April 26, 2021 9:08 AM

Appl'n 2: Connectivity in Undirected Graphs

Given undir. graph $G = (V, E)$, $s, t \in V$,
decide whether t is reachable from s.

Easy: $O(m)$ time by BFS/DFS
needs $O(n)$ space

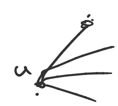
Question: space complexity?
$O(\log n)$ (in bits)?

by Savitch's Thm, $O(\log^2 n)$ space

Aleliunas et al. '79: $O(\log n)$ space using rand.!
("$SL \subseteq RL$")

idea - just by rand walk

Lemma Let G be connected undir graph with m edges.
Consider corresponding Markov chain
(with $P_{uv} = \frac{1}{\deg(u)}$ $\forall$ edge $uv \in G$).



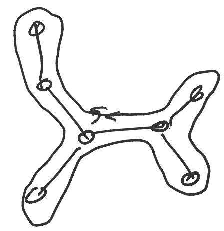
Then $h_{uv} \le O(m)$ $\forall$ edge $uv \in G$.

Cor C_s = expected cover time
$= E[\#$ steps to visit all vertices starting from $s]$

$\le O(mn)$.

Pf of Cor: Fix a walk $s v_1 v_2 \cdots v_\ell$ that
... vertices of length $\le 2n$

Pf of Cor: Fix a walk $s v_1 v_2 \cdots v_\ell$ that visits all vertices of length $\le 2n$ e.g. by doubling a spanning tree

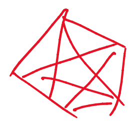


$$C_s \le h_{sv_1} + h_{v_1 v_2} + h_{v_2 v_3} + \cdots + h_{v_{\ell-1} v_\ell}$$

$$\le 2n \cdot O(m). \qquad \square$$

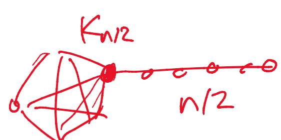
G: 

Cover time $= \Theta(n^2)$



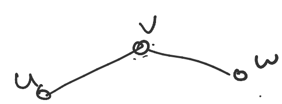
K_n

Cover time $= \Theta(n \lg n)$



Cover time $= \Theta(n^3)$.

Pf of Lemma: Define a diff. Markov chain where a state is a pair (u,v) with $uv \in E$ (# states $= 2m$)



$$P_{(u,v)(v,w)} = \frac{1}{\deg(v)}$$

Define $\pi_{(u,v)} = \frac{1}{2m}$ (the uniform distrib.)

Check π is a stationary distrib: $\sum_{(u,v)} \pi_{(u,v)} = 1$.

$$\forall (v,w), \quad \sum_u \pi_{(u,v)} P_{(u,v)(v,w)} \left(= \pi_{(v,w)} ?\right)$$

$$= \sum_{u:\text{ neighbor of } v} \frac{1}{2m} \cdot \frac{1}{\deg(v)}$$

$$= \deg(v) \cdot \frac{1}{2m} \cdot \frac{1}{\deg(v)} = \pi_{(v,w)}.$$

w: ~~is~~ neighbor of v

$$= \cancel{\deg(v)} \cdot \frac{1}{2m} \cdot \frac{1}{\cancel{\deg(v)}} = \pi_{(v,w)}.$$

by Fund Thm

$$\Rightarrow \quad h_{(u,v)(u,v)} = 2m.$$

$\Rightarrow$ $E($ # steps starting ~~at u~~, then v, ... & go back u, ~~then~~ v $) \leq 2m$

$$\Rightarrow \quad h_{vu} \leq 2m \qquad \forall \; uv \in E$$

Similarly, $h_{uv} \leq 2m$.. $\square$

$\Rightarrow$ Apply Cor to connected comp. of s

$\Rightarrow$ s-t connectivity in $O(mn)$ time

$O(\log n)$ space rand.

just remember curr. vertex + time counter

1-sided Monte Carlo w. err prob $\leq \frac{1}{2}$

Rmk - Reingold '08 deterministic $O(\log n)$ space (derand. using expanders)

("SL = L")

- major open problem: dir. case (is "NL = L"?)

When do Rand. Walks Mix Rapidly?

