

Chapter 28

Random Walks I

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“A drunk man will find his way home; a drunk bird may wander forever.”

Anonymous,

28.1. Definitions

Let $G = G(V, E)$ be an undirected connected graph. For $v \in V$, let $\Gamma(v)$ denote the set of neighbors of v in G ; that is, $\Gamma(v) = \{u \mid vu \in E(G)\}$. A **random walk** on G is the following process: Starting from a vertex v_0 , we randomly choose one of the neighbors of v_0 , and set it to be v_1 . We continue in this fashion, in the i th step choosing v_i , such that $v_i \in \Gamma(v_{i-1})$. It would be interesting to investigate the random walk process. Questions of interest include:

- (A) How long does it take to arrive from a vertex v to a vertex u in G ?
- (B) How long does it take to visit all the vertices in the graph.
- (C) If we start from an arbitrary vertex v_0 , how long the random walk has to be such that the location of the random walk in the i th step is uniformly (or near uniformly) distributed on $V(G)$?

Example 28.1.1. In the complete graph K_n , visiting all the vertices takes in expectation $O(n \log n)$ time, as this is the coupon collector problem with $n - 1$ coupons. Indeed, the probability we did not visit a specific vertex v by the i th step of the random walk is $\leq (1 - 1/n)^{i-1} \leq e^{-(i-1)/n} \leq 1/n^{10}$, for $i = \Omega(n \log n)$. As such, with high probability, the random walk visited all the vertex of K_n . Similarly, arriving from u to v , takes in expectation $n - 1$ steps of a random walk, as the probability of visiting v at every step of the walk is $p = 1/(n - 1)$, and the length of the walk till we visit v is a geometric random variable with expectation $1/p$.

28.1.1. Walking on grids and lines

Lemma 28.1.2 (Stirling's formula). For any integer $n \geq 1$, it holds $n! \approx \sqrt{2\pi n} (n/e)^n$.

28.1.1.1. Walking on the line

Lemma 28.1.3. Consider the infinite random walk on the integer line, starting from 0. Here, the vertices are the integer numbers, and from a vertex k , one walks with probability $1/2$ either to $k - 1$ or $k + 1$. The expected number of times that such a walk visits 0 is unbounded.

Proof: The probability that in the $2i$ th step we visit 0 is $\frac{1}{2^{2i}} \binom{2i}{i}$, As such, the expected number of times we visit the origin is

$$\sum_{i=1}^{\infty} \frac{1}{2^{2i}} \binom{2i}{i} \geq \sum_{i=1}^{\infty} \frac{1}{2\sqrt{i}} = \infty,$$

since $\frac{2^{2i}}{2\sqrt{i}} \leq \binom{2i}{i} \leq \frac{2^{2i}}{\sqrt{2i}}$ [MN98, p. 84]. This can also be verified using the Stirling formula, and the resulting sequence diverges. ■

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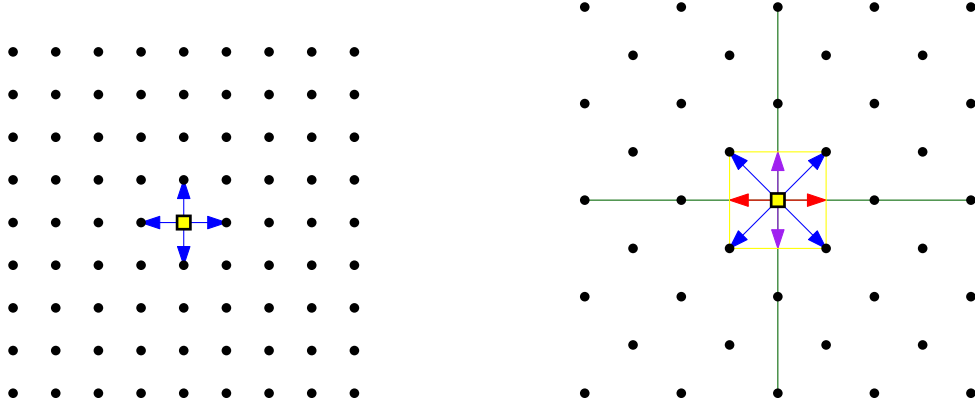


Figure 28.1: A walk in the integer grid, when rotated by 45 degrees, results, in two independent walks on one dimension.

28.1.1.2. Walking on two dimensional grid

A random walk on the integer grid \mathbb{Z}^d , starts from a point of this integer grid, and at each step if it is at point (i_1, i_2, \dots, i_d) , it chooses a coordinate and either increases it by one, or decreases it by one, with equal probability.

