Chapter 10

Conditional Expectation and Concentration

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> "You see, dogs aren't enough any more. People feel so damned lonely, they need company, they need something bigger, stronger, to lean on, something that can really stand up to it all. Dogs aren't enough, what we need is elephants..."
>
> The roots of heaven, Romain Gary

10.1. Conditional expectation

Definition 10.1.1. For two random variables X and Y, let $\mathbb{E}[X \mid Y]$ denote the expected value of X, if the value of Y is specified. Formally, we have

$$\mathbb{E}[X \mid Y = y] = \sum_{x \in \Omega} x \, \mathbb{P}[X = x \mid Y = y] .$$

The expression $\mathbb{E}[X \mid Y]$, which is a shorthand for $\mathbb{E}[X \mid Y = y]$, is the ***conditional expectation*** of X given Y.

As such, the conditional expectation is a function from the value of y, to the average value of X. As such, one can think of conditional expectation as a function $f(y) = \mathbb{E}[X \mid Y = y]$.

Lemma 10.1.2. *For any two random variables X and Y, we have $\mathbb{E}\big[\mathbb{E}[X \mid Y]\big] = \mathbb{E}\big[X\big]$.*

Proof: $\mathbb{E}\big[\mathbb{E}[X \mid Y]\big] = \mathbb{E}_Y\big[\mathbb{E}[X \mid Y = y]\big] = \sum_y \mathbb{P}[Y = y] \, \mathbb{E}[X \mid Y = y]$

$$= \sum_y \mathbb{P}\big[Y = y\big] \frac{\sum_x x \, \mathbb{P}[X = x \cap Y = y]}{\mathbb{P}[Y = y]}$$

$$= \sum_y \sum_x x \, \mathbb{P}\big[X = x \cap Y = y\big] = \sum_x x \sum_y \mathbb{P}\big[X = x \cap Y = y\big]$$

$$= \sum_x x \, \mathbb{P}\big[X = x\big] = \mathbb{E}\big[X\big] . \qquad \blacksquare$$

Lemma 10.1.3. *For any two random variables X and Y, we have $\mathbb{E}\big[Y \cdot \mathbb{E}[X \mid Y]\big] = \mathbb{E}\big[XY\big]$.*

Proof: We have that $\mathbb{E}\big[Y \cdot \mathbb{E}\big[X \,\big|\, Y\big]\big] = \sum_y \mathbb{P}[Y = y] \cdot y \cdot \mathbb{E}\big[X \,\big|\, Y = y\big]$

$$= \sum_y \mathbb{P}[Y = y] \cdot y \cdot \frac{\sum_x x \, \mathbb{P}[X = x \cap Y = y]}{\mathbb{P}[Y = y]} = \sum_x \sum_y xy \cdot \mathbb{P}\big[X = x \cap Y = y\big] = \mathbb{E}\big[XY\big] . \qquad \blacksquare$$

10.1.1. Concentration from conditional expectation

Lemma 10.1.4. *Let $X_1, \ldots, X_n$ be independent random variables, that with equal probability are* 0 *or* 1*. We have that $\mathbb{P}[\sum_i X_i < n/4] < 0.9^n$ and $\mathbb{P}[\sum_i X_i > (3/4)n] < 0.9^n$.*

Proof: Let $Y_0 = 1$. If $X_i = 1$, then we set $Y_i = Y_{i-1}$, and if $X_i = 0$, then we set $Y_i = Y_{i-1}/2$. We thus have that

$$\mathbb{E}[Y_i \mid Y_{i-1}] = \frac{1}{2}\frac{Y_{i-1}}{2} + \frac{1}{2}Y_{i-1} = \frac{3}{4}Y_{i-1}.$$

As such, by Lemma 10.1.2 we have

$$\mathbb{E}[Y_i] = \mathbb{E}\Big[\mathbb{E}[Y_i \mid Y_{i-1}]\Big] = \mathbb{E}\left[\frac{3}{4}Y_{i-1}\right] = \frac{3}{4}\,\mathbb{E}[Y_{i-1}] = \left(\frac{3}{4}\right)^i.$$

In particular, $\mathbb{E}[Y_n] = (3/4)^n$. Now, if $\sum_i X_i > (3/4)n$, then we have

$$Y_n \geq (1/2)^{n/4}.$$

We are now ready for our conclusions:

$$\mathbb{P}\Big[\sum\nolimits_i X_i > (3/4)n\Big] = \mathbb{P}\Big[Y_n \geq (1/2)^{n/4}\Big] \leq \frac{\mathbb{E}[Y_n]}{(1/2)^{n/4}} \leq \frac{(3/4)^n}{(1/2)^{n/4}} = \left(\frac{2^{1/4}3}{4}\right)^n \leq 0.9^n.$$

By symmetry, we have $\mathbb{P}[\sum_i X_i < (1/4)n] = \mathbb{P}[\sum_i X_i > (3/4)n] < 0.9^n$. ■