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- Method of conditional expectations
- Martingales.

$f(X_1, X_2, \dots, X_n)$
$X_1, X_2, \dots, X_n \in \{0,1\}$

F 3SAT formula n variables
$f(X_1, \dots, X_n) = \#$ of clauses in F that are satisfied by this assignment.

$Ef = E[f(X_1, X_2, \dots, X_n)] = \frac{7}{8}m.$

$X_{17} \vee \overline{X_{13}} \vee X_{27}$ $\quad K=3$

$P\left[\begin{matrix}\text{clause with } k \text{ variables}\\ \text{to be satisfied}\end{matrix}\right] = 1 - \frac{1}{2^k}$

m clauses in $F \Rightarrow$ in expectation # of satisfied clauses in random assignment is $\frac{7}{8}m$.

How to derandomize

$Ef(v_1, \dots, v_k) = E\left[f(X_1, X_2, \dots X_n) \,\middle|\, X_1 = v_1, X_2 = v_2, \dots X_k = v_k\right]$

$F[X_1 = v_1, X_2 = v_2, \dots X_k = v_k]$

$X_7 \to v_7 \qquad \overline{X_9} \to v_9$

$\underbrace{(1 \vee \cdots)}_{1} \qquad (0 \vee \overline{X_{17}} \vee X_{19}) \quad (\overline{X_{17}} \vee X_9)$

$X_{k+1}, \dots, X_n$

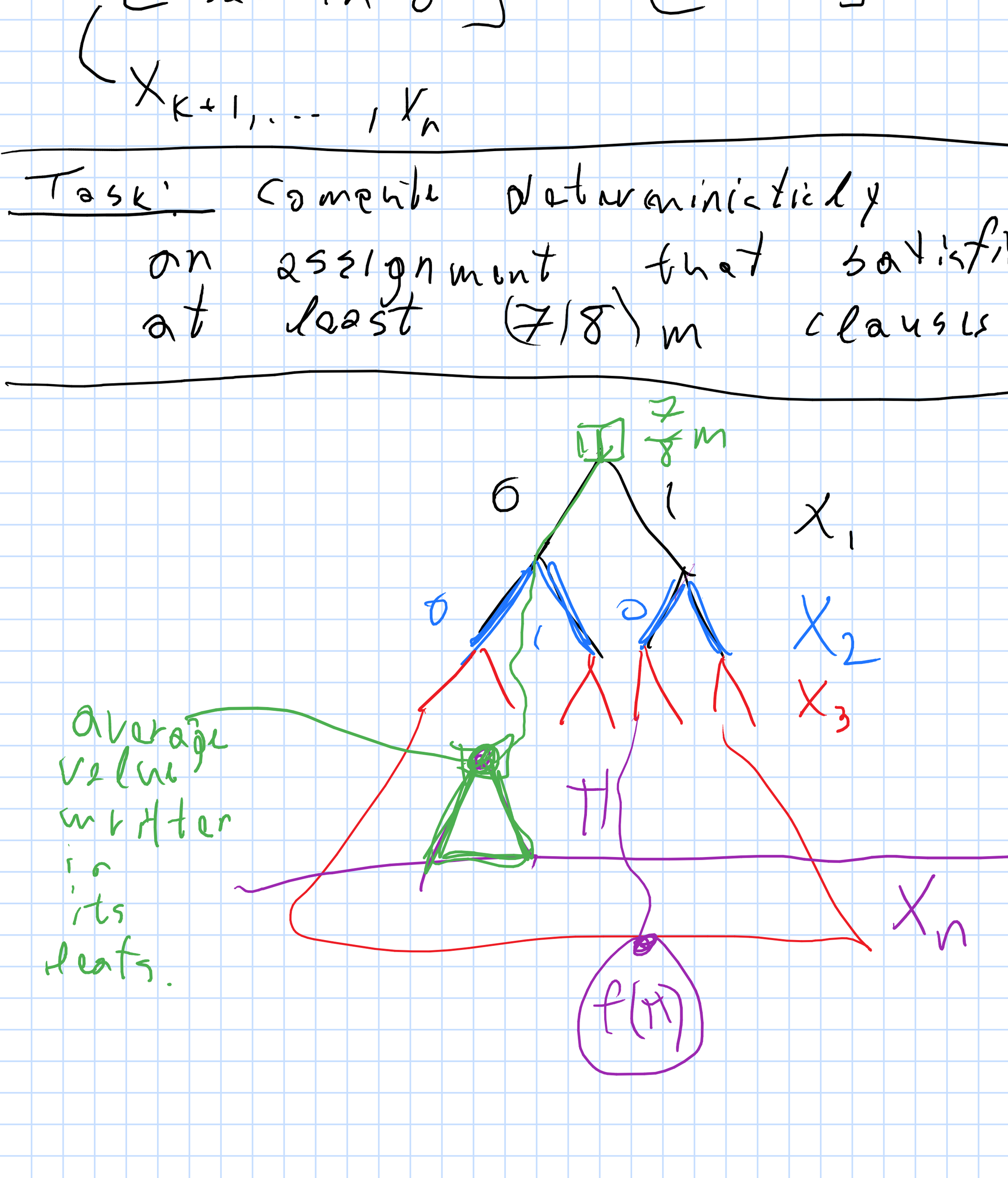
$G \Leftarrow F[v_1, \dots, v_k]$
$+$
clauses already satisfied

