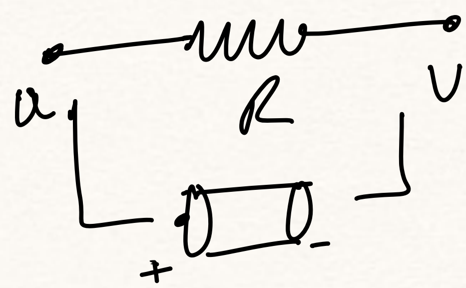
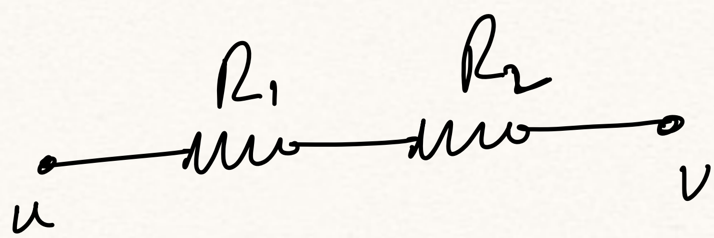
Electrical Networks and Random Walks

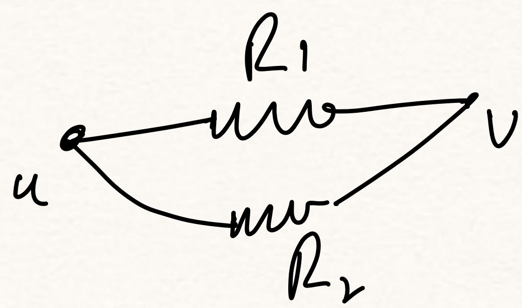


Ohm's law $V = IR$

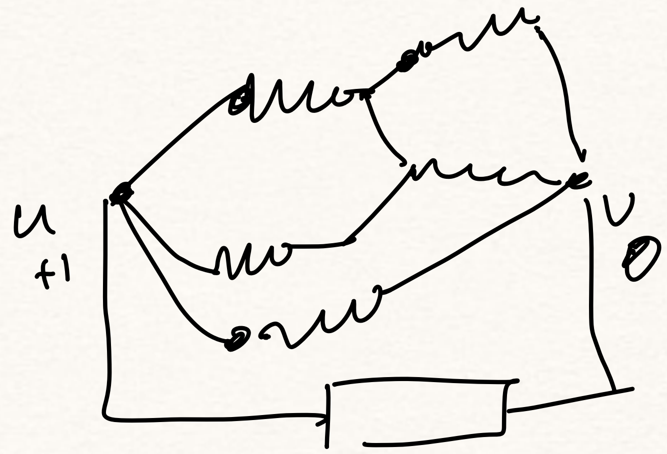
voltage = current times resistance.



effective resistance is $R_1 + R_2$



effective resistance is $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.



?

Let $R_{u,v}$ be "effective resistance" between u and v which is defined as $\frac{1}{i_{u,v}}$ where $i_{u,v}$ is the current from u to v if voltage diff of 1 is applied.

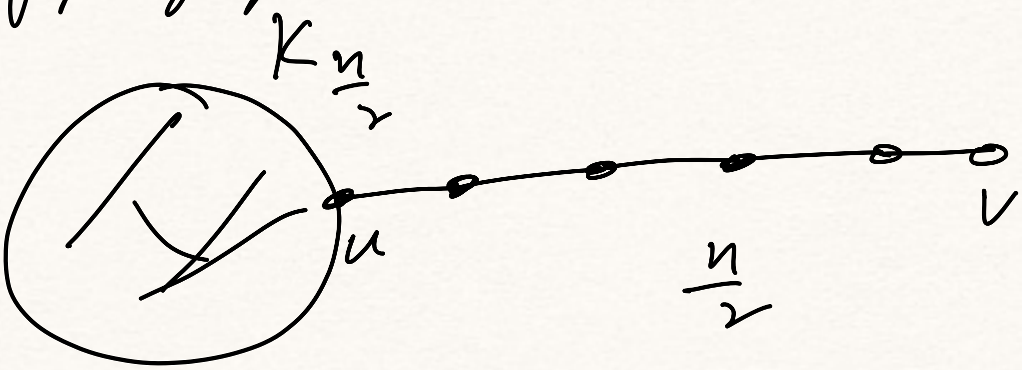
Theorem: $\forall u,v \quad C_{uv} = h_{u,v} + h_{v,u}$
$$= 2m R_{u,v}.$$

