Electrical Networks and Random Walks

1 Ohm's Law

Ohm's Law: $V = IR$

where V is voltage, I is current, and R is resistance.

For effective resistance:

$$R_{\text{series}}: \quad R = R_1 + R_2$$
$$R_{\text{parallel}}: \quad R = \frac{R_1 R_2}{R_1 + R_2}$$

Let R_{uv} be the effective resistance between vertices u and v, defined as the resistance between u and v when a voltage difference of 1 is applied, with current i_{uv} flowing from u to v.

Corollary 1. *If $(u, v) \in E$, then*

$$C_{uv} = \frac{2m}{R_{uv}}$$

Proof. Because $R_{uv} = I$ [from Ohm's law definition] □

2 Example: Lollipop Graph

For a lollipop graph, we can compute the effective resistance using series and parallel resistance formulas.

3 Electrical Flow

Let $G = (V, E)$ be an undirected graph. We work with electrical flow, as opposed to standard flows in directed graphs.

3.1 Standard Flow (Directed Graphs)

Basic Setup: Suppose $d : V \to \mathbb{R}$ is a demand vector, where $d(v)$ represents the demand at v.

- $d(v) > 0$: flow coming into v
- $d(v) < 0$: flow leaving v

For a directed graph $G = (V, E)$, flow conservation requires:

$$\sum_{e \text{ out of } u} f_e - \sum_{e \text{ into } u} f_e = d(u)$$

This requires $\sum_{u \in V} d(u) = 0$.
A special case is when $d(v) = 0$ except at source s and sink t.
Capacity Constraints: Typically we have capacity constraints C_e for $e \in E$ and want $0 \leq f_e \leq C_e$.
Note: If there exists a feasible flow, then there exists an acyclic feasible flow.
Feasibility as LP: Feasibility of flow can be written as a linear program:

$$\sum_{e \text{ out of } u} f_e - \sum_{e \text{ into } u} f_e = d(u) \quad \forall u \in V$$
$$0 \leq f_e \leq C_e \quad \forall e \in E$$

3.2 Electrical Flow (Undirected Graphs)

We now consider electrical flows in undirected graphs. For each edge e, we orient it arbitrarily and fix this orientation. Let f_e be positive or negative:

- If $f_e > 0$: flow goes along the chosen orientation
- If $f_e < 0$: flow goes in the reverse direction

Flow Conservation: With this notation:

$$\sum_{v \in N(u)} f_{u,v} = d(u) \quad \forall u \in V$$

where $d(u)$ is the current injected into u (can be positive or negative).

Energy Minimization: Each edge $e \in E$ has a resistance $r_e \geq 0$. An electrical flow minimizes energy:

The energy consumed by flow f_e over edge with resistance r_e is $r_e f_e^2$ (voltage $\times$ current).

Thus, an electrical flow minimizes:

$$E(f) = \sum_{e \in E} r_e f_e^2$$

Optimization Problem:

$$\begin{aligned} \min \quad & \sum_{e \in E} r_e f_e^2 \\ \text{s.t.} \quad & \sum_{v \in N(u)} f_{u,v} = d(u) \quad \forall u \in V \end{aligned}$$

where f_e are unrestricted (no capacity constraints). The objective is convex quadratic and the constraint is a set of linear constraints.

4 Constrained Optimization and Lagrange Multipliers

Abstract Problem:

$$\min g(\bar{x}) \quad \text{s.t.} \quad A\bar{x} = b$$

where g is convex and $\bar{x} \in \mathbb{R}^n$, A is $m \times n$.

Theorem 1. *Let $\bar{x}^*$ be an optimal solution. Then $\nabla g(\bar{x}^*) = A^T y$ for some $y \in \mathbb{R}^m$, where $\nabla g(\bar{x}^*)$ is the gradient of g.*

Proof. Let $\text{Ker}(A) = \{\bar{x} : A\bar{x} = 0\}$.

We claim that $\nabla g(\bar{x}^*)$ is orthogonal to $\text{Ker}(A)$.