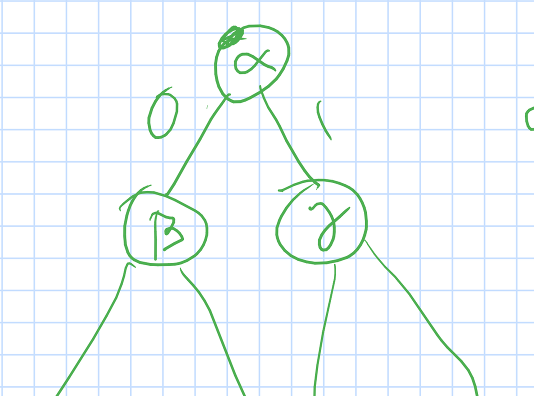
# vars	
1	$1 - 1/2 = 1/2$
2	$1 - 1/2^2 = \frac{3}{4}$
3	$7/8$

$Z_i = 1 \Leftrightarrow$ ith clause in G is satisfied
$P[Z_i = 1] = 1 - 1/2^{\#\text{ of literals in the clause}}$

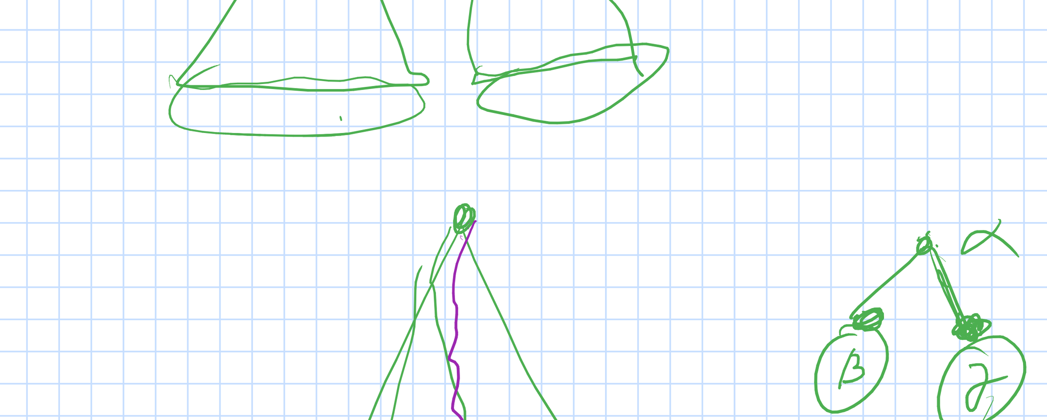
$E\left[\begin{matrix}\#\text{ clauses}\\ \text{sat in } G\end{matrix}\right] = E\left[\sum Z_i\right] = \sum E[Z_i] = \sum_i P[Z_i = 1]$
$X_{k+1}, \dots, X_n$

Task: compute deterministicly an assignment that satisfies at least $(7/8)m$ clauses F.





