CS 576 Lecture 9 2/14/17

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1. A Variant-Based $E \cup B$-unification (Semi-) Algorithm

By the Completeness Theorem of Folding Variant Narrowing (Lecture 8, pgs. 14-15), we know that if $(\Sigma, B, \vec{E})$ is a decomposition of $(\Sigma, E \cup B)$ and B has a finitary B-unification algorithm, the set

$$\langle\langle t \rangle\rangle_{\vec{E},B} = \{ (u, \eta|_{vars(t)}) \mid (t \overset{\eta}{\leadsto}{}^{*} u) \in FVN_{\vec{E},B}(t) \wedge u = u!_{\vec{E},B} \}$$

is a complete set of $\vec{E},B$-variants for any term $t \in T_\Sigma(X)$.

In particular since, by the Lemma in pg. 9 of Lecture 6, we then also have that $(\Sigma^{\equiv}, B, \vec{E}^{\equiv})$ is a decomposition of $(\Sigma^{\equiv}, B, E^{\equiv})$, we immediately get that

$$\langle\langle u \equiv v \rangle\rangle_{\vec{E}^{\equiv},B} = \{ (w, \eta|_{vars(u \equiv v)}) \mid (u \equiv v \overset{\eta}{\leadsto}{}^{*} w) \in FVN_{\vec{E}^{\equiv},B}(u \equiv v) \wedge w = w!_{\vec{E}^{\equiv},B} \}$$

is a complete set of $\vec{E},B$-variants for any $u \equiv v \in T_{\Sigma^{\equiv}}(X)$.

But then, it follows immediately from the Completeness Lemma in pages 11-13 of Lecture 8, and the Lemma in pg. 10 of Lecture 6 that

$$VarUnif_{E \cup B}(u = v) = \{ \eta|_{vars(u \equiv v)} \mid (tt, \eta|_{vars(u \equiv v)}) \in \langle\langle u \equiv v \rangle\rangle_{\vec{E}^{\equiv},B} \}$$

is a complete set of $E \cup B$-unifiers of the equation $u = v$.

We call each $\theta \in VarUnif_{E \cup B}(u = v)$ a <u>variant unifier</u> of $u = v$.

Of course, since the unifiers $\theta \in VarUnif_{EVB}(u=v)$ are computed by narrowing, if we denote by $NarrUnif_{EVB}(u=v)$ the set of unifiers computed by the unification semi-algorithm defined in Df. 11 of lecture 6, we always have:

$$VarUnif_{EVB}(u=v) \subseteq NarrUnif_{EVB}(u=v)$$

So, what's the difference? The difference is huge because:

(i) $VarUnif_{EVB}(u=v)$ will typically be much smaller than $NarrUnif_{EVB}(u=v)$. This might seem ludicrous to say when $VarUnif_{EVB}(u=v)$ is infinite, but it is not, even in that case. The point is that by drastically reducing the narrowing tree, we will be able to generate many more unifiers in the same amount of time, by not exploring a huge amount of useless dead ends and redundant solutions. Abstractly, $VarUnif_{EVB}(u=v)$ and $NarrUnif_{EVB}(u=v)$ can both have cardinality ω, but the finite set $VarUnif^{\leq k}_{EVB}(u=v)$ of unifiers computed at depth $\leq k$ in the narrowing tree will typically be much smaller than the corresponding set

$NarrUnif^{k\leq}_{EVB}(u=v)$.

(ii) $VarUnif_{E \cup B}(u=v)$ will <u>terminate</u> with either failure of a finite complete set $E \cup B$-unifiers in a huge number of cases where $NarrUnif_{E \cup B}(u=v)$ will typically <u>not</u> terminate (includig the case where $NarrUnif_{E \cup B}(u=v) = \emptyset$ but the semi-algorithm does <u>not</u> terminate).

Claim (ii) is a fundamental claim, but at the moment seems both vague and unjustified. Can we make good on it? Yes! But we first need a lemma:

<u>Lemma</u>. Let $(\Sigma, B, \vec{E})$ be an FVP decomposition of $(\Sigma, E \cup B)$ and assume B has a finitary B-unification algorithm (extensible by additional free function symbols). Then $(\Sigma^{\equiv}, B, \vec{E}^{\equiv})$ is also FVP.