Corollary: If $uv \in E$ then

$$C_{uv} \leq 2m.$$

Proof: Because $R_{u,v} \leq 1$

Ex: Lollipop graph L_n



$R_{u,v} = \frac{n}{2}$

$\Rightarrow C_{u,v} = \Theta(n^3) \Rightarrow$ Either $h_{u,v}$ or $h_{v,u} = \Theta(n^3)$.

We saw previously that $C(G) \leq O(mn) = O(n^3)$.

So tight.

Electrical Flows

Let $G=(V,E)$ be an undirected graph. We will work with electrical flows. Typically we work with standard flows in directed graphs.

Background in regular flows:

Suppose $d: V \to \mathbb{R}$ is a demand vector. $d(v)$ is demand at v.

$d(v) < 0 \Rightarrow$ flow coming into v.

$d(v) > 0 \Rightarrow$ flow leaving.

$G=(V,E)$ is a directed graph

We have $\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = d(v)$.

This requires $\sum_v d(v) = 0$.

Special case is when
$d(v) = 0$ except at s, t.

Typically we have capacity constraints $c(e)$ $e \in E$
and want $0 \leq f(e) \leq c(e)$.

We know that if $\exists$ a feasible flow then $\exists$ a flow that is acyclic.

Feasibility of flow can be written as an LP.

$$\sum_{e \in \delta^+(v)} f(e) - \sum_{e \in \delta^-(v)} f(e) = d(v) \quad \forall v \in V$$

$$f(e) \leq c(e) \quad \forall e \in E$$

$$f(e) \geq 0 \quad \forall e \in E.$$

We will now consider electrical flows in undirected graphs.

For this we orient each edge e arbitrarily. We will fix this orientation and let $f(e)$ be positive or negative.

If $f(e) > 0 \Rightarrow f(e)$ amount of flow is going along the chosen orientation, otherwise $-f(e)$ flow is going in the reverse direction.

With this notation we have

$$\sum_{v \in N(u)} f(u,v) = d(u) \quad \forall u \in V$$

where $d(v)$ is the current injected into u (+ve or negative).

Each edge $e \in E$ has a resistance $r(e) > 0$. Electrical flow minimizes energy.

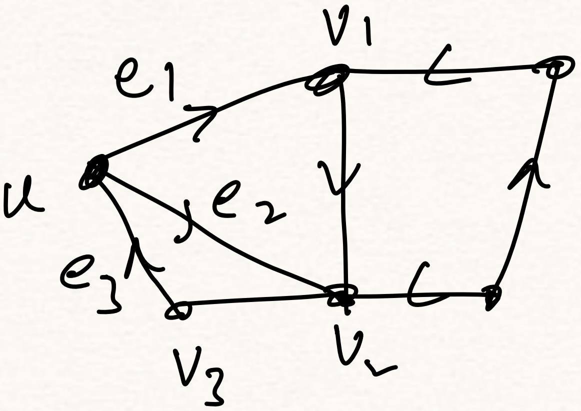
Energy consumed by $f(e)$ flowing over $r(e)$ is $r(e)(f(e))^2$ = voltage × current

Thus an electrical flow is one that

$$\min \sum_e r(e)(f(e))^2$$

$$\sum_{v \in N(u)} f(u,v) = d(u) \quad \forall u \in V$$

Example:



fix orientation

for u we write

$$f(e_1) + f(e_2) - f(e_3) = d(u)$$

Note that there are no capacity constraints. This is a constrained optimization problem. Objective is convex quadratic and constraint is a set of linear constraints.

Abstractly we have a problem

of the form

$$\min g(\bar{x})$$

s.t.

$$A\bar{x} = \bar{b}$$

where g is convex. and $\bar{x} \in R^n$
A is $m \times n$.

Theorem:

Let x^* be an opt solution.

