = \{y_1 \mapsto w_1, \ldots, y_k \mapsto w_k\} \cup \sigma'$, we have $|\sigma| \gg |\gamma|$, and hence, by induction hypothesis, $\overline{s\sigma} = \overline{(\overline{s\{x \mapsto q(y_1, \ldots, y_k)\}})\gamma} \succ_p \overline{(\overline{t\{x \mapsto q(y_1, \ldots, y_k)\}})\gamma} = \overline{t\sigma}$. ■

LEMMA 22. *$\succ_p$ is incremental.*

Proof. We prove that if $s \succ_p t$ wrt the precedence $\succ_{\mathcal{F}}$ then $s \succ_p t$ wrt the precedence $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ where f' and g' are symbols in $\mathcal{F}$ not related by $\succ_{\mathcal{F}}$. We proceed by induction on $|s| + |t|$ and case analysis on the proof of $s \succ_p t$ with the precedence $\succ_{\mathcal{F}}$.

1. $s = f(s_1, \ldots, s_n) \succ_p t$ by Case 1. Then $s_i \succ_p t$ and by induction hypothesis $s_i \succ_p t$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, which implies $s \succ_p t$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, by Case 1.

2. $s = f(s_1, \ldots, s_n) \succ_p g(t_1, \ldots, t_m) = t$ by Case 2. Then $f \succ_{\mathcal{F}} g$ and $s \succ_p t_i$ for all $i \in \{1 \ldots m\}$. Then, by induction hypothesis, $s \succ_p t_i$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ for all $i \in \{1 \ldots m\}$, and hence $s \succ_p t$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, by Case 2.

3. $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by Case 3. Then $\langle s_1, \ldots, s_n \rangle (\succ_p)_{lex} \langle t_1, \ldots, t_m \rangle$ and $s \succ_p t_i$ for all $i \in \{1 \ldots m\}$, and by induction hypothesis, $s \succ_p t$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, by Case 3.

4. $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by Case 4. Then we have $\{s_1, \ldots, s_n\} \gg_p \{t_1, \ldots, t_m\}$, and by induction hypothesis, $s \succ_p t$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, by Case 4.

5. $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by Case 5. Then there is a term $s' \in EmbNoBig(s)$ s.t. $s' \succeq_p t$. Then by induction hypothesis $s' \succeq_p t$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$. Now, by monotonicity and the subterm property $s \succ_p s'$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, and hence, by transitivity, $s \succ_p t$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$.

6. $s = f(s_1, \ldots, s_n) \succ_p f(t_1, \ldots, t_m) = t$ by Case 6. Then we have $s \succ_p t'$ for all $t' \in$ *EmbNoBig*(t) and *NoSmallHead*$(s) \succeq_{pf}$ *NoSmallHead*(t). First, since *EmbNoBig*(t) wrt $\succ_{\mathcal{F}}$ includes *EmbNoBig*(t) wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, by induction hypothesis $s \succ_p t'$ for all $t' \in$ *EmbNoBig*(t) wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$. Second, *NoSmallHead*$(s) \succeq_{pf}$ *NoSmallHead*(t) wrt $\succ_{\mathcal{F}}$ implies by Lemma 1 that *NoSmallHead*$(s) = X \cup \{s'_1, \ldots, s'_p\}$, *NoSmallHead*$(t) = Y \cup N_1 \cup \cdots \cup N_p$, $p \geq 1$, $X =_{AC} Y$, $\{s'_1, \ldots, s'_p\} \cap_{AC} (N_1 \cup \cdots \cup N_p) = \emptyset$, and $s'_i \succ t'$ and if $top(s'_i) \not\preceq_{\mathcal{F}} f$ then $top(s'_i) \succeq_{\mathcal{F}} top(t')$ for all $i \in \{1 \ldots p\}$ and $t' \in N_i$. Then *NoSmallHead*(s) wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ is $X' \cup \{s'_{i_1}, \ldots, s'_{i_k}\}$ where $X' \subseteq X$ and $1 \leq i_1 < \cdots < i_k \leq p$. Therefore for all $j \in (\{1 \ldots p\} \backslash \{i_1, \ldots, i_k\}$, $top(s'_j) \not\preceq_{\mathcal{F}} f$ wrt $\succ_{\mathcal{F}}$ and $f \succ_{\mathcal{F}} top(s'_j)$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, but then, since $top(s'_i) \succeq_{\mathcal{F}} top(t')$ for all $t' \in N_i$, we have $f \succ_{\mathcal{F}} top(t')$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ for all $t' \in N_i$, and hence N_i is not in *NoSmallHead*(t) wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$. Therefore we have that *NoSmallHead*(t) wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ is included in $Y' \cup N_{i_1} \cup \cdots \cup N_{i_k}$, where $Y' \subseteq Y$ and $Y' =_{AC} X'$ (note that it is inclusion since some terms in $N_{i_1} \cup \cdots \cup N_{i_k}$ may be removed). Then, by Lemma 1, *NoSmallHead*$(s) \succeq_{pf}$ *NoSmallHead*(t) wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$.