<u>Proof</u>. By the extensibility assumption, the finitary B-unification algorithm with symbols Σ extends to one with symbols $\Sigma^{\equiv}$, and in fact coincides with it for all sorts except the new Eqn sort. By the Completeness Theorem for FVP in Lecture 8, pgs. 14-15, we will be done if we prove that $\langle\langle f(x_1,..,x_n)\rangle\rangle_{\vec{E}^{\equiv},B}$ is finite for each $f \in \Sigma^{\equiv}$. But since for each $f \in \Sigma$ we have $\langle\langle f(x_1,..,x_n)\rangle\rangle_{\vec{E}^{\equiv},B} = \langle\langle f(x_1,..,x_n)\rangle\rangle_{\vec{E},B}$ and $(\Sigma, B, \vec{E})$ is FVP, we only need to prove that

$\langle\langle x \equiv y \rangle\rangle_{\vec{E}^{\equiv}, B}$ for each $x : s$, $y : s$ with s a top sort in Σ. But this follows trivially from the fact that the narrowing tree for $x \equiv y$ is <u>finite</u>, namely:

$$x \equiv y \xrightarrow[\vec{E}^{\equiv}, B]{\left\{\begin{array}{l} x \mapsto x'' \\ y \mapsto x'' \\ x' \mapsto x'' \end{array}\right\}} tt$$

This is so because the only rule that can B-unify at a non-variable position is $x' : s \equiv x' : s \longrightarrow tt$, and we have

$$Unif_B(x \equiv y,\ x' \equiv x') = \{\{x \mapsto x'', y \mapsto x'', x' \mapsto x''\}\}$$

We just need to check that, indeed, this singleton B-unifier is a complete set of B-unifiers. Let θ be any such B-unifier, so that $\theta(x) \equiv \theta(y) =_B \theta(x') \equiv \theta(x')$. Since $\_\equiv\_$ is a free function symbol not involving the axioms in B, this holds if and only if

$$\theta(x) =_B \theta(x'), \text{ and } \theta(y) =_B \theta(x').$$

But then

$$\theta =_B \left(\{x \mapsto x'', y \mapsto x'', x' \mapsto x''\}\{x'' \mapsto \theta(x)\} \overset{+}{\cup} \theta|_{dom(\theta)-\{x,y,x'\}}\right)\Big|_{dom(\theta) \cup \{x,y,x'\}}$$

since we have:

$$\boxed{\begin{array}{rcl} x' \longmapsto \theta(x') & =_B & \theta(x) \longleftarrow x'' \longleftarrow x' \\ x \longmapsto \theta(x) & = & \theta(x) \longleftarrow x'' \longleftarrow x \\ y \longmapsto \theta(y) & =_B & \theta(x) \longleftarrow x'' \longleftarrow y \\ dom(\theta)-\{x,y,x'\} \ni z \longmapsto \theta(z) & =_B & \theta(z) \longleftarrow z \end{array}}$$

showing that $(\Sigma^{\equiv}, B, \vec{E}^{\equiv})$ is FVP, as desired. □

Here is now the main theorem about variant unification:

<u>Termination Theorem</u>. Let $(\Sigma, B, \vec{E})$ be a decomposition of $(\Sigma, E \cup B)$ and assume B has a unification algorithm for Σ-terms extensible with free function symbols. Then $VarUnif_{E \cup B}(u = v)$ terminates with either failure or a finite set of $E \cup B$-unifiers iff $u \equiv v$ has a finite complete set of $\vec{E}^{\equiv}, B$-variants.

<u>Proof</u>. Obviously, $VarUnif_{E \cup B}(u = v)$ terminates iff $FVN_{\vec{E}^{\equiv}, B}(u \equiv v)$ terminates iff (by the Completeness Theorem in Lecture 8, pps. 14-15) $u \equiv v$ has a finite complete set of variants. $\square$

<u>Corollary</u>. Let $(\Sigma, B, \vec{E})$ be as in the above Termination Theorem. Then $VarUnif_{E \cup B}(u = v)$ terminates for all inputs $u = v$ iff $(\Sigma, B, \vec{E})$ is FVP.

