March 29, 2021 9:07 AM

Perfect Matching in Bipartite Graphs

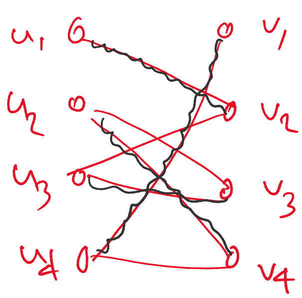
Given bipartite graph $G = (U \cup V, E)$

$U = \{u_1, .., u_n\}$, $V = \{v_1, .., v_n\}$

decide if $\exists$ perfect matching $M \subseteq E$

i.e. no 2 edges in M share a vertex

& $|M| = n$



special case of max flow

Hopcroft-Karp '73: $O(m\sqrt{n}) \leq O(n^{2.5})$

disadv: these alg'ms inherently sequential.

open: parallel alg'm with $O(\text{polylog } n)$ time & $O(\text{poly}(n))$ processors?

(i.e. in NC).

Fact (Schwartz-Zippel)

A nonzero polynomial Q in k vars of deg n over $\mathbb{Z}_p$ (p prime)

has $\leq n p^{k-1}$ roots.

(e.g. $Q(x_1,x_2,x_3) = x_1 x_2^2 + 3x_2 x_3 + 4x_1 x_3$)

(Thus, for rand $x_1, .., x_k \in \mathbb{Z}_p$,

$$\Pr[Q(x_1,..,x_k) \equiv 0 \pmod p] \leq \frac{np^{k-1}}{p^k} = \frac{n}{p}$$)

Pf: by induction on k.

$k=1$: done

Pf: by induction

$k=1$: done

Write $Q(x_1,\dots,x_k) = Q_m x_k^m + Q_{m-1} x_k^{m-1} + \dots + Q_0$

where Q_i: polynomials in $x_1,\dots,x_{k-1}$

$m \le n$.

Q_m nonzero polynomial of deg $\le n-m$.

$$\begin{aligned}
\#\text{ roots of } Q &\le p^{k-1}\cdot m \\
&\quad + (\#\text{ roots of } Q_m)\cdot p \\
&= p^{k-1}\cdot m + \left((n-m)\,p^{k-2}\right)\cdot p \\
&= p^{k-1}\cdot n. \qquad \square
\end{aligned}$$

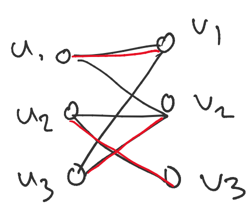
Define a polynomial in n^2 vars, with deg n:

$$Q_G(x_{11}, x_{12}, \dots, x_{nn}) = \det(A)$$

$$\text{where } A_{ij} = \begin{cases} x_{ij} & \text{if } (u_i, v_j) \in E \\ 0 & \text{else} \end{cases}$$

variable

Ex



$$\begin{aligned}
Q_G(x_{11},\dots,x_{33}) &= \begin{vmatrix} x_{11} & x_{12} & 0 \\ 0 & x_{22} & x_{23} \\ x_{31} & x_{32} & 0 \end{vmatrix} \\
&= x_{11}\begin{vmatrix} x_{22} & x_{23} \\ x_{32} & 0 \end{vmatrix} - x_{12}\begin{vmatrix} 0 & x_{23} \\ x_{31} & 0 \end{vmatrix} \\
&= -x_{11}x_{23}x_{32} + x_{12}x_{23}x_{31}
\end{aligned}$$

$2^{w(\text{matching})}$

Obs (Edmonds '67)

... of matching

Obs (Edmonds '67)

G has a perfect matching $\iff$ Q_G is nonzero polynomial.

Rand. Alg'm: (Lovasz '79)

fix prime $p \in [n^c, 2n^c]$

for each i,j, pick rand. $r_{ij} \in \{0, \dots, p-1\}$

if $Q_G(r_{11}, \dots, r_{nn}) = 0$ ← determinant reduces to matrix mult.

return no

else return yes.

