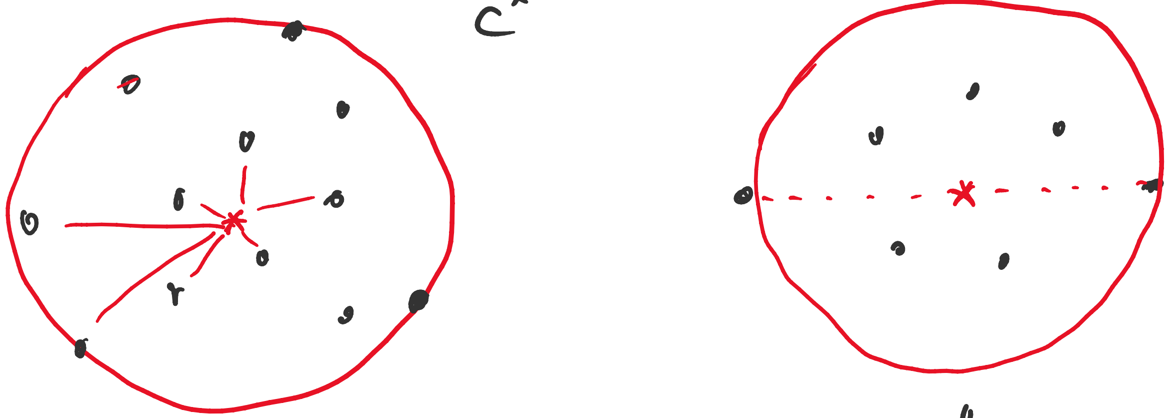
February 22, 2021 9:16 AM

VI

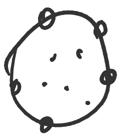
Min Enclosing Circle

Given set S of n pts in 2D,
find smallest circle enclosing S .
C^*

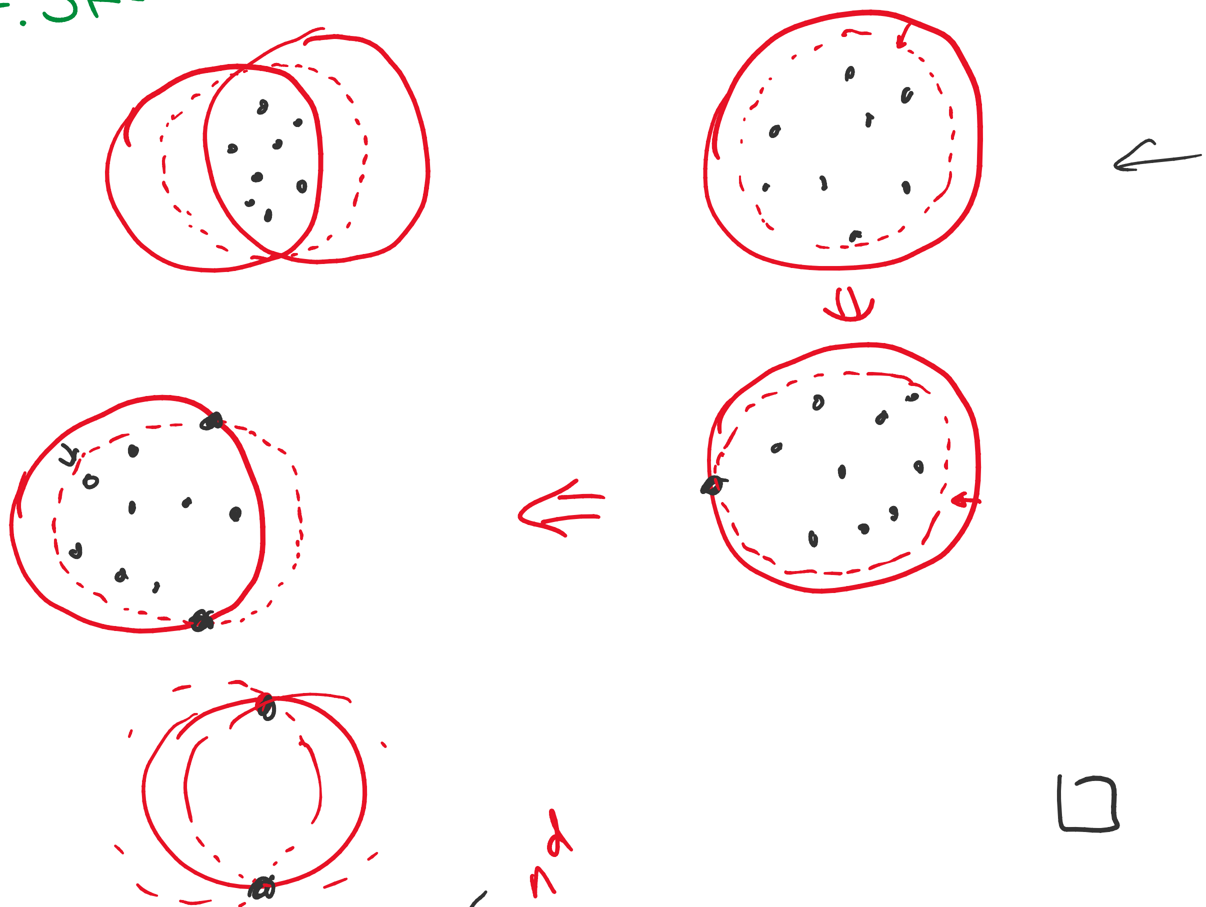


(appl'ns: bounding volume, "center point", facility location, ...)

Obs: C^* is unique & has 3 pts on boundary or 2 pts as diameter ⇒ brute force $O(n^3 \cdot n)$ time



Pf: Sketch:



□

brute force $O(n^3 \cdot n) = O(n^4)$ time

farthest-point Voronoi diagram $O(n \log n)$ time

$O(n)$ time

Voronoi diagram

Megiddo/Dyer '83 $O(n)$ time but complicated...

Seidel/Welzl '90: $O(n)$ expected time but simpler

by rand. incremental alg'm ...

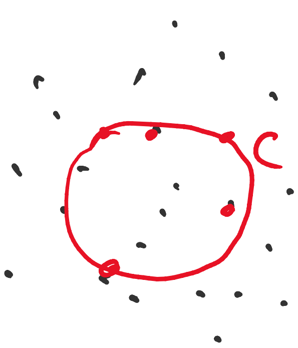
