

Chapter 45

Expanders I

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“Mr. Matzerath has just seen fit to inform me that this partisan, unlike so many of them, was an authentic partisan. For - to quote the rest of my patient’s lecture - there is no such thing as a part-time partisan. Real partisans are partisans always and as long as they live. They put fallen governments back in power and over throw governments that have just been put in power with the help of partisans. Mr. Matzerath contended - and this thesis struck me as perfectly plausible - that among all those who go in for politics your incorrigible partisan, who undermines what he has just set up, is closest to the artiest because he consistently rejects what he has just created.”

Gunter Grass, The tin drum

45.1. Preliminaries on expanders

45.1.1. Definitions

Let $G = (V, E)$ Be an undirected graph, where $V = \{1, \dots, n\}$. A ***d -regular graph*** is a graph where all vertices have degree d . A d -regular graph $G = (V, E)$ is a δ -edge expander (or just, ***δ -expander***) if for every set $S \subseteq V$ of size at most $|V|/2$, there are at least $\delta d |S|$ edges connecting S and $\bar{S} = V \setminus S$; that is

$$e(S, \bar{S}) \geq \delta d |S|, \quad (45.1)$$

where

$$e(X, Y) = |\{uv \mid u \in X, v \in Y\}|.$$

A graph is ***$[n, d, \delta]$ -expander*** if it is a n vertex, d -regular, δ -expander.

A ***(n, d) -graph*** G is a connected d -regular undirected (multi) graph. We will consider the set of vertices of such a graph to be the set $\llbracket n \rrbracket = \{1, \dots, n\}$.

For a (multi) graph G with n nodes, its ***adjacency matrix*** is a $n \times n$ matrix M , where M_{ij} is the number of edges between i and j . It would be convenient to work the ***transition matrix*** Q associated with the random walk on G . If G is d -regular then $Q = M(G)/d$ and it is doubly stochastic.

A vector x is ***eigenvector*** of a matrix M with ***eigenvalue*** μ , if $xM = \mu x$. In particular, by taking the dot product of both size by x , we get $\langle xM, x \rangle = \langle \mu x, x \rangle$, which implies $\mu = \langle xM, x \rangle / \langle x, x \rangle$. Since the adjacency matrix M of G is symmetric, all its eigenvalues are real numbers (this is a special case of the spectral theorem from linear algebra). Two eigenvectors with different eigenvalues are orthogonal to each other.

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We denote the eigenvalues of \mathbf{M} by $\widehat{\lambda}_1 \geq \widehat{\lambda}_2 \geq \dots \widehat{\lambda}_n$, and the eigenvalues of \mathbf{Q} by $\widehat{\lambda}_1 \geq \widehat{\lambda}_2 \geq \dots \widehat{\lambda}_n$. Note, that for a d -regular graph, the eigenvalues of \mathbf{Q} are the eigenvalues of \mathbf{M} scaled down by a factor of $1/d$; that is $\widehat{\lambda}_i = \widehat{\lambda}_i/d$.

Lemma 45.1.1. *Let \mathbf{G} be an undirected graph, and let Δ denote the maximum degree in \mathbf{G} . Then, $|\widehat{\lambda}_1(\mathbf{G})| = |\widehat{\lambda}_1(\mathbf{M})| = \Delta$ if and only one connected component of \mathbf{G} is Δ -regular. The multiplicity of Δ as an eigenvector is the number of Δ -regular connected components. Furthermore, we have $|\widehat{\lambda}_i(\mathbf{G})| \leq \Delta$, for all i .*

