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Independent set – Turán’s theorem

I don’t know why it should be, I am sure; but the sight of another man asleep in bed when I am up, maddens me. It seems to me so shocking to see the precious hours of a man’s life - the priceless moments that will never come back to him again - being wasted in mere brutish sleep.

Jerome K. Jerome, Three men in a boat

19.1. Turán’s theorem

19.1.1. Some silly helper lemmas

We will need the following well-known inequality.

Lemma 19.1.1 (AM-GM inequality: Arithmetic and geometric means inequality).

For any $x_1, \dots, x_n \geq 0$ we have $\frac{x_1 + x_2 + \dots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \dots x_n}$.

This inequality readily implies the “inverse” inequality: $\frac{1}{\sqrt[n]{x_1 x_2 \dots x_n}} \geq \frac{n}{x_1 + x_2 + \dots + x_n}$

Lemma 19.1.2. Let $x_1, \dots, x_n \geq 0$ be n numbers. We have that $\sum_{i=1}^n \frac{1}{x_i} \geq \frac{n}{(\sum_i x_i)/n}$.

Proof: By the SM-GM inequality and then its “inverse” form, we have

$$\frac{\sum_{i=1}^n \frac{1}{x_i}}{n} = \frac{1/x_1 + 1/x_2 + \dots + 1/x_n}{n} \geq \sqrt[n]{(1/x_1)(1/x_2) \dots (1/x_n)} = \frac{1}{\sqrt[n]{x_1 x_2 \dots x_n}} \geq \frac{n}{x_1 + x_2 + \dots}$$


Lemma 19.1.3. *Let $G = (V, E)$ be a graph with n vertices, and let d_G be the average degree in the graph. We have that $\sum_{v \in V} \frac{1}{1 + d(v)} \geq \frac{n}{1 + d_G}$.*

Proof: Let the i th vertex in G be v_i . Set $x_i = 1 + d(v_i)$, for all i . By [Lemma 19.1.2](#), we have

$$\sum_{i=1}^n \frac{1}{1 + d(v_i)} = \sum_{i=1}^n \frac{1}{x_i} \geq \frac{n}{(\sum_i x_i)/n} = \frac{n}{[\sum_i (1 + d(v_i))]/n} = \frac{n}{1 + d_G}. \quad \color{blue}{\text{🐼}}$$

19.1.2. Statement and proof

Theorem 19.1.4 (Turán’s theorem). *Let $G = (V, E)$ be a graph with n vertices. The graph G has an independent set of size at least $\frac{n}{1 + d_G}$, where d_G is the average vertex degree in G .*

Proof: Let $\pi = (\pi_1, \dots, \pi_n)$ be a random permutation of the vertices of G . Pick the vertex π_i into the independent set if none of its neighbors appear before it in π . Clearly, v appears in the independent set if and only if it appears in the permutation before all its $d(v)$ neighbors. The probability for this is $1/(1 + d(v))$. Thus, the expected size of the independent set is (exactly)

$$\tau = \sum_{v \in V} \frac{1}{1 + d(v)}, \quad (19.1)$$

by linearity of expectations. Thus, by the probabilistic method, there exists an independent set in G of size at least τ . The claim now readily follows from [Lemma 19.1.3](#). 

19.1.3. An alternative proof of Turán’s theorem

Following a post of this write-up on my blog, readers suggested two modifications. We present an alternative proof incorporating both suggestions.

Alternative proof of Theorem 19.1.4: We associate a charge of size $1/(d(v) + 1)$ with each vertex of G . Let $\gamma(G)$ denote the total charge of the vertices of G . We prove, using induction, that there is always an

independent set in G of size at least $\gamma(G)$. If G is the empty graph, then the claim trivially holds. Otherwise, assume that it holds if the graph has at most $n - 1$ vertices, and consider the vertex v of lowest degree in G . The total charge of v and its neighbors is

$$\frac{1}{d(v) + 1} + \sum_{uv \in E} \frac{1}{d(u) + 1} \leq \frac{1}{d(v) + 1} + \sum_{uv \in E} \frac{1}{d(v) + 1} = \frac{d(v) + 1}{d(v) + 1} = 1,$$

since $d(u) \geq d(v)$, for all $uv \in E$. Now, consider the graph H resulting from removing v and its neighbors from G . Clearly, $\gamma(H)$ is larger (or equal) to the total charge of the vertices of $V(H)$ in G , as their degree had either decreased (or remained the same). As such, by induction, we have an independent set in H of size at least $\gamma(H)$. Together with v this forms an independent set in G of size at least $\gamma(H) + 1 \geq \gamma(G)$. Implying that there exists an independent set in G of size

$$\tau = \sum_{v \in V} \frac{1}{1 + d(v)}, \quad (19.2)$$

Now, set $x_v = 1 + d(v)$, and observe that

$$(n + 2|E|)\tau = \left(\sum_{v \in V} x_v \right) \left(\sum_{v \in V} \frac{1}{x_v} \right) \geq \left(\sum_{v \in V} \sqrt{x_v} \frac{1}{\sqrt{x_v}} \right)^2 = n^2,$$

using Cauchy-Schwartz inequality. Namely, $\tau \geq \frac{n^2}{n + 2|E|} = \frac{n}{1 + 2|E|/n} = \frac{n}{1 + d_G}$. 

Lemma 19.1.5 (Cauchy-Schwartz inequality). *For positive numbers $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$, we have*

$$\sum_i \alpha_i \beta_i \leq \sqrt{\sum_i \alpha_i^2} \sqrt{\sum_i \beta_i^2}.$$

19.1.4. An algorithm for the weighted case

In the weighted case, we associate weight $w(v)$ with each vertex of G , and we are interested in the maximum weight independent set in G . Deploying the algorithm described in the first proof of [Theorem 19.1.4](#), implies the following.

Lemma 19.1.6. *The graph $G = (V, E)$ has an independent set of size*

$$\geq \sum_{v \in V} \frac{w(v)}{1 + d(v)}.$$

Proof: By linearity of expectations, we have that the expected weight of the independent set computed is equal to

$$\sum_{v \in V} w(v) \cdot \mathbb{P}[v \text{ in the independent set}] = \sum_{v \in V} \frac{w(v)}{1 + d(v)}, \quad \color{red}{\mathfrak{z}}$$

References

- [MR95] R. Motwani and P. Raghavan. *Randomized Algorithms*. Cambridge, UK: Cambridge University Press, 1995.