closely related to linear programming (LP) in $d+1$ vars

Clarkson's Alg'm ('88)

idea - by rand. sampling!

MEC(S):

1. pick rand. subset $R \subseteq S$ of size r
2. repeat 4 times {
3. $C = MEC(R)$
4. add $\{p \in S : p \text{ outside } C\}$ to R.

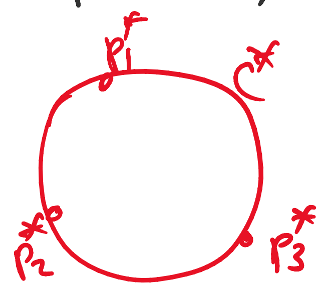


} (until no new pts are added to R)

5. return C

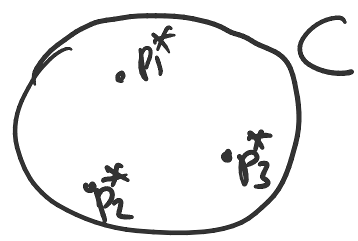
Correctness & # repeats ≤ 4:

let C^* be opt circle, defined by 3 pts $\{p_1^*, p_2^*, p_3^*\}$



at each iteration.

Case 1. if all p_1^*, p_2^*, p_3^* are inside C,



then

$$\text{radius}(\underbrace{MEC(S)}_{C^*}) = \text{radius}(MEC(p_1^* p_2^* p_3^*)) \le \text{radius}(C)$$

$$\leq \text{radius}(C)$$
$$= \text{radius}(MEC(R))$$
$$\leq \text{radius}(MEC(S))$$

$\Rightarrow C = C^*$: done.