Proof: The i th entry of $\mathbf{M}\mathbf{1}_n$ is the degree of the i th vertex v_i of \mathbf{G} (i.e., $\mathbf{M}\mathbf{1}_n = d(v_i)$, where $\mathbf{1}_n = (1, 1, \dots, 1) \in \mathbb{R}^n$). So, let x be an eigenvector of \mathbf{M} with eigenvalue λ , and let $x_j \neq 0$ be the coordinate with the largest (absolute value) among all coordinates of x corresponding to a connect component H of \mathbf{G} . We have that

$$|\lambda| |x_j| = |(\mathbf{M}x)_j| = \left| \sum_{v_i \in N(v_j)} x_i \right| \leq \Delta |x_j|,$$

where $N(v_j)$ are the neighbors of v_j in \mathbf{G} . Thus, all the eigenvalues of \mathbf{G} have $|\widehat{\lambda}_i| \leq \Delta$, for $i = 1, \dots, n$. If $\lambda = \Delta$, then this implies that $x_i = x_j$ if $v_i \in N(v_j)$, and $d(v_j) = \Delta$. Applying this argument to the vertices of $N(v_j)$, implies that H must be Δ -regular, and furthermore, $x_j = x_i$, if $x_i \in V(H)$. Clearly, the dimension of the subspace with eigenvalue (in absolute value) Δ is exactly the number of such connected components. ■

The following is also known. We do not provide a proof since we do not need it in our argumentation.

Lemma 45.1.2. *If \mathbf{G} is bipartite, then if λ is eigenvalue of $\mathbf{M}(\mathbf{G})$ with multiplicity k , then $-\lambda$ is also its eigenvalue also with multiplicity k .*

45.2. Tension and expansion

Let $\mathbf{G} = (V, E)$, where $V = \{1, \dots, n\}$ and \mathbf{G} is a d regular graph.

Definition 45.2.1. For a graph \mathbf{G} , let $\gamma(\mathbf{G})$ denote the *tension* of \mathbf{G} ; that is, the smallest constant, such that for any function $f : V(\mathbf{G}) \rightarrow \mathbb{R}$, we have that

$$\mathbb{E}_{x,y \in V} [|f(x) - f(y)|^2] \leq \gamma(\mathbf{G}) \mathbb{E}_{xy \in E} [|f(x) - f(y)|^2]. \quad (45.2)$$

Intuitively, the tension captures how close is estimating the variance of a function defined over the vertices of \mathbf{G} , by just considering the edges of \mathbf{G} . Note, that a disconnected graph would have infinite tension, and the clique has tension 1.

Surprisingly, tension is directly related to expansion as the following lemma testifies.

Lemma 45.2.2. *Let $\mathbf{G} = (V, E)$ be a given connected d -regular graph with n vertices. Then, \mathbf{G} is a δ -expander, where $\delta \geq \frac{1}{2\gamma(\mathbf{G})}$ and $\gamma(\mathbf{G})$ is the tension of \mathbf{G} .*

Proof: Consider a set $S \subseteq V$, where $|S| \leq n/2$. Let $f_S(v)$ be the function assigning 1 if $v \in S$, and zero otherwise. Observe that if $(u, v) \in (S \times \bar{S}) \cup (\bar{S} \times S)$ then $|f_S(u) - f_S(v)| = 1$, and $|f_S(u) - f_S(v)| = 0$ otherwise. As such, we have

$$\frac{2|S|(n-|S|)}{n^2} = \mathbb{E}_{x,y \in V} [|f_S(x) - f_S(y)|^2] \leq \gamma(\mathsf{G}) \mathbb{E}_{xy \in E} [|f_S(x) - f_S(y)|^2] = \gamma(\mathsf{G}) \frac{e(S, \bar{S})}{|E|},$$

by [Lemma 45.2.4](#). Now, since G is d -regular, we have that $|E| = nd/2$. Furthermore, $n - |S| \geq n/2$, which implies that

$$e(S, \bar{S}) \geq \frac{2|E| \cdot |S|(n-|S|)}{\gamma(\mathsf{G})n^2} = \frac{2(nd/2)(n/2)|S|}{\gamma(\mathsf{G})n^2} = \frac{1}{2\gamma(\mathsf{G})} d|S|.$$

which implies the claim (see [Eq. \(45.1\)](#)). ■

Now, a clique has tension 1, and it has the best expansion possible. As such, the smaller the tension of a graph, the better expander it is.

