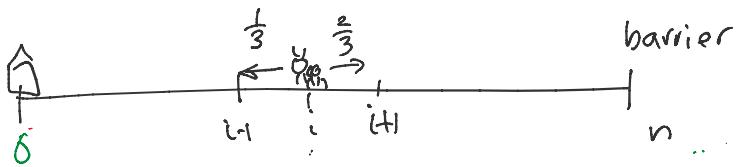


Set $t = 2n^2 \xrightarrow{\text{by Markov's inequality}} \Pr(\text{err}) \leq 2$.

Poly time for 2SAT?
(not new, for 2SAT)

What about 3SAT?



$$t_i = 1 + \frac{1}{3}t_{i-1} + \frac{2}{3}t_{i+1}$$

$$3t_i = 3 + t_{i-1} + 2t_{i+1}$$

$$t_{i-1} - t_{i+1} = 3 + 2(t_{i+1} - t_i)$$

$$d_i = \underline{3 + 2d_{i+1}}$$

$$\Rightarrow \Theta(2^n) \text{ bad!!}$$

Schöning's Alg'm ('99)

start with a random assignment A

do same as Papadimitriou
but for $t = 3n$ steps

Claim $\Pr[\text{correct}] \geq \left(\frac{3}{4}\right)^n$ for 3SAT.

Repeat $c\left(\frac{4}{3}\right)^n$ times

$$\Rightarrow \Pr[\text{err}] \leq \left(1 - \left(\frac{3}{4}\right)^n\right)^{c\left(\frac{4}{3}\right)^n} \leq e^{-c}$$

$$\Rightarrow O^*(\left(\frac{4}{3}\right)^n) = \boxed{O^*(1.33...^n)}$$

Pf of Claim:

Let $p_i = \Pr(\text{(# steps to reach state } 0) \leq 3n \mid \text{start at state } i)$

Then $p_i \geq \Pr(\text{make } i \text{ increments & } 2i \text{ decrements} \mid \text{start at state } i)$

$$\geq \binom{3i}{i} \left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{2i}$$

$$= \frac{(3i)!}{i! (2i)!} \left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{2i}$$

Stirling's formula

$$\approx \frac{\left(\frac{3i}{e}\right)^{3i}}{\left(\frac{i}{e}\right)^i \left(\frac{2i}{e}\right)^{2i}} \left(\frac{2}{3}\right)^i \left(\frac{1}{3}\right)^{2i}$$

$$= \left(\frac{3^3}{2^2} \cdot \frac{2}{3} \cdot \frac{1}{3^2} \right)^i$$

$$= \frac{1}{2^i}.$$

$\Pr(\text{Start at state } i) = \Pr(\text{A matches exactly } i \text{ values of } A^*)$

$$= \frac{\binom{n}{i}}{2^n}$$

$\Rightarrow \Pr(\text{correct}) \geq \sum_{i=0}^n \frac{1}{2^i} \cdot \frac{\binom{n}{i}}{2^n}$

$$= \frac{1}{2^n} \sum_{i=0}^n \binom{n}{i} \left(\frac{1}{2}\right)^i \cdot 1^{n-i}$$

$$= \frac{1}{2^n} \left(\frac{1}{2} + 1\right)^n = \left(\frac{1}{2} \cdot \frac{3}{2}\right)^n - 127^n$$

browsing thru:
 $(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}$

$$\sum_{k=0}^{2^n} \binom{n}{k} = \frac{1}{2^n} \left(\frac{1}{2} + 1 \right)^n = \left(2 - \frac{1}{2} \right)^n = \left(\frac{3}{4} \right)^n.$$

□

Rmk - current record for 3SAT: $O(1.308^n)$ (Hertli '14)

extends to kSAT $\tilde{O}\left(\left(2 - \frac{2}{k}\right)^n\right)$ time
 $= \left(2 - O\left(\frac{1}{k}\right)\right)^n$

Strong Exp Time Hypothesis (SETH):

no $\tilde{O}(1.999^n)$ for all k.

General Framework: Markov Chains

Given set S of states

At each step, if current state is i,
 go to state j with prob. p_{ij}

for given values $p_{ij} \geq 0$ s.t. $\sum_j p_{ij} = 1$ ∀i.