Suppose not. Then there exists $\bar{z} \in \text{Ker}(A)$ such that $\nabla g(\bar{x}^*) \cdot \bar{z} > 0$.

For small $\epsilon > 0$, let $\bar{x} = \bar{x}^* + \epsilon \bar{z}$. Then:

- $A\bar{x} = A(\bar{x}^* + \epsilon \bar{z}) = A\bar{x}^* + \epsilon A\bar{z} = b + 0 = b$
- $g(\bar{x}) = g(\bar{x}^* + \epsilon \bar{z}) \approx g(\bar{x}^*) + \epsilon \nabla g(\bar{x}^*) \cdot \bar{z} < g(\bar{x}^*)$

This contradicts optimality of $\bar{x}^*$.

Geometrically: If $a_1, \ldots, a_m \in \mathbb{R}^n$ are the rows of A, then $\text{Ker}(A)$ is the set of all vectors orthogonal to the space spanned by $a_1, \ldots, a_m$.

Any vector orthogonal to $\text{Ker}(A)$ is in the span of $a_1, \ldots, a_m$, and is of the form:

$$\sum_{i=1}^{m} y_i a_i = A^T y \quad \text{for some } y \in \mathbb{R}^m$$

Therefore, $\nabla g(\bar{x}^*) = A^T y$ for some $y \in \mathbb{R}^m$. □

5 Application to Electrical Flow

Theorem: Suppose $f^* = (f_e^*)_{e \in E}$ is an optimum solution to the electrical flow problem. Then there exists a potential function $p : V \to \mathbb{R}$ such that:

$$f_e^* = \frac{p(u) - p(v)}{r_e}$$

for each edge $e = (u, v)$.

In other words, there exist voltages that induce the electrical current via Ohm's law.

Note: Shifting all potentials by a constant Δ does not change currents. Hence, we can assume $p(t) = 0$ for some vertex t.

6 Connecting Electrical Flow and Hitting Times

Fix a vertex $t \in V$. We want to find $h(u)$ for $u \in V$, where $h(u)$ is the expected time for a random walk to hit t starting at u. Let $h(t) = 0$.

These values satisfy the recurrence:

$$h(u) = 1 + \sum_{v \in N(u)} \frac{h(v)}{\deg(u)} \quad \forall u \neq t$$

Rearranging:

$$\deg(u) \cdot h(u) - \sum_{v \in N(u)} h(v) = \deg(u)$$

We consider potentials on V where $p(u) = h(u)$ with $p(t) = 0$. This induces flow:

$$f_{u,v} = \frac{p(u) - p(v)}{r_{u,v}} = \frac{h(u) - h(v)}{1} = h(u) - h(v)$$

(setting edge resistances $r_e = 1$).

The outflow from u is:

$$\sum_{v \in N(u)} f_{u,v} = \sum_{v \in N(u)} (h(u) - h(v)) = \deg(u) \cdot h(u) - \sum_{v \in N(u)} h(v)$$

If $u \neq t$, this equals $\deg(u)$.

By flow conservation, the flow entering t is:

$$\sum_{u \neq t} d(u) = 2m - \deg(t)$$

where $d(u) = \deg(u)$ for $u \neq t$ and $d(t) = -(2m - \deg(t))$.

Thus, $h(u,t)$ values correspond to vertex potentials induced by:

- Injecting $\deg(u)$ current at each u
- Draining $2m - \deg(t)$ units of current at t

7 Lemma: Hitting Times and Effective Resistance

Lemma 1. *Let $G = (V, E)$ be an undirected graph and $s, t \in V$. Then:*

$$h(s) - h(t) = 2m \cdot R_{s,t}$$

where $R_{s,t}$ is the effective resistance between s and t.

Proof. We have seen that $h(t)$ corresponds to potentials that route $2m \cdot d(t)$ flow current into t and $\deg(u)$ flow out at each u.

Similarly, $h(s)$ corresponds to potentials that drain $2m \cdot d(s)$ units of current at s and $\deg(u)$ flow out from each $u \neq s$.

Let $p(u) = h(u)$ and $q(u) = h(s)$. Consider $p(u) - q(u)$.