Definition 45.2.3. Given a random walk matrix Q associated with a d -regular graph, let $\mathcal{B}(\mathsf{Q}) = \langle v_1, \dots, v_n \rangle$ denote the *orthonormal eigenvector basis* defined by Q . That is, v_1, \dots, v_n is an orthonormal basis for \mathbb{R}^n , where all these vectors are eigenvectors of Q and $v_1 = \mathbf{1}^n/\sqrt{n}$. Furthermore, let $\widehat{\lambda}_i$ denote the i th eigenvalue of Q , associated with the eigenvector v_i , such that $\widehat{\lambda}_1 \geq \widehat{\lambda}_2 \geq \dots \geq \widehat{\lambda}_n$.

Lemma 45.2.4. *Let $\mathsf{G} = (V, E)$ be a given connected d -regular graph with n vertices. Then $\gamma(\mathsf{G}) = \frac{1}{1-\widehat{\lambda}_2}$, where $\widehat{\lambda}_2 = \lambda_2/d$ is the second largest eigenvalue of Q .*

Proof: Let $f : V \rightarrow \mathbb{R}$. Since in [Eq. \(45.2\)](#), we only look on the difference between two values of f , we can add a constant to f , and would not change the quantities involved in [Eq. \(45.2\)](#). As such, we assume that $\mathbb{E}[f(x)] = 0$. As such, we have that

$$\begin{aligned} \mathbb{E}_{x,y \in V} [|f(x) - f(y)|^2] &= \mathbb{E}_{x,y \in V} [(f(x) - f(y))^2] = \mathbb{E}_{x,y \in V} [(f(x))^2 - 2f(x)f(y) + (f(y))^2] \quad (45.3) \\ &= \mathbb{E}_{x,y \in V} [(f(x))^2] - 2 \mathbb{E}_{x,y \in V} [f(x)f(y)] + \mathbb{E}_{x,y \in V} [(f(y))^2] \\ &= \mathbb{E}_{x \in V} [(f(x))^2] - 2 \mathbb{E}_{x \in V} [f(x)] \mathbb{E}_{y \in V} [f(y)] + \mathbb{E}_{y \in V} [(f(y))^2] = 2 \mathbb{E}_{x \in V} [(f(x))^2]. \end{aligned}$$

Now, let \mathcal{I} be the $n \times n$ identity matrix (i.e., one on its diagonal, and zero everywhere else). We have that

$$\begin{aligned} \rho &= \frac{1}{d} \sum_{xy \in E} (f(x) - f(y))^2 = \frac{1}{d} \left(\sum_{x \in V} d(f(x))^2 - 2 \sum_{xy \in E} f(x)f(y) \right) = \sum_{x \in V} (f(x))^2 - \frac{2}{d} \sum_{xy \in E} f(x)f(y) \\ &= \sum_{x,y \in V} (\mathcal{I} - \mathsf{Q})_{xy} f(x)f(y). \end{aligned}$$

Note, that $\mathbf{1}^n$ is an eigenvector of Q with eigenvalue 1, and this is the largest eigenvalue of Q . Let $\mathcal{B}(\mathsf{Q}) = \langle v_1, \dots, v_n \rangle$ be the orthonormal eigenvector basis defined by Q , with eigenvalues $\widehat{\lambda}_1 \geq \widehat{\lambda}_2 \geq \dots \geq \widehat{\lambda}_n$, respectively. Write $f = \sum_{i=1}^n \alpha_i v_i$, and observe that

$$0 = \mathbb{E}[f(x)] = \sum_{i=1}^n \frac{f(i)}{n} = \left\langle f, \frac{v_1}{\sqrt{n}} \right\rangle = \left\langle \sum_i \alpha_i v_i, \frac{v_1}{\sqrt{n}} \right\rangle = \frac{1}{\sqrt{n}} \langle \alpha_1 v_1, v_1 \rangle = \frac{\alpha_1}{\sqrt{n}},$$

since $v_i \perp v_1$ for $i \geq 2$. Hence $\alpha_1 = 0$, and we have

$$\begin{aligned} \rho &= \sum_{x,y \in V} (\mathcal{I} - \mathcal{Q})_{xy} f(x) f(y) = \sum_{x,y \in V} (\mathcal{I} - \mathcal{Q})_{xy} \sum_{i=2}^n \alpha_{i=1}^n \alpha_i v_i(x) \sum_{j=1}^n \alpha_j v_j(y) \\ &= \sum_{i,j} \alpha_i \alpha_j \sum_{x \in V} v_i(x) \sum_{y \in V} (\mathcal{I} - \mathcal{Q})_{xy} v_j(y). \end{aligned}$$