$\alpha = \frac{\beta + \gamma}{2}$



$n \cdot 2 \cdot O(m) = O(nm)$

$E[f(X_1, \dots, X_n)]$

$E[f(\underbrace{v_1 \dots v_k}_{0/1 \text{ concrete values}}, X_{k+1} \dots X_{k+1}]$

Martingales

$X_0, X_1, X_2, \dots, X_n$
$X_i \equiv$ amount of money after the ith round in a betting game.

$X_0 = 1$
B_i fraction of X_{i-1}

$X_i = \begin{cases} (1+B_i)X_{i-1} & \text{You won the bet} \\ (1-B_i)X_{i-1} & \text{You lost the bet} \end{cases}$

Such a sequence is a martingale if

$E[X_i \mid X_1, X_2, \dots, X_{i-1}] = X_{i-1}$

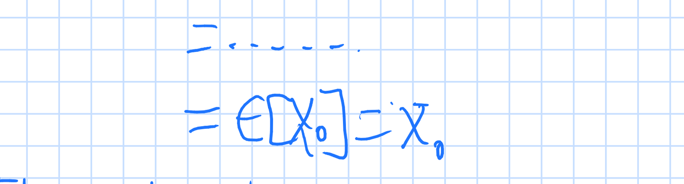
In the example
$E[X_i \mid X_{i-1}] = X_{i-1}$
$X_1, X_2, X_3, \dots X_n$ they are not independent

$X_n = 2^n$
$E[X_n] = E[E[X_n \mid X_1, \dots, X_{n-1}]]$
$= E[X_{n-1}]$
$= \dots$
$= E[X_0] = X_0$

There is strong concentration.!

Martingales

Urn



put back c copies of the ball you took

X_i what is the ratio of black balls in the ith iteration

$n_i := \#$ of balls after the ith round
$n_i := n + (c-1)i$

$E[X_i \mid X_1, \dots, X_{i-1}] = X_{i-1}$ ← martingale!

$= X_{i-1} \frac{(c-1) + X_{i-1} n_{i-1}}{n_i} + \overbrace{(1 - X_{i-1})}^{\text{Prob red ball}} \frac{X_{i-1} n_{i-1}}{n_i}$

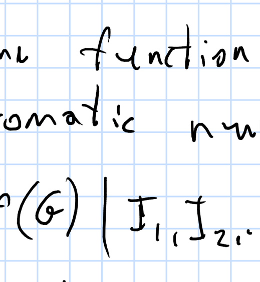
$= \frac{X_{i-1}(c-1) + X_{i-1}^2 n_{i-1} + (1 - X_{i-1}) X_{i-1} n_{i-1}}{n_i}$

$= \frac{X_{i-1}(c-1) + \cancel{X_{i-1}^2 n_i} + X_{i-1} n_{i-1} - \cancel{X_{i-1}^2 n_{i-1}}}{n_i}$

$= \frac{X_{i-1}(c-1) + X_{i-1} n_{i-1}}{n_i} = X_{i-1} \frac{c - 1 + n_{i-1}}{n_i} = X_{i-1} \frac{n_i}{n_i}$
$= X_{i-1}.$

Exposure martingales.
Edge exposure

G graph over n vertices
In the ith step tell you if the ith edge is in G.



$\binom{n}{2}$ possible edges

$f(G)$: some function of a graph
Chromatic number

$X_i = E[f(G) \mid I_1, I_2, \dots I_i]$
$I_i = 1 \Leftrightarrow$ ith edge is in the graph.

Assume every edge in G is present with probability p.

$G \sim G_{n,p}$

$X_0 = E[f(G)]$ - some number
$X_i = E[f(G) \mid I_1, I_2, \dots I_n]$
$X_{\binom{n}{2}} = E[f(G) \mid I_1, \dots I_n] =$ chromatic number of G

Fact: $X_0, X_1, \dots, X_m$ martingale sequence.
$m = \binom{n}{2}$
$\boxed{|X_i - X_{i-1}| \leq 1}$ bounded difference.

Azuma's ineq
$X_0, X_1, \dots, X_n$ is a martingale and
$|X_{i-1} - X_i| \leq 1$

$\Rightarrow P[|X_0 - X_m| \geq \lambda\sqrt{m}] \leq \exp\left(-\frac{\lambda^2}{2}\right).$