Then $g'(\bar{x}^*) = A^T y$ for some $\bar{y} \in R^n$

where $g'(\bar{x}^*)$ is the derivative of g. To see this we let

$$\mathrm{Ker}(A) = \{\bar{x} \mid A\bar{x} = \bar{0}\}$$

We claim that $g'(x^*)$ is orthogonal

to Ker(A). Suppose not. $\exists\ \bar{z} \in \text{Ker}(A)$
such that $\langle g'(x), \bar{z} \rangle \neq 0$, assume < 0
wlog.

then $x^* + \varepsilon \bar{z}$ satisfies

$$A(x^* + \varepsilon z) = Ax^* = b.$$

$$g(x^* + \varepsilon \bar{z}) \approx g(x^*) + \langle g'(x^*), \varepsilon \bar{z}^* \rangle$$
$$< g(x^*)$$

a contradiction to optimality of x^*.

Geometrically, if $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m \in \mathbb{R}^n$
are the rows of A then Ker(A) is
the set of all vectors that are orthogonal
to the space spanned by $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$.
$\Rightarrow$ any vector orthogonal to Ker(A) is
in the span$(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m)$.

$\Rightarrow$ Such a vector is of the form $\sum_{i=1}^{m} y_i \bar{a}_i = A^T \bar{y}$ for some $\bar{y} \in \mathbb{R}^m$.

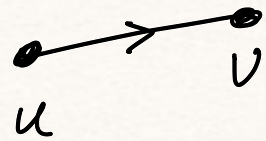
$$A = \begin{bmatrix} — & \bar{a}_1 & — \\ — & \bar{a}_2 & — \\ & \vdots & \\ — & \bar{a}_m & — \end{bmatrix} \qquad A^T = \begin{bmatrix} | & | & & | \\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_m \\ | & | & & | \end{bmatrix}$$

$$A^T \bar{y} = \sum_{i=1}^{m} y_i \bar{a}_i .$$

Therefore $g'(x^*) = A^T \bar{y}$ for some $\bar{y} \in \mathbb{R}^m$.

Applying this to our electrical flow problem. Suppose $f^*(e)$ $e \in E$ is an optimum/min-energy electrical flow. Then $\exists\ p : V \to \mathbb{R}$ such that

$$f(u,v) = \frac{p(u) - p(v)}{r(e)}.$$



In other words $\exists$ voltages that induce the current.

Ohm's law!

Thus for any demands $\exists$ potentials that induce the electrical flow that meets the demands via Ohm's law.

Note: shifting all potentials by Δ does not charge currents. Hence can assume one is 0.

Connecting Electrical Flow and Hitting times

Fix a vertex $t \in V$. Want to find $h_{u,t}$ $\forall u \in V - \{t\}$. Recall $h_{u,t}$ is the expected time ~~to~~ for the random walk to hit t starting at u. We let $h_{t,t} = 0$.

These values satisfy the following equations:

$$h_{u,t} = 1 + \frac{1}{deg(u)} \cdot \sum_{v \in N(u)} h(v,t).$$

We consider potentials on V
where $p(u) = h_{u,t}$ with $p(t) = 0$

This induces flow

$$f(u,v) = \frac{p(u) - p(v)}{r(u,v)} = \frac{h(u,t) - h(v,t)}{1}$$

$$= h(u,t) - h(v,t).$$

We see that

$$\sum_{v \in N(u)} f(u,v) = -\sum_{v \in N(u)} h(v,t) + \deg(u)\, h(u,t)$$

$$= \deg(u).$$

if $u \neq t$

$\Rightarrow$ current leaving u is $\deg(u)$.

By flow conservation flow <u>entering</u>

t is $\sum_{u \neq t} deg(u) = 2m - deg(t)$.

Thus $h(u,t)$ values correspond to vertex potentials induced by injecting $deg(u)$ current at each $u \neq t$ and draining $2m - deg(t)$ units at t.

<u>Lemma</u>: Let $G = (V, E)$ be an undirected graph and $s \neq t$.

$h_{s,t} + h_{t,s} = 2m R_{s,t}$ where $R_{s,t}$ is the effective resistance between s and t.