Now, we have that

$$\sum_{y \in V} (\mathcal{I} - \mathcal{Q})_{xy} v_j(y) = \left\langle \begin{bmatrix} \text{xth row of} \\ (\mathcal{I} - \mathcal{Q}) \end{bmatrix}, v_j \right\rangle = ((\mathcal{I} - \mathcal{Q})v_j)(x) = \left((1 - \widehat{\lambda}_j) v_j \right)(x) = (1 - \widehat{\lambda}_j) v_j(x),$$

since v_j is eigenvector of \mathcal{Q} with eigenvalue $\widehat{\lambda}_j$. Since v_1, \dots, v_n is an orthonormal basis, and $f = \sum_{i=1}^n \alpha_i v_i$, we have that $\|f\|^2 = \sum_j \alpha_j^2$. Going back to ρ , we have that

$$\begin{aligned} \rho &= \sum_{i,j} \alpha_i \alpha_j \sum_{x \in V} v_i(x) (1 - \widehat{\lambda}_j) v_j(x) = \sum_{i,j} \alpha_i \alpha_j (1 - \widehat{\lambda}_j) \sum_{x \in V} v_i(x) v_j(x) \\ &= \sum_{i,j} \alpha_i \alpha_j (1 - \widehat{\lambda}_j) \langle v_i, v_j \rangle = \sum_{j=1}^n \alpha_j^2 (1 - \widehat{\lambda}_j) \langle v_j, v_j \rangle \\ &\geq (1 - \widehat{\lambda}_2) \sum_{j=2}^n \alpha_j^2 \sum_{x \in V} (v_j(x))^2 = (1 - \widehat{\lambda}_2) \sum_{j=2}^n \alpha_j^2 = (1 - \widehat{\lambda}_2) \|f\|^2 = (1 - \widehat{\lambda}_2) \sum_{j=1}^n (f(x))^2 \quad (45.4) \\ &= n(1 - \widehat{\lambda}_2) \mathbb{E}_{x \in V} [(f(x))^2], \end{aligned}$$

since $\alpha_1 = 0$ and $\widehat{\lambda}_1 \geq \widehat{\lambda}_2 \geq \dots \geq \widehat{\lambda}_n$.

We are now ready for the kill. Indeed, by Eq. (45.3), and the above, we have that

$$\begin{aligned} \mathbb{E}_{x,y \in V} [|f(x) - f(y)|^2] &= 2 \mathbb{E}_{x \in V} [(f(x))^2] \leq \frac{2}{n(1 - \widehat{\lambda}_2)} \rho = \frac{2}{dn(1 - \widehat{\lambda}_2)} \sum_{xy \in E} (f(x) - f(y))^2 \\ &= \frac{1}{1 - \widehat{\lambda}_2} \cdot \frac{1}{|E|} \sum_{xy \in E} (f(x) - f(y))^2 = \frac{1}{1 - \widehat{\lambda}_2} \mathbb{E}_{xy \in E} [|f(x) - f(y)|^2]. \end{aligned}$$

This implies that $\gamma(\mathbf{G}) \leq \frac{1}{1 - \widehat{\lambda}_2}$. Observe, that the inequality in our analysis, had risen from Eq. (45.4),

but if we take $f = v_2$, then the inequality there holds with equality, which implies that $\gamma(\mathbf{G}) \geq \frac{1}{1 - \widehat{\lambda}_2}$, which implies the claim. \blacksquare

Lemma 45.2.2 together with the above lemma, implies that the expansion δ of a d -regular graph \mathbf{G} is at least $\delta = 1/2\gamma(\mathbf{G}) = (1 - \lambda_2/d)/2$, where λ_2 is the second eigenvalue of the adjacency matrix of \mathbf{G} . Since the tension of a graph is direct function of its second eigenvalue, we could either argue about the tension of a graph or its second eigenvalue when bounding the graph expansion.