HW 2

Due on **Monday, February 26, 2024 at 10am**

CS 574: Randomized Algorithms, Spring 2022 Version: **1.2**

Instructions: As in first homework.

4 (100 PTS.) Skip in peace.

You are given a set of n elements $e_1, \ldots, e_n$. Consider the following randomized algorithm – in step one, it creates a node n_1 that stores e_1. In the ith step, it creates a node n_i (that stores e_i), randomly picks a random number $\alpha_i \in [\![i-1]\!] = \{1, 2, \ldots, i-1\}$, and creates a directed edge from n_i to n_{α_i}.

This algorithm computes a random directed tree T with n nodes (the tree is a *reverse tree*). Let L be the length of the longest path in T.

4.A. Prove an upper bound on the expected length of L.

4.B. Prove a lower bound on the expected length of L (it should match the bound from (A) up to a constant).

4.C. Prove that the bounds above hold with high probability on L itself (i.e., not on the expectation). That is your upper/lower bound has to hold with probability $\geq 1 - 1/n^c$, where $c \geq 4$.

(The upper bound is not difficult, the lower bound requires some cleverness.)

5 (100 PTS.) Chernoff inequality is tight by direct calculations.

For this question use only basic argumentation – do not use Stirling's formula, Chernoff inequality or any similar "heavy" machinery. This question is more tedious than hard.

5.A. Prove that $\displaystyle\sum_{i=0}^{n-k} \binom{2n}{i} \leq \frac{n}{4k^2} 2^{2n}$.

Hint: Consider flipping a coin $2n$ times. Write down explicitly the probability of this coin to have at most $n-k$ heads, and use Chebyshev inequality.

5.B. Using (A), prove that $\binom{2n}{n} \geq 2^{2n}/4\sqrt{n}$ (which is a pretty good estimate).

5.C. Prove that $\displaystyle\binom{2n}{n+i+1} = \left(1 - \frac{2i+1}{n+i+1}\right)\binom{2n}{n+i}$.

5.D. Prove that $\displaystyle\binom{2n}{n+i} \leq \exp\left(\frac{-i(i-1)}{2n}\right)\binom{2n}{n}$.

5.E. Prove that $\displaystyle\binom{2n}{n+i} \geq \exp\left(-\frac{8i^2}{n}\right)\binom{2n}{n}$.

5.F. Using the above, prove that $\displaystyle\binom{2n}{n} \leq c\frac{2^{2n}}{\sqrt{n}}$ for some constant c (I got $c = 0.824...$ but any reasonable constant will do).

5.G. Using the above, prove that

$$\sum_{i=t\sqrt{n}+1}^{(t+1)\sqrt{n}} \binom{2n}{n-i} \leq c 2^{2n} \exp\bigl(-t^2/2\bigr).$$

In particular, conclude that when flipping fair coin $2n$ times, the probability to get less than $n - t\sqrt{n}$ heads (for t an integer) is smaller than $c' \exp(-t^2/2)$, for some constant c'.

5.H. Let X be the number of heads in $2n$ coin flips. Prove that for any integer $t > 0$ and any $\delta > 0$ sufficiently small, it holds that $\mathbb{P}[X < (1-\delta)n] \geq \exp(-c''\delta^2 n)$, where c'' is some constant. Namely, the Chernoff inequality is tight in the worst case.

6 (100 PTS.) Tail inequality for geometric variables.

Let $X_1, \ldots, X_m$ be m independent random variables with geometric distribution with probability p (i.e., $\mathbb{P}[X_i = j] = (1-p)^{j-1}p$). Let $Y = \sum_i X_i$, and let $\mu = \mathbf{E}[Y] = m/p$. Prove that $\mathbb{P}\bigl[Y \geq (1+\delta)\mu\bigr] \leq \exp\left(-\frac{\delta^2}{8}m\right)$. Here $\delta \in (0, 1/4)$. (A proof with a different constant than 8 is also fine.)

Hint: Consider an infinite sequence B of IID (independent and identically distributed) random bits $b_1, b_2, \ldots$, where $b_i = 1$ with probability p.