Chapter 32

A Bit on Algebraic Graph Theory

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> "The Party told you to reject the evidence of your eyes and ears. It was their final, most essential command."
>
> 1984, George Orwell

32.1. Graphs and Eigenvalues

Consider an undirected graph $\mathsf{G} = \mathsf{G}(V, E)$ with n vertices. The adjacency matrix $\mathsf{M}(\mathsf{G})$ of G is the $n \times n$ symmetric matrix where $\mathsf{M}_{ij} = \mathsf{M}_{ji}$ is the number of edges between the vertices v_i and v_j. If G is bipartite, we assume that V is made out of two independent sets X and Y. In this case the matrix $\mathsf{M}(\mathsf{G})$ can be written in block form.

32.1.1. Eigenvalues and eigenvectors

A non-zero vector v is an eigenvector of M, if there is a value λ, known as the ***eigenvalue*** of v, such that $\mathsf{M}v = \lambda v$. That is, the vector v is mapped to zero by the matrix $\mathsf{N} = \mathsf{M} - \lambda I$. This happens only if N is not full ranked, which in turn implies that $\det(\mathsf{N}) = 0$. We have that $f(\lambda) = \det(\mathsf{M} - \lambda I)$ is a polynomial of degree n. It has n roots (not necessarily real), which are the ***eigenvalues*** of M. A matrix $\mathsf{N} \in \mathbb{R}^{n \times n}$ is ***symmetric*** if $\mathsf{N}^T = \mathsf{N}$.

Lemma 32.1.1. *The eigenvalues of a symmetric real matrix $\mathsf{N} \in \mathbb{R}^{n \times n}$ are real numbers.*

Proof: Observe that for any real vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we have that $\sum_{i=1}^n v_i^2 = \langle v, v \rangle \geq 0$. As such, for a vector v with eigenvalue λ, we have

$$0 \leq \langle \mathsf{N}v, \mathsf{N}v \rangle = (\mathsf{N}v)^T \mathsf{N}v = (\lambda v)^T \lambda v = \lambda^2 \langle v, v \rangle .$$

Namely, λ^2 is a non-negative number, which implies that the λ is a real number. ■

Lemma 32.1.2. *Let $\mathsf{N} \in \mathbb{R}^{n \times n}$ be a matrix. Consider two eigenvectors v_1, v_2 that corresponds to two eigenvalues λ_1, λ_2, where $\lambda_1 \neq \lambda_2$. Then v_1 and v_2 are orthogonal.*

Proof: Indeed, $v_1^T \mathsf{N} v_2 = \lambda_2 v_1^T v_2$. Similarly, we have $v_1^T \mathsf{N} v_2 = (\mathsf{N}^T v_1)^T v_2) = \lambda_1 v_1^T v_2$. We conclude that either $\lambda_1 = \lambda_2$, or v_1 and v_2 are orthogonal (i.e., $v_1^T v_2 = 0$). ■

32.1.2. Eigenvalues and eigenvectors of a graph

Since $\mathsf{N} = \mathsf{M}(\mathsf{G})$ the adjacency matrix of an undirected graph is symmetric, all its eigenvalues exists and are real numbers $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$, and their corresponding orthonormal basis vectors are $e_1, \ldots, e_n$.

We will need the following theorem.

Theorem 32.1.3 (Fundamental theorem of algebraic graph theory). *Let* $\mathsf{G} = \mathsf{G}(V, E)$ *be an undirected (multi)graph with maximum degree* d *and with* n *vertices. Let* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *be the eigenvalues of* $\mathsf{M}(\mathsf{G})$ *and the corresponding orthonormal eigenvectors are* $e_1, \ldots, e_n$*. The following holds.*

(i) *If* G *is connected then* $\lambda_2 < \lambda_1$*.*

(ii) *For* $i = 1, \ldots, n$*, we have* $|\lambda_i| \leq d$*.*

(iii) d *is an eigenvalue if and only if* G *is regular.*

(iv) *If* G *is* d*-regular then the eigenvalue* $\lambda_1 = d$ *has the eigenvector* $e_1 = \frac{1}{\sqrt{n}}(1, 1, 1, \ldots, 1)$*.*

(v) *The graph* G *is bipartite if and only if for every eigenvalue* λ *there is an eigenvalue* $-\lambda$ *of the same multiplicity.*

(vi) *Suppose that* G *is connected. Then* G *is bipartite if and only if* $-\lambda_1$ *is an eigenvalue.*

(vii) *If* G *is* d*-regular and bipartite, then* $\lambda_n = d$ *and* $e_n = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1, -1, \ldots, -1)$*, where there are equal numbers of* 1*s and* -1*s in* e_n*.*

32.2. Bibliographical Notes

A nice survey of algebraic graph theory appears in [Wes01] and in [Bol98].

References

[Bol98] B. Bollobas. *Modern graph theory*. Springer-Verlag, 1998.

[Wes01] D. B. West. *Intorudction to graph theory*. 2ed. Prentice Hall, 2001.