March 8, 2021 9:06 AM

Last Time:

finding simple k-cycle $\tilde{O}(\underbrace{5.44}_{(2e)}{}^{k} n^{\omega})$

Rmks: for simple k-path, $\tilde{O}(5.44^k m)$

$\downarrow$

Williams '09: $\tilde{O}(2^k \mathrm{poly}(n))$.

deterministic?

Alon, Yuster, Zwick 95: $O(c^k \mathrm{poly}(n))$ for some large c

⋮

Tsur '19: $O(2.554^k \mathrm{poly}(n))$

Concentration Bds / Tail Inequalities

Markov's ineq: If $X \geq 0$, $\mu = E[X]$,

$$\Pr[X \geq c\mu] \leq \frac{1}{c} \qquad \left(\text{i.e. } \Pr[X \geq t] \leq \frac{\mu}{t}\right)$$

Chebyshev's ineq: If $\mu = E[X]$, $\sigma^2 = \mathrm{Var}[X]$,

$$\Pr(|X - \mu| \geq c\sigma) \leq \frac{1}{c^2} \qquad \left(\text{i.e. } \Pr[|X-\mu| \geq t] \leq \left(\frac{\sigma}{t}\right)^2\right)$$

Chernoff Bds

Thm 1 If $X = \sum_{i=1}^{n} X_i$ where

X_i's are indep. 0-1 rand vars

& $\mu = E[X]$,

(i) $\Pr(X \geq (1+\delta)\mu) < \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$

(i) $\Pr(X > (1+\delta)\mu) < \left(\dfrac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$

for all $\delta > 0$.

(ii) $\Pr(X < (1-\delta)\mu) < \left(\dfrac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^\mu$.

Pf:

idea - apply Markov to s^X for some const $s > 1$.

(i) $\Pr(X > (1+\delta)\mu) = \Pr(s^X > s^{(1+\delta)\mu})$

$$\le \frac{E(s^X)}{s^{(1+\delta)\mu}}$$

$$E(s^X) = E\left(s^{\sum_{i=1}^n X_i}\right)$$

$$= E\left[\prod_{i=1}^n s^{X_i}\right]$$

$$= \prod_{i=1}^n E(s^{X_i}) \quad \text{by indep.}$$

let $p_i = \Pr(X_i = 1)$

$\mu = \sum_{i=1}^n p_i$,

$$= \prod_{i=1}^n \left((1-p_i)\cdot 1 + p_i \cdot s\right)$$

$$= \prod_{i=1}^n (1 + (s-1)p_i)$$

$(1+x \le e^x)$

$$\le \prod_{i=1}^n e^{(s-1)p_i}$$

$\sum^n (s-1)p_i$

$$= e^{\sum_{i=1}^{n} (s-1) p_i}$$

$$= e^{(s-1)\mu}$$

$$\Rightarrow \Pr(X > (1+\delta)\mu) \leq \frac{e^{(s-1)\mu}}{s^{(1+\delta)\mu}}$$

$$= \left(\frac{e^{s-1}}{s^{1+\delta}} \right)^{\mu}$$

Set $s = 1+\delta$. $\square$

(ii) choose $s < 1$ instead.

$$\Pr(X \leq (1-\delta)\mu] = \Pr[s^X \geq s^{(1-\delta)\mu}]$$

$$\leq \frac{E[s^X]}{s^{(1-\delta)\mu}}.$$

$$\leq \left(\frac{e^{s-1}}{s^{1-\delta}} \right)^{\mu}.$$

Set $s = 1-\delta$. $\square$

Cor 2 (simplified form)

Under same assumption,

a) $\Pr[\,|X-\mu| > \delta\mu] \leq e^{-\Theta(\delta^2\mu)}$ if $\delta < 1$.

$\rightarrow$ (i.e. $\Pr[\,|X-\mu| > t] \leq e^{-\Theta(t^2/\mu)}$ if $t \leq \mu$)

b) $\Pr[X > c\mu] \leq c^{-\Theta(c\mu)}$ if $c \geq 2$.

(i.e. $\Pr[X > t] \leq \left(\frac{t}{\mu}\right)^{-\Theta(t)}$ if $t \geq 2\mu$)

Pf: a) as $\delta \to 0$,

$$\frac{e^{\delta}}{(1+\delta)^{1+\delta}} = \frac{e^{\delta}}{e^{(1+\delta)\ln(1+\delta)}}$$

$$= \frac{e^{\delta}}{e^{(1+\delta)(\delta - O(\delta^2))}}$$

$$= e^{-\delta^2 + O(\delta^3)}$$

similar for $\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}$ ... $\square$

b) Set $\delta = c-1$.

$$\frac{e^{\delta}}{(1+\delta)^{1+\delta}} = \frac{e^{c-1}}{c^c} = \frac{1}{c^{\Theta(c)}}. \quad \square$$

Consequences:

1. $X = O(\mu)$ with prob $\geq 1 - \frac{1}{N}$
   if $\mu \geq \log N$
   (by Cor 2b)

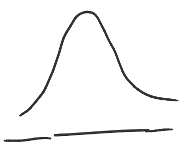
2. $X \in \mu \pm O(\sqrt{\mu \log N})$ with prob $\geq 1 - \frac{1}{N}$
   if $\mu \gg \log N$
   (by Cor 2a with $t = \Theta(\sqrt{\mu \log N})$)

3. if $\mu = O(1)$,
   $X = O\left(\frac{\log N}{\log\log N}\right)$ with prob $\geq 1 - \frac{1}{N}$
   $t^{\Theta(t)} \simeq N$

$\ldots \left(\overline{\log\log N}\right)$

(by Cor 2b with $t^{\Theta(t)} \approx N$
$t \log t \approx \log N$
$t \approx \dfrac{\log N}{\log\log N}$).

Rmks: for k-wise indep, back to k^{th} moment
$\Rightarrow$ weaker bds ...



- for i.i.d. case (all p_i's same),
  could directly analyze using bin coeffs ...

- proof technique works for many other distributions
  $\Rightarrow$ many diff. variants of Chernoff ...

Thm (Hoeffding's variant)

If $X = \sum_{i=1}^{n} X_i$ where X_i's are indep "bounded" vars $X_i \in [a_i, b_i]$,

$\mu = E[X]$,

$$\Pr(|X-\mu| > t) \leq e^{-\Theta\left(\frac{t^2}{\sum_{i=1}^{n}(b_i-a_i)^2}\right)}$$

for all t.

Pf: apply Markov to $s^{X-\mu}$ ...

$$E\left[s^{X-\mu}\right] = \prod E\underbrace{\left[s^{X_i-\mu_i}\right]}.$$