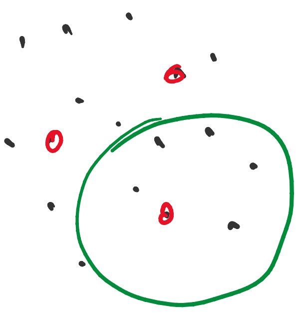
Case 2: Otherwise, at least one of p_1^*, p_2^*, p_3^* is outside C. & is added to R.

$\Rightarrow$ 4 iterations suffice. $\square$

Runtime Analysis:

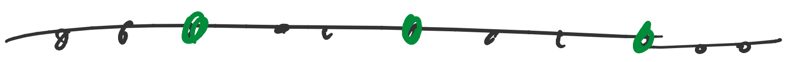
Def A subset $R \subseteq S$ is called an <u>ε-net</u>

if $\forall$ circle C (exterior of) containing $\geq \varepsilon n$ pts of S,
$\Rightarrow$ (exterior of) C contains ≥ 1 pt of R.



(i.e. R is a hitting set of all large sets defined by circles)

1D :



εn

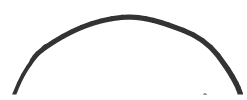
ε-net of size $\lceil \frac{1}{\varepsilon} \rceil$

Lemma $\exists$ ε-net R of size $r := O(\frac{1}{\varepsilon} \log n)$.

In fact, rand sample R works w.h.p. ("with high prob." $\geq 1 - \frac{1}{n^c}$)

Pf: Fix circle C with $\geq \varepsilon n$ pts of S in exterior

Pr[exterior of C contains no pt of R]



$\Pr[\overset{\text{exterior of}}{C} \text{ contains no pt of } R]$

$\leq (1-\varepsilon)^r$

(assume sampling w. replacement)

choose $r = \frac{c}{\varepsilon} \ln n$

if we flip coins
$\leq \left(1-\frac{r}{n}\right)^{\varepsilon n}$
$\leq e^{-\varepsilon r}$

$\leq e^{-\varepsilon r}$

$1-\varepsilon \leq e^{-\varepsilon}$

$= e^{-c \ln n} = \frac{1}{n^c}$.

"distinct" circles $= O(n^3)$

$\Rightarrow \Pr[\text{err}] \leq O(n^3) \cdot \frac{1}{n^c}$ by union bd

$= O\left(\frac{1}{n^{c-3}}\right)$. $\square$

Rmk: size can be reduced to $O\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)$ . . .

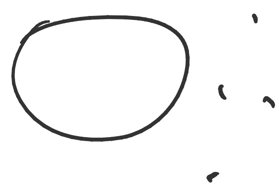
for any "range space of bounded VC dims"

(for circles in 2D, $\exists$ ε-net of size $O\left(\frac{1}{\varepsilon}\right)$)

Back to Clarkson's alg'm . . .

In line 3, we know C contains no pts of R in exterior.

$\Rightarrow$ C contains $\leq \varepsilon n$ pts of S in exterior



(by contrapositive of ε-net prop)

$A \Rightarrow B$

with $\varepsilon \approx \left(\frac{1}{r}\right) \log n$

$r \approx \frac{1}{\varepsilon} \log n$

$= O\left(\frac{n}{r} \log n\right)$ pts of S

$A \Rightarrow B$
$\neg B \Rightarrow \neg A$

$= O\left(\frac{n}{r} \log n\right)$ pts of S.

$\Rightarrow$ in each iter,

$|R| = O\left(r + 3 * \frac{n}{r} \log n\right)$

Set $r = \sqrt{n}$ $\Rightarrow O\left(\sqrt{n} \log n\right)$

$\Rightarrow T(n) \le 4\, T\left(\sqrt{n}\, \cancel{\log} n\right) + O(n)$

ignore

$\Rightarrow O\left(n + 4\sqrt{n} + 16\, n^{1/4} + \cdots\right)$

$= \boxed{O(n)}$