<u>Proof</u>. $(\Leftarrow)$ Follows from the Termination Theorem and the Lemma in pg. 3. We can prove $(\Rightarrow)$ by contradiction. Assume that $(\Sigma, B, \vec{E})$ is not FVP. By the Completeness Theorem in Lecture 8, pages 14-15 this means that there is

a term $t \in T_\Sigma(X)$ such that $\langle\!\langle t \rangle\!\rangle_{\vec{E},B}$ is an infinite complete set of variants and no finite subset of it is complete set of variants. But this can only happen if $vars(t) \neq \emptyset$. Let ρ rename all the variables in $vars(t)$ by fresh new ones. By the above Termination Theorem we will be done if we show that $t \equiv t\rho$ does not have a finite complete set of variants. Suppose it did. Since we can compute them by narrowing and t and $t\rho$ share no variables, they must be of the form

$$(u_1 \equiv u'_1,\ \theta_1 \dot{\cup} \theta'_1), \ldots, (u_n \equiv u'_n,\ \theta_n \dot{\cup} \theta'_n),$$

$$(tt, \gamma_1), \ldots, (tt, \gamma_m)$$

with $n \geq 1$, $m \geq 1$, and (u_i, θ_i), resp (u'_i, θ'_i), $\vec{E},B$-variants of t resp. $t\rho$. But this is impossible, because we obviously can find $(v, \alpha), (w, \beta) \in \langle\!\langle t \rangle\!\rangle_{\vec{E},B}$ such that $ran(\alpha) \cap ran(\beta) = \emptyset$, $(u_i, \theta) \not\geq_B (v, \alpha)$, $1 \leq i \leq n$, $(u', \theta'_i) \not\geq_B (w, \beta)$

But this shows that $(u_i \equiv u'_i, \theta_i \dot{\cup} \theta'_i) \not\models_B (v \equiv w, \alpha \dot{\cup} \beta)$ violating the assumption that $t \equiv t_3$ had a finite set of variants. $\square$

In summary, therefore, the variant unification semi-algorithm $VarUnif_{E \cup B}(u = v)$ terminates for all inputs and therefore becomes a finitary $E \cup B$-unification algorithm iff $(\Sigma, B, \vec{E})$ is FVP.

From the satisfiability point of view this then gives us:

Theorem. Let $(\Sigma, B, \vec{E})$ be an FVP decomposition of $(\Sigma, E \cup B)$ and let B have a finitary unification algorithm extensible with free function symbols. Assume furthermore that Σ has non-empty sorts. Then satisfiability of a positive QF formula $\varphi \in QF^+(\Sigma)$ in:

$$\begin{cases} 1. & T_{\Sigma/E \cup B}(X) \\ 2. & T_{\Sigma/E \cup B} \end{cases}$$

is decidable. $\square$

One last, useful remark is the following:

<u>Lemma</u>. Let $(\Sigma, B, \vec{E})$ be FVP with B having a finitary B-unification algorithm extensible by free function symbols. Then $VarUnif^{\circ}(u=v) = \{\theta \mid (tt, \theta) \in [\![u \equiv v]\!]_{\vec{E}^{\equiv}, B}\}$ is a complete set of <u>most general</u> unifiers.

<u>Proof</u>. We already know that it is a finite complete set of $E \cup B$-unifiers, because $VarUnif(u=v)$ is obtained from $\langle\langle u \equiv v \rangle\rangle$ and $[\![u \equiv v]\!] = max_{\sqsupseteq_B}(\langle\langle u \equiv v \rangle\rangle)$.

But we cannot have $\theta, \theta' \in VarUnif^{\circ}(u=v)$ with $\theta \sqsupseteq_B \theta'$ without $\theta = \theta'$, since then we would have $(tt, \theta) \sqsupseteq_B (tt, \theta')$ without $\theta = \theta'$ in $[\![u \equiv v]\!]_{\vec{E}^{\equiv}, B}$, which is impossible. $\Box$

Note that when $(\Sigma, B, \vec{E})$ is FVP and B any combination of assoc, comm, id axioms, Maude's <u>variant unify</u> command computes exactly $VarUnif^{\circ}(u=v)$.