Now if we have applied Case 6b it holds trivially and if we have applied Case 6c then it holds by induction hypothesis. If we have applied Case 6a then we have *BigHead*$(s) \gg_p$ *BigHead*(t) wrt $\succ_{\mathcal{F}}$. By Lemma 1 we have *BigHead*$(s) = X \cup \{s'_1, \ldots, s'_p\}$, *BigHead*$(t) = Y \cup N_1 \cup \cdots \cup N_p$, $p \geq 1$, $X =_{AC} Y$, $\{s'_1, \ldots, s'_p\} \cap_{AC} (N_1 \cup \cdots \cup N_p) = \emptyset$ and $s'_i \succ t'$ for all $i \in \{1 \ldots p\}$ and $t' \in N_i$. Since *NoSmallHead*$(s) \succeq_{pf}$ *NoSmallHead*(t), wrt $\succ_{\mathcal{F}}$, by Lemma 1, *NoSmallHead*$(s) = X \cup X' \cup \{s'_1, \ldots, s'_p\} \cup \{s''_1, \ldots, s''_q\}$ and *NoSmallHead*$(t) = Y \cup Y' \cup N_1 \cup \cdots \cup N_p \cup N'_1 \cup \cdots \cup N'_q$, with $X' =_{AC} Y'$ and $top(t') \not\succ_{\mathcal{F}} f$ for all $t' \in Y'$, and $top(s''_i) \not\succ_{\mathcal{F}} f$, $s''_i \succ_p t'$ and $top(s''_i) \succeq_{\mathcal{F}} top(t')$ for all $i \in \{1 \ldots q\}$ and $t' \in N'_i$.

Therefore, if some term $t' \in N'_i$ has $top(t') \succ_{\mathcal{F}} f$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ then $top(s'_i) \succ_{\mathcal{F}} f$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, and hence *BigHead*$(s)$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ is $X \cup X'' \cup \{s'_1, \ldots, s'_n\} \cup \{s''_{i_1}, \ldots, s''_{i_k}\}$ with $X'' \subseteq X'$ and $1 \leq i_1 < \cdots < i_k \leq p$, and *BigHead*$(t)$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ is included in $Y \cup Y'' \cup N_1 \cup \cdots \cup N_p \cup N'_{i_1} \cup \cdots \cup N'_{i_k}$ with $X'' =_{AC} Y''$, we have, by Lemma 1, *BigHead*$(s) \gg_p$ *BigHead*(t) wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$, and hence $s \succ_p t$ wrt $\succ_{\mathcal{F}} \cup \{f' \succ g'\}$ by Case 6a. ■

7. EFFICIENCY AND IMPLEMENTATION

Regarding efficiency of an implementation of the given ordering, the main problem is due to the use of the set *EmbNoBig* in Cases 5 and 6.

The aim of this section is to show several simplifications in the use of this set without changing the ordering.