Lemma 28.1.4. *Consider the infinite random walk on the two dimensional integer grid \mathbb{Z}^2 , starting from $(0, 0)$. The expected number of times that such a walk visits the origin is unbounded.*

Proof: Rotate the grid by 45 degrees, and consider the two new axes X' and Y' , see Figure 28.1.. Let x_i be the projection of the location of the i th step of the random walk on the X' -axis, and define y_i in a similar fashion. Clearly, x_i are of the form $j/\sqrt{2}$, where j is an integer. By scaling by a factor of $\sqrt{2}$, consider the resulting random walks $x'_i = \sqrt{2}x_i$ and $y'_i = \sqrt{2}y_i$. Clearly, x_i and y_i are random walks on the integer grid, and furthermore, they are *independent*. As such, the probability that we visit the origin at the $2i$ th step is $\mathbb{P}[x'_{2i} = 0 \cap y'_{2i} = 0] = \mathbb{P}[x'_{2i} = 0]^2 = \left(\frac{1}{2^{2i}} \binom{2i}{i}\right)^2 \geq 1/4i$. We conclude, that the infinite random walk on the grid \mathbb{Z}^2 visits the origin in expectation

$$\sum_{i=0}^{\infty} \mathbb{P}[x'_i = 0 \cap y'_i = 0] \geq \sum_{i=0}^{\infty} \frac{1}{4i} = \infty,$$

as this sequence diverges. ■

28.1.1.3. Walking on three dimensional grid

In the following, let $\binom{i}{a \ b \ c} = \frac{i!}{a!b!c!}$ denote the multinomial coefficient. The multinomial theorem states that

$$(x_1 + x_2 + \dots + x_m)^n = \sum_{k_1+k_2+\dots+k_m=n} \sum_{k_1, k_2, \dots, k_m \geq 0} \binom{n}{k_1, k_2, \dots, k_m} \prod_{t=1}^m x_t^{k_t}.$$

In particular, we have

$$1^n = (1/3 + 1/3 + 1/3)^n = \sum_{a+b+c=n, a, b, c \geq 0} \binom{n}{a \ b \ c} \frac{1}{3^n}. \quad (28.1)$$

Lemma 28.1.5. *Consider the infinite random walk on the three dimensional integer grid \mathbb{Z}^3 , starting from $(0, 0, 0)$. The expected number of times that such a walk visits the origin is bounded.*

Proof: The probability of a neighbor of a point (x, y, z) to be the next point in the walk is $1/6$. Assume that we performed a walk for $2i$ steps, and decided to perform $2a$ steps parallel to the x -axis, $2b$ steps parallel to the y -axis, and $2c$ steps parallel to the z -axis, where $a + b + c = i$. Furthermore, the walk on each dimension is balanced, that is we perform a steps to the left on the x -axis, and a steps to the right on the x -axis. Clearly, this corresponds to the only walks in $2i$ steps that arrives to the origin.

Next, the number of different ways we can perform such a walk is $\frac{(2i)!}{a!a!b!b!c!c!}$, and the probability to perform such a walk, summing over all possible values of a, b and c , is

$$\alpha_i = \sum_{\substack{a+b+c=i \\ a,b,c \geq 0}} \frac{(2i)!}{a!a!b!b!c!c!} \frac{1}{6^{2i}} = \binom{2i}{i} \frac{1}{2^{2i}} \sum_{\substack{a+b+c=i \\ a,b,c \geq 0}} \left(\frac{i!}{a!b!c!} \right)^2 \left(\frac{1}{3} \right)^{2i} = \binom{2i}{i} \frac{1}{2^{2i}} \sum_{\substack{a+b+c=i \\ a,b,c \geq 0}} \left(\binom{i}{a \ b \ c} \left(\frac{1}{3} \right)^i \right)^2$$

Consider the case where $i = 3m$. We have that $\binom{i}{a \ b \ c} \leq \binom{i}{m \ m \ m}$. As such, we have

$$\alpha_i \leq \binom{2i}{i} \frac{1}{2^{2i}} \left(\frac{1}{3} \right)^i \underbrace{\binom{i}{m \ m \ m} \sum_{\substack{a+b+c=i \\ a,b,c \geq 0}} \left(\binom{i}{a \ b \ c} \left(\frac{1}{3} \right)^i \right)}_{=1 \text{ by Eq. (28.1)}}.$$

By the Stirling formula, we have

$$\binom{i}{m \ m \ m} \approx \frac{\sqrt{2\pi i} (i/e)^i}{\left(\sqrt{2\pi i/3} \left(\frac{i}{3e} \right)^{i/3} \right)^3} = c \frac{3^i}{i},$$

for some constant c . As such, $\alpha_i = O\left(\frac{1}{\sqrt{i}} \left(\frac{1}{3} \right)^i \frac{3^i}{i} \right) = O\left(\frac{1}{i^{3/2}} \right)$. Thus,

$$\sum_{m=1}^{\infty} \alpha_{6m} = \sum_i O\left(\frac{1}{i^{3/2}} \right) = O(1).$$

Finally, observe that $\alpha_{6m} \geq (1/6)^2 \alpha_{6m-2}$ and $\alpha_{6m} \geq (1/6)^4 \alpha_{6m-4}$. Thus,

$$\sum_{m=1}^{\infty} \alpha_m = O(1). \quad \blacksquare$$

28.2. Bibliographical notes

The presentation here follows [Nor98].

References

- [MN98] J. Matoušek and J. Nešetřil. *Invitation to discrete mathematics*. Oxford Univ Press, 1998.
- [Nor98] J. R. Norris. *Markov chains*. Statistical and Probabilistic Mathematics. Cambridge Press, 1998.