Given prob. distrib. $\pi^{(0)}$ for start vertex,

let $\pi_v^{(t)} = \Pr[$ walk ends at v after t steps$)$

distrib.

gcd (lengths of cycles

let $\pi_v^{(t)} = \Pr[\text{walk } \ldots]$

Let π be stationary distrib.

Known: (part of Fund. Thm) If Markov chain is finite, irreducible, & aperiodic, (gcd(lengths of all cycles thru v) = 1 $\forall v$)

then for any $\pi^{(0)}$, $\lim_{t\to\infty} \pi_v^{(t)} = \pi_v \quad \forall v$

But how fast does it converge?

approach - linear algebra! ("spectral")

Think of $P = (p_{uv})_{u,v\in V}$ as a matrix.

Obs (i) $\pi^{(t)} = \pi^{(t-1)} P$ $\quad \left(\pi_v^{(t)} = \sum_u \pi_u^{(t-1)} p_{uv}\right)$

$\vdots$

$= \pi^{(0)} P^t$ $\leftarrow$ t^{th} matrix power

(ii) π stationary $\iff$ $\pi = \pi P$

i.e. π is an eigenvector with eigenvalue 1

$(A\tilde{x} = \lambda \tilde{x})$
(take transpose)

Consider a d-regular undir. graph G.

1. $P = \frac{1}{d} A$ where A is the adj matrix ("normalized adj matrix")

$\left(\text{since } p_{uv} = \begin{cases} 1/d & \text{if } uv \in E \\ 0 & \text{else} \end{cases}\right)$

2. Stationary distrib. is uniform:

$\pi = \frac{1}{n}(1, 1, \ldots, 1)$

3. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be eigenvalues of P

& $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ be their orthonormal eigenvectors

$\Rightarrow \quad \vec{e}_i P = \lambda_i \vec{e}_i$

Know $\lambda_1 = 1$, $\vec{e}_1 = \frac{1}{\sqrt{n}}(1,1,\dots,1)$

Def A d-regular graph is an α-expander $(\alpha<1)$ if $|\lambda_2|,\dots,|\lambda_n| \le \alpha$.

Rapid Mixing Lemma For an α-expander,

$$\|\pi^{(t)} - \pi\|_2 \le \alpha^t.$$

Pf: Write $\pi^{(0)} = c_1\vec{e}_1 + \sum_{i=2}^{n} c_i\vec{e}_i$

$$\Rightarrow \pi^{(t)} = c_1\vec{e}_1 P^t + \sum_{i=2}^{n} c_i\vec{e}_i P^t$$

$$= c_1\vec{e}_1 + \sum_{i=2}^{n} c_i\lambda_i^t\vec{e}_i$$

$$\Rightarrow \|\pi^{(t)} - \underbrace{c_1\vec{e}_1}_{=\pi}\|_2 \le \alpha^t\left\|\sum_{i=2}^{n} c_i\vec{e}_i\right\|_2$$

$$\le \alpha^t\|\pi^{(0)}\|_2$$

$$\le \alpha^t. \qquad \square$$

Facts about Expanders

$\exists$ α-expander with $\alpha = O\left(\frac{1}{\sqrt{d}}\right)$ in fact, const factor is 1 ("Ramanujan graphs")

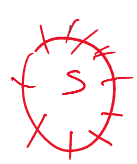
many constructions:

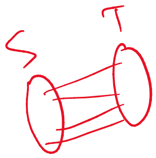
- random graph! (Probabilistic method)
- algebraic methods
- graph products

⋮

- Combinatorial characterization:

$\forall S \subseteq V$, with $|S| \le n/2$,





$$\left(\#\text{ edges out of } S\right) \geq \Omega\left(d(1-\alpha)|S|\right).$$

When do Rand. Walks Resemble Indep. Sampling?

again, expanders!

Lemma Take rand walk $v_0, \ldots, v_{t-1}$ from rand. start vertex.

Fix subset $B \subseteq V$, $|B| = \beta n$.

For an α-expander,

$$\Pr\left[v_0 \in B \wedge \cdots \wedge v_{t-1} \in B\right] \leq (\beta + \alpha)^t$$

(if indep. it'd be β^t)

Pf:

next time ...