$\Rightarrow \Pr(err) \leq \frac{n}{p} \leq \frac{1}{n^{c-1}}$

runtime: $\boxed{O(n^{\omega})} \leq O(n^{2.38})$

faster than Hopcroft-Karp

& can be parallelized!

one issue - if yes, how to output matching?

first idea -

for every $ij \in E$ in parallel

run alg'm on $G - \{i,j\}$ (remove i,j)

if ans is yes, include ij in output

naively, $O(mn^{\omega}) \leq O(n^{\omega+2})$ time

but can compute all $O(n^2)$ determinants faster

related to matrix adjoints & inverse

more importantly, parallelizable in $O(\text{polylog } n)$ time

$O(n^{\omega+2})$ processors

but not correct when perfect matching isn't unique!!

Mulmuley, Vazirani, & Vazirani's Alg'm ('87)

idea - Solve a tougher problem:

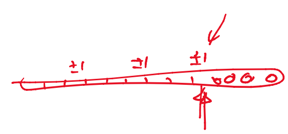
find min-weight perfect matching, for a weighted bipartite graph.

Let w_{ij} = weight of $u_i v_j$, W = max w_{ij}.

Obs Set $r_{ij} = 2^{w_{ij}}$.

If min perfect matching is unique, it can be determined from its weight $Q_G(r_{11}, \ldots, r_{nn})$.

Pf: $\displaystyle\sum_{\text{perf. matching } M} (\pm 1) \cdot 2^{\text{weight}(M)}$



least significant 1-bit. $\square$

$\Rightarrow$ by earlier idea, if unique, can find min perfect matching in $O(\text{polylog } n)$ time

$\tilde{O}(n^{\omega+2} W)$ processors (need W-bit #s)

How to assign edge weights to make ans unique??

just set w_{ij} randomly in $\{1, \ldots, n^c\}$ $(W = n^c)$

$\Rightarrow$ err prob $\le \frac{m}{n^c} \le \frac{1}{n^{c-2}}$.

Just set w_{ij} ...

$\underset{\text{by Iso. Lem}}{\Longrightarrow}$ err prob $\leq \frac{m}{n^c} \leq \frac{1}{n^{c-2}}$.

Isolation Lemma Given m elements X,

Given collection $\mathcal{C}$ of sets over X,

assign wts to elems randomly in $\{1, .., U\}$ (unif., indep).

$\Rightarrow$ $\Pr($ min-weight set in $\mathcal{C}$ is not unique$)$

$$\leq \frac{m}{U}.$$

Pf: Fix elem. $i \in X$.

Assume weights of all elems have been assigned, except i.

Let $f_i = \min_{S \in \mathcal{C} \text{ containing } i} w(S - \{i\})$.

$g_i = \min_{S \in \mathcal{C} \text{ not containing } i} w(S)$

Then min-weight set S^* has weight

$\min\{ f_i + w(i), \; g_i \}$.

If we choose $w(i)$ to be $< g_i - f_i$,
then $f_i + w(i)$ wins, i.e. S^* contains i.

If we choose $w(i)$ to be $> g_i - f_i$,
then g_i wins, i.e. S^* does not contain i.

If $w(i) = g_i - f_i$,
i is bad.

If no bad, S^* is unique.

$$\Pr[\, i \text{ bad} \,] \leq \frac{1}{U}$$

$$\Rightarrow \Pr(\, S^* \text{ not unique} \,) \leq m \cdot \frac{1}{U}. \qquad \square$$

We conclude that perfect matching $\in$ "RNC".

(derand. open).

Rmks: extend to non-bipartite, max matching, ...

sequential algms:

Mucha, Sankowski '04 $\tilde{O}(n^{\omega})$ time rand. for finding a perfect matching (or max).

Madry '13 $\tilde{O}(m^{10/7})$

Liu, Sidford '20 $\tilde{O}(m^{4/3})$

van den Brand et al. '20 $\tilde{O}(m + n^{3/2})$.

($>$ 100 pp)