Questions:

- expected # steps to reach state j from state i
 (hitting time)
- expected # steps to visit all states given start state i
 (cover time)
- in a walk with t steps,
 prob of ending in state i as $t \rightarrow \infty$ ↪
 (or frequency of state i) ↪
 (stationary distribution)
- how long to get close to stationary distribution
 (mixing time)
- in a walk with t steps,
 how close does sequence of visited states
 $\dots, s_t \dots$ get?

- in a walk with τ steps,
how close does sequence of visited states
behave like independent?
- ⋮
⋮
⋮

Def $h_{ij} =$ expected hitting time from i to j
 $=$ expected # steps to reach j starting at state i .

Def A vector $\pi = (\pi_i)_{i \in S}$ is a stationary distribution

if $\forall j \in S, \quad \boxed{\pi_j = \sum_i \pi_i P_{ij}}$

(in matrix form)
 $\pi = \pi P$
fixed pt

and $\sum_i \pi_i = 1, \quad \pi_i \geq 0.$

Thm ("Fundamental Thm of Markov Chains")
(i.e. graph is strongly connected)

For any finite, irreducible Markov chain,
a stationary distribution π exists with all $\pi_i > 0$
& is unique,

Furthermore, $h_{ii} = \frac{1}{\pi_i}$.

Pf: (Uniqueness) Suppose π, π' are both stationary.

Let j minimize $\frac{\pi_j}{\pi'_j}$. Let $c = \frac{\pi_j}{\pi'_j}$,

(know $\pi_i, \frac{\pi_i}{\pi'_i} \geq c$).

$$\Rightarrow \pi_j = \sum_i \pi_i P_{ij} \geq \sum_i c \pi'_i P_{ij} = c \pi'_j = \pi_j$$

$$\Rightarrow \pi = c \pi' \text{ and } c = 1.$$

(Existence) Fix a state s .

Talk random walk from s & stop when we first
go back to s .
~ - r ... : visited in this walk)

Talk round until we go back to s .

Define $\tilde{\pi}_i = E(\# \text{times } i \text{ is visited in this walk})$

(special case $\tilde{\pi}_s = 1$)
 (check $\tilde{\pi}_i < \infty$ by irreducibility...)

$$= \sum_{t=0}^{\infty} \Pr(t^{\text{th}} \text{ state in walk is } i \text{ & } \#\text{ steps} > t)$$

$$\begin{aligned} H_j, \sum_i \tilde{\pi}_i \cdot p_{ij} &= \sum_{t=0}^{\infty} \underbrace{\sum_i \Pr(t^{\text{th}} \text{ state is } i \text{ & } \#\text{ steps} > t)}_{\tilde{\pi}_j} \cdot p_{ij} \\ &= \sum_{t=0}^{\infty} \Pr((t+1)^{\text{th}} \text{ state is } j \text{ & } \#\text{ steps} > t). \\ &= \tilde{\pi}_j \quad (\text{for } j \neq s \text{ (check also for } j=s)). \end{aligned}$$

Let $\pi_i = \frac{\tilde{\pi}_i}{\sum_j \tilde{\pi}_j}$

(Hitting time)

$$\pi_s = \frac{\tilde{\pi}_s}{\sum_j \tilde{\pi}_j} = \frac{1}{h_{ss}}$$

true for all s by uniqueness. \square

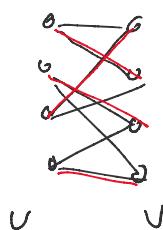
Appln: Perfect matching in Bipartite Graphs

Given bipartite graph $G = (U \cup V, E)$, $|U| = |V| = n$, $|E| = m$.

classical algm for perfect matching:

$O(mn)$ time by augmenting paths

$O(m\sqrt{n})$ by Hopcroft, Karp



but faster algm possible for d -regular case (all vertices have deg d)

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Gabow-Kariv '82: $\tilde{O}(m)$ time for $d = 2^k$.

:
Cole-Ost-Schirra '01: $\tilde{O}(m)$ time for any d
(related to edge coloring)

Surprisingly, can beat $\tilde{O}(m)$ using rand!!

by rand walk ...