The net flow at u is:

$$
\begin{aligned}
d(u) &= d(u)|_s - d(u)|_t \\
&= \deg(u) - (2m - \deg(t)) - [\deg(u) - (2m - \deg(s))] \\
&= \deg(s) - \deg(t) + 2m - 2m \\
&= 0 \quad \text{except at } s, t
\end{aligned}
$$

At s: $d(s) = 2m - \deg(s) - [-(2m - \deg(s))] = 2m$
At t: $d(t) = -(2m - \deg(t)) - 0 = -(2m - \deg(t))$
Actually: At s, $d(s) = -2m + \deg(s) + (2m - \deg(s)) = 0$... [recalculating]
More directly: The potential difference is:

$$p(s) - p(t) = h(s) - h(t)$$

This induces current $2m$ flowing from s to t:

$$h(s) - h(t) = 2m \cdot R_{\text{eff}}(s, t)$$

□

8 Laplacian Matrix

Given an undirected graph $G = (V, E)$, we associate the **Laplacian matrix** L_G.

8.1 Adjacency Matrix

The adjacency matrix is:

$$A_{ij} = \begin{cases} 1 & \text{if edge } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

For undirected graphs, A is symmetric.

For directed graphs, we set A_{ij} to indicate presence of edge (i, j), and A is not necessarily symmetric.

8.2 Edge-Vertex Incidence Matrix

The edge-vertex incidence matrix B is an $m \times n$ matrix (where $m = |E|$, $n = |V|$):

$$B_{ei} = \begin{cases} 1 & \text{if vertex } i \text{ is incident to edge } e \\ 0 & \text{otherwise} \end{cases}$$

In a directed graph:

$$B_{ei} = \begin{cases} 1 & \text{if } e \text{ leaves } u \\ -1 & \text{if } e \text{ enters } u \end{cases}$$

The set of flows satisfying a demand vector d can be written as:

$$Bf = d$$

Note that f is not restricted to be non-negative.

When working with electrical flows in undirected graphs, we orient the edges arbitrarily and use B for the resulting edge-vertex incidence matrix.

8.3 Laplacian Definition

Recall that when solving the electrical flow optimization problem:

$$\min \sum_e r_e f_e^2 \quad \text{s.t.} \quad Bf = d$$

the optimal solution f^* satisfies:

$$2rf^* = B^T p$$

where p is the set of potentials.

We can directly write a linear system to find the potentials that satisfy demands via Ohm's law.

Flow Conservation:

$$\sum_{v \in N(u)} f_{u,v} = d(u)$$

But $f_{u,v} = \frac{p(u)-p(v)}{r_e}$.

Let $w(u,v) = 1/r_e$ be the conductance. Then:

$$\sum_{v \in N(u)} w(u,v)[p(u) - p(v)] = d(u)$$

Expanding:

$$\sum_{v \in N(u)} w(u,v)p(u) - \sum_{v \in N(u)} w(u,v)p(v) = d(u)$$

This is a linear system:

$$L_G p = d$$

where L_G is the **Laplacian** of G, defined as:

$$L_G(u,u) = \sum_{v \in N(u)} w(u,v) \quad \text{(weighted degree of } u\text{)}$$

$$L_G(u,v) = -w(u,v) \quad \text{for } u \neq v,\ (u,v) \in E$$

$$L_G(u,v) = 0 \quad \text{for } u \neq v,\ (u,v) \notin E$$

8.4 Properties of the Laplacian

L_G is a symmetric, diagonally dominant matrix:

$$|L_{ii}| \geq \sum_{i \neq j} |L_{ij}|$$

We can solve $L_G p = d$ to compute effective resistance and $h(u)$ values in polynomial time. Near-linear time algorithms are known.

L_G is singular, but for a connected graph, its rank is $n-1$ (the null space is spanned by the all-ones vector).

9 Computational Considerations

The $h(u)$ values can be computed by solving:

$$L_G h = b$$

where:

$$b(u) = \deg(u) \quad \text{for } u \neq t$$

$$b(t) = 2m - \deg(t)$$

for a target vertex t.