Let f be an AC-symbol and let t be a term of the form $f(t_1, \ldots, t_n)$ such that $t' = f(t_1, \ldots, \mathrm{tf}_f(v_j), \ldots, t_n) \in$ *EmbNoBig*(t) for some $t_i = h(v_1, \ldots, v_r)$ with $h \not\succ_{\mathcal{F}} f$.

1. We only need to consider t' if v_j is maximal in $v_1, \ldots, v_r$, since, otherwise, by monotonicity, there is a greater term in *EmbNoBig*(t).
2. We only need to consider t' if v_j contains some variable or some symbol g s.t. $f \not\succ_{\mathcal{F}} g$ and $h \not\preceq_{\mathcal{F}} g$, since otherwise this embedding does not help to obtain (even with further embedding in v_j) a multiset greater than *NoSmallHead*(t) or *BigHead*(t) or an expression greater than $\#(t)$, and hence is better to keep t_i.
3. When checking $s \succ_p t$ by Case 6, by Lemma 16, we only need to consider t' if *BigHead*$(s) \not\gg_p$ *BigHead*$(t) \cup \{t_i\}$.
4. When checking $s = f(s_1, \ldots, s_m) \succ_p t$ by Case 6, by Lemma 18, we only need to consider t' if t_i is in the multiset $\{t_1, \ldots, t_n\} \backslash \{s_1, \ldots, s_m\}$.

Note that the first two simplifications apply to both Cases 5 and 6.

We are currently studying other ways to further simplify the set, but more in the spirit of the first two, with the aim of detecting a single relevant term in *EmbNoBig*(t) at least in the ground case with total precedences, which will imply that the ordering can be checked in polynomial time in that case.

8. CONCLUSIONS

We have presented the first fully syntactic AC-compatible *recursive path ordering* (RPO). The ordering is AC-total on ground terms (if the precedence is total and all non-AC-symbols have lexicographic status) and defined uniformly for both ground and nonground terms, as well as for partial precedences, being the first incremental one (note that due to this, we can allow as well signature extensions without any restriction).

8.1. Related Work

Our ordering does not coincide (even for ground terms and total precedences) with any of the ones given in [9], [14], and [10] (with *fcount* abstraction). With the precedence $h \succ_{\mathcal{F}} f \succ_{\mathcal{F}} g \succ_{\mathcal{F}} a$ and $f \in \mathcal{F}_{AC}$ the terms $f(h(a), g(a))$ and $f(g(h(a)), a)$ are compared in a different way (only in our case the first one is greater than the second one). The reason is that in our approach the arguments headed by big symbols are more important than in the others. However, it could be the case that by taking another abstraction function for [10], the orderings coincide. On the other hand we have also found another syntactic definition in which the number of arguments of an AC-symbol is more important than its arguments headed by big symbols, which we believe to coincide with the orderings in [14] and [10] (with *fcount* abstraction). A weakness of this new definition is that it is only monotonic for ground terms, although, in fact, this is not a problem for practical applications.

8.2. More General Precedences

A simple improvement within the ordering is obtained by allowing the precedence to equate some symbols. Equating non-AC-symbols trivially holds, provided that equal symbols have the same status.

If we want to equate AC-symbols among themselves and with non-AC-symbols, then we have to extend the flattening by normalizing with the new rule

$$f(x_1, \ldots, x_n, g(y_1, \ldots, y_m), z_1, \ldots, z_r) \to f(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_r)$$

for every $f, g \in \mathit{Flat}$, $f \sim_{\mathcal{F}} g$, and $n, m, r \geq 0$, where *Flat* is a new status, that includes at least all AC-symbols and represents all symbols that are flattened. Additionally all symbols that are in *Flat* are treated by the ordering as being AC, i.e., applying Cases 5 and 6. The only restriction is that, as before, equal symbols must have the same status and that all flattened symbols must have arity greater than or equal to 2, otherwise we cannot prove the subterm property or monotonicity.

Finally it is easy to prove that the ordering can deal with infinite signatures, provided that the precedence is well-founded.

8.3. Constraint Solving

As a future development, due to its simplicity and, mainly, the fact that it is not interpretation based, it opens the door to finding practically feasible ordering constraint solvers for the AC-case [4].

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