HW 1

Due on **Monday, February 14, 2022 at 10am**

CS 574: Randomized Algorithms, Spring 2022 Version: **1.5**

The homeworks can be submitted in groups of any size, as long as the size is an integer of value at most 3. The solution should be typed using latex (with no errors), and the pdf should be submitted electronically on gradescope.

If you get a considerable amount of information from somebody you should mention it in your solution (which is quite alright). As for google policy, a lot of homework questions might have solutions on the web (hopefully not, but I can not be sure). I would *prefer* if you try to solve them without searching on the web, but if you use solutions found on the web, you should **explicitly** say so in your solution (and provide the link!). In any case, I expect you to write your solution on your own (i.e., cut and paste is not acceptable), since the homeworks are there to demonstrate that you know the material and understand it.

If you have any questions, feel free to email me.

Grade policy. Homeworks would be 25% of the final grade. The main purpose of the homeworks is to prepare you for the midterms. I would recommend that you individually spend significant time trying to solve the homeworks on your own, but this is up to you.

Solutions. No solutions to the homeworks would be posted. If you want to know a solution to a question, ask me in person.

Planned # of homeworks would be in the range 7-10.

1 (100 PTS.) A coloring of trees.

Let $p \in (0, 1/2]$ be some fixed constant, and let $h > 0$ be some fixed integer. For an integer $t > 1$, let $T_{h,t}$ denote a rooted tree of height h, where every internal node has exactly t children (i,e., all the leafs are in distance h [in the number of edges] from the root).

Consider the following coloring game – we color each edge independently by either red or blue. The probability for red is p and the probability for blue is $1 - p$. Let $\mathcal{E}$ be the event that there is a path from the root to a leaf, such that all the edges on this path are red.

Let $f(p)$ be the minimum value of t, such that $\mathbf{Pr}[\mathcal{E}] \geq 1/2$. Provide an upper bound on $f(p)$ as tight as possible. What is the value of $f(p)$ for the values $p = 1/10$, $p = 1/8$,$p = 1/6$, $p = 1/2$?

2 (100 PTS.) Some probability required.

2.A. (25 PTS.) Let π be a random permutation of $[\![n^2]\!] = \{1, \ldots, n^2\}$. Let $\tau(i, j) = n(i - 1) + j$. Consider the $n \times n$ matrix M, where $M[i, j] = \pi_{\tau(i,j)}$. (Alternative interpretation is that we fill M by numbers $1, \ldots, n$, and then randomly permute its entries.).

Let $\mathcal{E}_{i,j}$ be the event that $M[i, j] = \min_{\alpha \in [\![i]\!], \beta \in [\![j]\!]} M[\alpha, \beta]$. Let $X_{i,j}$ be the indicator variable for $\mathcal{E}_{i,j}$.

Prove that $\mathbf{E}\left[\sum_{i,j} X_{i,j}\right] = \Theta(\log^2 n)$.

2.B. (25 PTS.) Prove or disprove that $X_{1,2}$ and $X_{2,1}$ are independent.

2.C. (25 PTS.) Prove that $X_{i,j}$ and $X_{i',j'}$ are *independent* random variables, for any $i' \in [\![i]\!]$, $j' \in [\![j]\!]$, such that $i' + j' < i + j$.

2.D. (25 PTS.) Consider another random permutation (picked uniformly out of the set of permutations) σ of $[\![n^2]\!]$. Let $Z_{i,j}$ be the indicator variable to the event that

$$\sigma_{\tau(i,j)} = \min_{(\alpha,\beta) \in [\![i]\!] \times [\![j]\!]} \sigma_{\tau(\alpha,\beta)} \qquad \text{and} \qquad \pi_{\tau(i,j)} = \min_{(\alpha,\beta) \in [\![i]\!] \times [\![j]\!]} \pi_{\tau(\alpha,\beta)}$$

Provide an upper and lower bounds (hopefully equal up to a constant) for

$$f(n) = \mathbf{E}\left[\sum_{i=1}^{n} \sum_{j=1}^{n} Z_{i,j}\right].$$

What is $\lim_{n=1}^{\infty} f(n)$?

Hint: Read Section 11.1 from `https://courses.engr.illinois.edu/cs574/sp2022/lec/old_notes/rand_alg_fa19.pdf`.

3 (100 PTS.) Binom games.

The following requires some tedious calculations, but hopefully would provide you with some intuition about the binomial distribution. Please use only elementary calculations in solving this problem (so, no Stirling's formula, or similar frightening stuff). You would probably need the following two formulas:

(i) $1 - x \leq \exp(-x)$ and $1 + x \leq \exp(x)$, for all $x \geq 0$.
(ii) $(1 - x)(1 - y) \geq 1 - x - y$, for all $x, y \geq 0$.

Assume n is some positive even integer, such that $\sqrt{n}/2$ is also an integer number.

3.A. (10 PTS.) Prove that $\binom{2n}{n+k} = \binom{2n}{n-k}$.

It is also easy to verify that for any $k \geq 0$, we have that $\binom{2n}{n+k} \geq \binom{2n}{n+k+1}$ and $\binom{2n}{n-k} \geq \binom{2n}{n-k-1}$. Namely, the middle binomial coefficient $\binom{2n}{n}$ is the largest, and the binomial coefficients decreases as you move away from the middle binomial coefficient.

3.B. (20 PTS.) For any $k \leq n$, prove that $\alpha(k) = \frac{\binom{2n}{n+k}}{\binom{2n}{n}} \geq \frac{(n-k+1)^k}{(n+k)^k} \geq 1 - \frac{2k^2}{n}$.

3.C. (10 PTS.) Prove from (B) that $\alpha(\sqrt{n}/2) \geq 1/2$.

3.D. (10 PTS.) Prove from (C) that $\binom{2n}{n} \leq c\frac{2^{2n}}{\sqrt{n}}$, for some absolute constant c.

[Hint: Use Chebychev to argue that $\sum_{i=-\sqrt{n}}^{\sqrt{n}} \binom{2n}{n+i} \geq c' 2^{2n}$, where $c' > 0$ is some absolute constant. Now, using (C) argue that the $\sqrt{n}$ largest elements in this summation are roughly equal, and thus the largest term is bounded from above by the average (up to a constant).]

3.E. (10 PTS.) Prove that $\alpha(k) \leq \exp(-k^2/2n)$.

3.F. (10 PTS.) [No longer for submission because of mistake in earlier versions.] For any integer t, let

$$S(t) = \sum_{k=t\sqrt{n}}^{(t+1)\sqrt{n}-1} \alpha(k).$$

Prove that for any integer $t \geq 0$, we have

$$S(t) \leq c' \exp\left(-t^2/2\right) S(0).$$

(t is at most $\sqrt{n} - 1$), for some absolute constant c' (Hint: Use (D) and (E)).

3.G. (20 PTS.) Prove that $2^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} \leq c'' \sqrt{n} \binom{2n}{n}$, for some absolute constant c''.

3.H. (10 PTS.) (Really easy.) Prove that $\binom{2n}{n} \geq \frac{2^{2n}}{c''\sqrt{n}}$.

3.I. (Not for submission, but follows from the above [essentially Chernoff inequality].) Let $X_1, X_2, \ldots, X_n$ be independent random variables that are 0 or 1 with equal probability. Let $Y = \sum_{i=1}^{n} X_i$. For any $\Delta \geq 0$, prove (using the above) that

$$\mathbf{Pr}[Y \geq n/2 + \Delta] \leq c_3 \exp\left(-\Delta^2/2n\right),$$

where c_3 is some absolute constant.