

A "hybrid" alg'm:

for  $i = 1$  to  $r$  do BoruvkaStep()  
Prim()

$\Rightarrow$  time  $O(\underline{rm} + \underbrace{\left(\frac{n}{2^r} \log n + m\right)}_{O(n)})$

Set  $r = \log \log n$

$= \boxed{O(m \log \log n)}$  time !!

⋮

History of "modern" MST alg's:

	Yao '75	$O(m \log \log n)$	$\leftarrow$ iterated log
	Fredman, Tarjan '85 (Fib heaps)	$O(m \log^* n)$	$\leftarrow$
$\uparrow$	Gabow et al. '86	$O(m \log(\log^* n))$	$\leftarrow$
$\uparrow$	Chazelle '97	$O(m \alpha(n))$	$\leftarrow$ inverse Ackermann very complicated

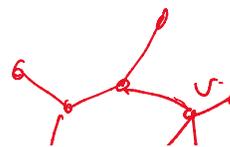
Karger's Rand. Alg'm '93

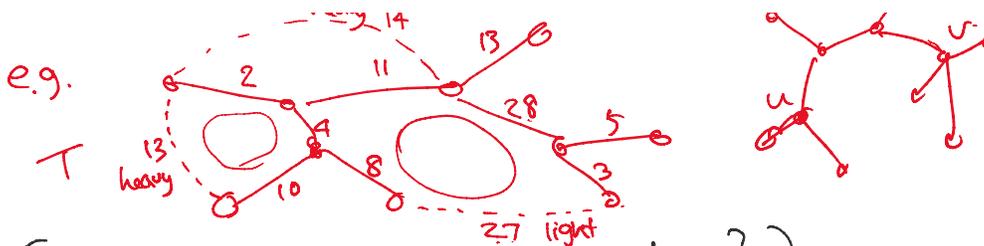
- idea - rand. sampling  
- use exclusion rule to reduce # edges

MST(E): (if G not connected, "min spanning-forest")

1. take rand subset  $R \subseteq E$  of size  $r$
2.  $T = \text{MST}(R)$
3. for each  $uv \in E$
4. classify  $uv$  heavy (w.r.t.  $T$ ) if  $uv \notin T$  and  $uv$  is heavier than all edges along path from  $u$  to  $v$  in  $T$   
light else

$O(m)$  time





5. return MST ( { all light edges } ).

Obs (i) If  $e$  is heavy w.r.t.  $T = \text{MST}(R)$ ,  
then  $\text{MST}(R \cup \{e\}) = \text{MST}(R)$ .  
else  $\text{MST}(R \cup \{e\})$  uses  $e$ . ←

(ii)  $T = \text{MST}(E)$  iff all edges  $e \notin T$   
are heavy w.r.t.  $T$ .

MST Verification Problem →

Implementation -

(line 4) known data structures  
for <sup>offline</sup> path max queries in trees

Tarjan '79  $O(m \alpha(n))$  time

Komlos '85/  $O(m)$  time

King '93/  
Buchstam et al. '98

←  $(nm)$

Sampling Lemma

For rand sample  $R \subseteq E$  of size  $r$ ,

$$E \left( \begin{array}{l} \# \text{ light edges} \\ \text{w.r.t. MST}(R) \end{array} \right) \leq \frac{nm}{r}$$

Pf: (C'98) Pick random  $e \in E$ .

Suffice to show  $\Pr[e \text{ is light}] \leq \frac{n}{r}$ .

idea - "backwards analysis"

Fix  $R' = R \cup \{e\}$ .

$\Pr[e \text{ is light} \mid \text{fixed } R']$

$= \Pr[\text{MST}(R \cup \{e\}) \text{ uses } e \mid \text{fixed } R']$   
by Obs (i).

$= \Pr[\text{MST}(R') \text{ uses } e \mid \text{fixed } R']$

$$= \Pr(\text{MST}(R') \text{ uses } e \mid \text{fixed } R')$$

$$\leq \frac{n-1}{|R'|} \leq \frac{n-1}{r} !!$$

unconditionally,

$$\Rightarrow \Pr(e \text{ is light}) \leq \frac{n-1}{r} \quad \square$$

(  $\Pr(A) = \sum \Pr(A|B_i) \Pr(B_i)$  )

Analysis: expected time

$$T(m, n) \leq T(r, n) + T\left(\frac{m}{r}, n\right) + O(m+n)$$

e.g. choose  $r = 2n$ :

$$T(m, n) \leq T(2n, n) + T\left(\frac{m}{2}, n\right) + O(m+n)$$

e.g. by Kruskal:  
 $O(n \log n)$

$$\Rightarrow T(m, n) \leq T\left(\frac{m}{2}, n\right) + O(m+n \log n)$$

$$\Rightarrow T(m, n) \leq O\left(m + n \log n + \frac{m}{2} + n \log n + \frac{m}{4} + n \log n + \dots\right)$$

(  $\log m = O(\log n)$  )

$$= O(m + n \log^2 n)$$

(linear when  $m > n \log^2 n$   
 without Fib heaps!)

final idea.

we have used exclusion rule to reduce  $m$

- but can use inclusion rule to reduce  $n$

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## Karger, Klein, and Tarjan's modified alg'm ('94)

MST( $E$ ):

0. for  $i=1$  to 3 do BoruvkaStep()

1. :

2. : Same as before

3.

4.

↙

$n \rightarrow \frac{n}{8}$

$$T(m, n) \leq T\left(\frac{n}{4}, \frac{n}{8}\right) + T\left(\frac{m}{2}, \frac{n}{8}\right) + c(m+n)$$

Guess  $T(m, n) \leq c'(m+n)$

Verify by induction:

$$T(m, n) \leq c'\left(\frac{n}{4} + \frac{n}{8}\right) + c'\left(\frac{m}{2} + \frac{n}{8}\right) + c(m+n)$$

$$= \left(\frac{c'}{2} + c\right)m + \left(\frac{c'}{2} + c\right)n$$

$$= \left(\frac{c'}{2} + c\right)(m+n)$$

$$\leq c'(m+n)$$

as long as  $\frac{c'}{2} + c \leq c'$

i.e.  $c' \geq 2c$ .

$\Rightarrow \boxed{O(m+n)}$  expected time: (linear for all  $m$ !)

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Pettie, Ramachandran:

reduce # rand bits ...