HW 3

Due on **Monday, March 21, 2022 at 10am**

CS 574: Randomized Algorithms, Spring 2022 Version: **1.1**

Submission guidelines same as previous homework.

7 (100 PTS.) Independence.

Using the method of conditional expectations describe an efficient deterministic algorithm, that given a graph G with n vertices and m edges, outputs an independent set in G of size at least $n/(1+2m/n)$. How fast is your algorithm? (Faster is better.)

8 (100 PTS.) Ballmania.

The following two (sub)problems are variants of the classical coupon collector problem, and both variants have somewhat surprising answers. Part (B) is easier, and you might want to try doing it first.

8.A. The weights decrease.

In the game Ballmania, you are given a holy urn with n balls $b_1, \ldots, b_n$, labeled by $1, \ldots, n$, respectively. Initially, the weights of the balls are all one (i.e., $w_i = 1$, for all i). At each iteration of the game, you randomly pick a ball, decrease its weight by half (i.e., $w_i \leftarrow w_i/2$) and put it back in the urn. Here, the probability of picking a ball b_i is $w_i / \sum_j w_j$.

Provide upper and lower bounds, as tight as possible, on the expected number of rounds, till all the n balls in the urn are encountered.

8.B. The weights increase.

Consider the same settings as above, except that the weights increase. Specifically, if you pick the ball b_i, its weight becomes $w_i \leftarrow w_i + 1$.

Similar in spirit to the above, provide upper and lower bounds, as tight as possible, on the number of rounds, till all the n balls in the urn are encountered, and this happens with probability at least half. (The expectation in this case is somewhat harder to bound.)

9 (100 PTS.) Up and down

Given a sequence of numbers $x_1, x_2, \ldots, x_n$, a set of k ***canonical intervals*** is a sequence of indices $i_1 < i_2 < \ldots < i_k < j_k < j_{k-1} < \ldots < j_1$, such that $\left[x_{i_t}, x_{j_t}\right] \subseteq \left[x_{i_{t-1}}, x_{j_{t-1}}\right]$, for all t (we also require that $x_{i_k} < x_{j_k}$).

9.A. Let K be the random variable of the maximum number of canonical intervals, when $x_1, \ldots, x_n$ are randomly and uniformly chosen from $[0, 1]$.

Prove (as tight as possible) upper and lower bounds on the expected value of K.

9.B. Prove that K is strongly concentrated.