Proof: We have seen that $h_{u,t}$ values corresponds to potentials that route $2m - deg(t)$ flow/current into t and $deg(u)$ flow out at each u.

Similarly $h_{u,s}$ corresponds to potentials that drain $2m - deg(s)$ units of current at s and $deg(u)$ flow out from each $v \neq s$.

Let $p(u) = h_{u,t}$ and $q(u) = h_{u,s}$ $\forall u \in V$

Consider $p(u) - q(u)$.

This induces a subtraction of the

two flows. Let $d(u)$ be net flow at u.

$\forall u \notin \{s,t\}$ $\quad d(u) = 0$.

$$d(s) = \deg(s) - (-(2m - \deg(s)))$$

$$= 2m.$$

$$d(t) = -(2m - \deg(t)) - \deg(t)$$

$$= -2m.$$

$$p(s) - q(s) = h_{s,t} - h_{s,s} = h_{s,t}$$

$$p(t) - q(t) = h_{t,t} - h_{t,s} = -h_{t,s}$$

$\Rightarrow$ potential difference is $h(s,t) + h(t,s)$.

and this induces current $2m$ from s to t.

$\Rightarrow \quad h(s,t) + h(t,s) = 2m\, R_{eff}(s,t)$.

$\square$

Graph Laplacian

Given an (undirected) graph we can associate several matrices with G.

Adjacency matrix is the natural one

$A_{ji} = A_{ij} = 1$ if edge ij 0 otherwise. Symmetric if G is undir

For directed graphs we set A_{ij} is 1 if $(i,j) \in E$. Not necessarily symmetric.

Edge - vertex incident graph.

$m \times n$. matrix

$B_{e,u} = 1$ if $v \in e$, 0 otherwise

directed $B_{(u,v),u} = 1$ $B_{(u,v),v} = -1$.

In a directed graph the set of flows that satisfy a demand vector d can be conveniently written as $B\bar{f} = \bar{d}$.

Note that f is not restricted to be non-negative.

When working with electrical flows we orient the edges arbitrarily and use B for the resulting edge-vertex incident matrix.

Recall that when we solved the electrical flow optimization problem we wrote it as

$$\min \sum_{e} r(e) f(e)^2$$

$$B\bar{f} = \bar{d}.$$

and opt. solution f^* satisfies

$$2 \langle R^T, f^* \rangle = B^T p \quad \text{where}$$

p is the set of potentials.

We can directly write a linear system to find the potentials that satisfy demands via Ohm's law.

What should we have.

$$\sum_{v \in N(u)} f(u,v) = d(u).$$

But $f(u,v) = \dfrac{p(u) - p(v)}{r(e)}$.

Let $w(e) = \dfrac{1}{r(e)}$.

$$w(e) \sum_{v \in N(u)} [p(u) - p(v)] = d(u). \quad \forall u \in V$$

$$\Rightarrow \quad -\sum_{v \in N(u)} w(u,v)\, p(v) + \Big(\sum_{v} w(u,v)\Big) p(u) = d(u)$$

$$\forall u \in V$$

This is a linear system

$$\mathcal{L}_G \bar{p} = \bar{d}$$

where $\mathcal{L}_G$ is the Laplacian of G.

$\mathcal{L}_G(u,u)$ is $\sum_{e \in \delta(u)} w(e)$ is the weighted degree of u

$$\mathcal{L}_G(u,v) = \mathcal{L}_G(v,u) = -w(u,v)$$

$\mathcal{L}_G$ is a symmetric diagonally dominant matrix

ie $|D_{ii}| \geq \sum_{j \neq i} |d_{ij}| \quad \forall \; i$.

Can solve $\mathcal{L}_G \bar{\phi} = \bar{d}$ to compute $R_{u,v}$ and $h_{u,v}$ values in polynomial time. Near linear time algorithms are known.

L_G is singular but for a connected graph its rank is $(n-1)$.

Computational considerations

$h_{u,t}$ values can be computed by solving $L_G \bar{p} = \bar{d}$

where $d(u) = deg(u) \ \forall u \neq t$
and $d(t) = -(2m - deg(t))$