Appl'n 2: Connectivity in Undirected Graphs

Given undir. graph $G = (V, E)$, $s, t \in V$, decide whether $t$ is reachable from $s$.

**Easy:** $O(m)$ time by BFS/DFS
needs $O(n)$ space

**Question:** space complexity?
$O(\log n)$ (in bits)?

by Savitch's Thm, $O(\log^2 n)$ space

Aleliunas et al. '79: $O(\log n)$ space using rand!

("
$SL \leq RL$
"
)

idea - just by rand walk

**Lemma** Let $G$ be undir. graph with $m$ edges.

Consider corresponding Markov chain
(with $P_{uv} = \frac{1}{\deg(u)}$ if $uv \in E$).

Then $P_{uv} \leq O(m)$ if edge $uv \in G$.

**Cor** $C_S = \text{expected cover time}$

$\leq E(\text{II steps to visit all vertices starting from } s)$

$\leq O(mn)$.

**Pf of Cor:** Fix a walk $s, u_1, v_2, \ldots, u_k$ that

.. 

ends at length $\leq 2n$
Pf of Cor: Fix a walk $v_1, v_2, \ldots, v_k$ that visits all vertices of length $\leq 2n$ e.g. by doubling a spanning tree

$$C_5 \leq h_{v_1} + h_{v_1, v_2} + h_{v_2, v_3} + \ldots + h_{v_{k-1}, v_k}.$$ 

$$\leq 2n \cdot O(m).$$

G: $\begin{array}{c}
\text{Cover time } = \Theta(n^2)
\end{array}$

$K_n$: $\begin{array}{c}
\text{Cover time } = \Theta(n \log n)
\end{array}$

$K_{n/2}$: $\begin{array}{c}
\text{Cover time } = \Theta(n^3)
\end{array}$

Pf of Lemma: Define a diff. Markov chain where a state is a pair $(u, v)$ with $uv \in E$ (2m states)

$$P(u, v)(v, w) = \frac{1}{\deg(v)}$$

Define $\pi(u, v) = \frac{1}{2m}$ (the uniform distr.)

Check $\pi$ is a stationary distr: $\sum_{(u, v)} \pi(u, v) = 1.$

$h(u, w), \sum_{u} \pi(u, v) P(u, v)(v, w) (= \bar{P}(u, w) ?)$

$$= \sum_{u: \text{ indegree of } v} \frac{1}{2m} \cdot \frac{1}{\deg(v)}$$

$$= \frac{1}{2m} \cdot \sum_{u: \text{ indegree of } v} \frac{1}{\deg(v)}$$

$$= \pi(v, u).$$
by Hand Then

\[ \frac{1}{2m} \cdot \deg(u) = \pi(u,v). \]

\[ \Rightarrow \quad h(u,v)(u,v) = 2m. \]

\[ \Rightarrow \quad \exists (s \text{ steps starting at } u, \text{ then } v, \text{ then } u) \leq 2m \]

\[ \Rightarrow \quad h_{uv} \leq 2m \quad \forall u,v \in E \]

Similarly, \( h_{uw} \leq 2m \)...

\[ \Rightarrow \quad \text{Apply Cor to connected comp. of } s \]

\[ \Rightarrow \quad \text{get connectivity in } O(mn) \text{ time } \]

\[ O(\log n) \text{ space w/ rand.} \]

1-sided Monte Carlo

\[ w. \text{ exp. prob } \leq \frac{1}{2} \]

\[ \text{just remember curr. vertex + time counter} \]

Remark - Reingold '08 deterministic \( O(\log n) \) space (demand. using expanders)

( "\( SL = L \)"

- major open problem: dir. case

( is "\( NL = L \)"? )

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When do Rand. Walks Mix Rapidly?

Given prob. distrib. \( \pi(0) \) for start vertex,

let \( \pi(t) = \Pr \left[ \text{walk ends at } u \text{ after } t \text{ steps} \right] \)

[add length of cycles]
Let $\pi_v^{(t)} \to \pi_v$ (as $t \to \infty$).

Let $\pi$ be stationary distrib.

Known: If Markov chain is finite, irreducible, & aperiodic,

\[ \lim_{t \to \infty} \pi_v^{(t)} = \pi_v \quad \forall v, \]

Then for any $\pi_v^{(0)}$,

\[ \lim_{t \to \infty} \pi_v^{(t)} = \pi_v \quad \forall v. \]

But how fast does it converge?

Approach: linear algebra! ("spectral")

Think of $P = (P_{uv})_{u,v}$ as a matrix.

1. $\pi_v^{(t)} = \pi_v^{(t-1)} P = \pi_v^{(t-2)} P^2 = \pi_v^{(t-3)} P^3 = \cdots = \pi_v^{(0)} P^t$ (i.e. $t$th matrix power)

2. $\pi$ stationary $\iff \pi = \pi P$

i.e. $\pi$ is an eigenvector with eigenvalue 1

\[ \begin{align*}
2. & \quad (A \vec{x} = \lambda \vec{x}) \\
& \quad \text{(take transpose)}
\end{align*} \]

Consider a d-regular undir. graph $G$.

1. $P = \frac{1}{d} A$ where $A$ is the adj matrix

$A_{uv} = \begin{cases} \frac{1}{d} & \text{if } uv \in E \\ 0 & \text{else} \end{cases}$

2. Stationary distrb. is uniform:

\[ \pi = \frac{1}{n} (1, 1, \ldots, 1) \]

3. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be eigenvalues of $P$ & $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ be their eigenvectors orthonormal

\[ \Rightarrow \quad \vec{e}_i P = \lambda_i \vec{e}_i \]
Know $\lambda_1 = 1$, $e_1 = \frac{1}{\sqrt{n}} (1, 1, \ldots, 1)$

**Def** A $d$-regular graph is an $\alpha$-expander if $|\lambda_2|, \ldots, |\lambda_n| \leq \alpha$.

Rapid Mixing Lemma. For an $\alpha$-expander, $\|\pi(t) - \pi\|_2 \leq \alpha^t$.

**Pr:** Write $\pi(0) = c_1 e_1 + \sum_{i=2}^{n} c_i e_i$.

$\Rightarrow \pi(t) = c_1 e_1 p_t + \sum_{i=2}^{n} c_i e_i p_t$.

$= c_1 e_1 + \sum_{i=2}^{n} c_i \lambda_i^t e_i$.

$\Rightarrow \|\pi(t) - c_1 e_1\|_2 \leq \alpha^t \|\sum_{i=2}^{n} c_i e_i\|_2$

$\leq \alpha^t \|\pi\|_2$

$\leq \alpha^t$. $\Box$

Facts about Expanders

$\exists \alpha$-expander with $\alpha = O\left(\frac{1}{\sqrt{d}}\right)$ (in fact, const factor is 1, “Ramanujan graphs”)

- many constructions:
  - random graph! (probabilistic method)
  - algebraic methods
  - graph products
  -

- combinatorial characterization:
  $\forall S \subseteq V$, with $1S \leq n/2$, 

\[ \|\pi(S) - \pi(S)\|_2 \leq \alpha^t \]
(\# edges out of S) \geq \Omega (d(1-\alpha)|S|).

When do Rand. Walks Resemble Indep. Sampling?

again, expanders!

Lemma: Talk rand walk \(v_0, \ldots, v_{t-1}\) from rand. start vertex.

Fix subset B \subseteq V, \quad |B| = \beta n.

For an \(\alpha\)-expander,

\[ \Pr(v_0 \in B \land \ldots \land v_t \in B) \leq (\beta + \alpha)^t \]

(if indep. it'd be \(\beta^t\)).

Pf:

next time ...