CS 573: Algorithms, Fall 2014

Approximate Max Cut

Lecture 24

November 19, 2014

Part I

Normal distribution

Normal distribution – proof

$$\tau^2 = \left(\int_{x=-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \right)^2$$

$$= \int_{(x,y)\in\mathbb{R}^2} \exp\left(-\frac{x^2+y^2}{2}\right) \mathrm{d}x\mathrm{d}y \qquad \text{Change of vars: } \begin{array}{l} x = r\cos\alpha, \\ y = r\sin\alpha \end{array}$$

$$= \int_{\alpha=0}^{2\pi} \int_{r=0}^{\infty} \exp\left(-\frac{r^2}{2}\right) \left| \det \begin{pmatrix} \frac{\partial r\cos\alpha}{\partial r} & \frac{\partial r\cos\alpha}{\partial \alpha} \\ \frac{\partial r\sin\alpha}{\partial r} & \frac{\partial r\sin\alpha}{\partial \alpha} \end{pmatrix} \right| \mathrm{d}r\,\mathrm{d}\alpha$$

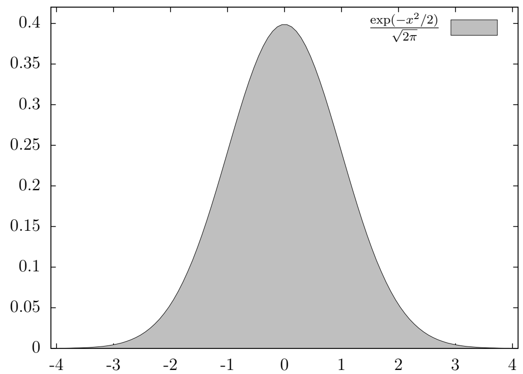
$$= \int_{\alpha=0}^{2\pi} \int_{r=0}^{\infty} \exp\left(-\frac{r^2}{2}\right) \left| \det \begin{pmatrix} \cos\alpha & -r\sin\alpha \\ \sin\alpha & r\cos\alpha \end{pmatrix} \right| \mathrm{d}r\,\mathrm{d}\alpha$$

$$= \int_{\alpha=0}^{2\pi} \int_{r=0}^{\infty} \exp\left(-\frac{r^2}{2}\right) r\,\mathrm{d}r\,\mathrm{d}\alpha$$

$$= \int_{\alpha=0}^{2\pi} \left[-\exp\left(-\frac{r^2}{2}\right) \right]_{r=0}^{\infty} \mathrm{d}\alpha = \int_{\alpha=0}^{2\pi} 1\,\mathrm{d}\alpha = 2\pi$$

One dimensional normal distribution

1. A random variable X has ***normal distribution*** if $\Pr[X = x] = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right)$.
2. $X \sim N(0,1)$.



Multidimensional normal distribution

1. A random variable X has ***normal distribution*** if $\Pr[X = x] = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right)$.
2. $X \sim N(0,1)$.
3. $\mathbf{x} = (x_1, \ldots, x_n)$ has d-dimensional normal distributed (i.e., $\mathbf{v} \sim N^n(0,1)$ $\iff$ $v_1, \ldots, v_n \sim N(0,1)$
4. $\mathbf{v} \in \mathbb{R}^n$, such that $\|\mathbf{v}\| = 1$.
5. Let $\mathbf{x} \sim N^n(0,1)$. Then $z = \langle \mathbf{v}, \mathbf{x} \rangle$ has...
6. ...normal distribution!

Part II

Approximate Max Cut

The movie so far...

Summary: It sucks.

1. Seen: Examples of using rounding techniques for approximation.
2. So far: Relaxed optimization problem is LP.
3. But... We know how to solve ***convex programming***.
4. Convex programming $\gg$ LP.
5. Convex programming can be solved in polynomial time.
6. Solving convex programming is outside scope: assume doable in polynomial time.
7. Today's lecture:
   7.1 Revisit **MAX CUT**.
   7.2 Show how to relax it into semi-definite programming problem.
   7.3 Solve relaxation.
   7.4 Show how to round the relaxed problem.

Problem Statement: **MAX CUT**

Since this is a theory class, we will define our problem.

1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: undirected graph.
2. $\forall ij \in \mathbf{E}$: nonnegative weights ω_{ij}.
3. **MAX CUT** (***maximum cut problem***): Compute set $S \subseteq \mathbf{V}$ maximizing weight of edges in cut $\left(S, \overline{S}\right)$.
4. $ij \notin \mathbf{E} \implies \omega_{ij} = O$.
5. ***weight*** of cut: $w\left(S, \overline{S}\right) = \sum_{i \in S,\, j \in \overline{S}} \omega_{ij}$.
6. Known: problem is **NP-Complete**.
   Hard to approximate within a certain constant.

Max cut as integer program

because what can go wrong?

1. Vertices: $\mathbf{V} = \{1, \ldots, n\}$.
2. ω_{ij}: non-negative weights on edges.
3. max cut $w(S, \overline{S})$ is computed by the integer quadratic program:

$$(Q) \qquad \max \quad \frac{1}{2}\sum_{i<j} \omega_{ij}(1 - y_i y_j)$$
$$\text{subject to:} \quad y_i \in \{-1, 1\} \qquad \forall i \in \mathbf{V}.$$

4. Set: $S = \left\{ i \,\middle|\, y_i = 1 \right\}$.
5. $\omega\left(S, \overline{S}\right) = \frac{1}{2}\sum_{i<j} \omega_{ij}(1 - y_i y_j)$.

Relaxing $-1, 1$...

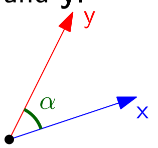
Because 1 and -1 are just vectors.

1. Solving quadratic integer programming is of course **NP-Hard**.
2. Want a relaxation...
3. 1 and -1 are just roots of unity.
4. FFT: All roots of unity are a circle.
5. In higher dimensions: All unit vectors are points on unit sphere.
6. y_i are just unit vectors.
7. $y_i * y_j$ is replaced by dot product $\langle y_i, y_j \rangle$.

Quick reminder about dot products

Everybody knows, thats how it goes

1. $\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{y} = (y_1, \ldots, y_d)$.
2. $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{d} x_i y_i$.
3. For a vector $\mathbf{v} \in \mathbb{R}^d$: $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle$.
4. $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos \alpha$.
   α: Angle between $\mathbf{x}$ and $\mathbf{y}$.



5. $\mathbf{x} \perp \mathbf{y}$: $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.
6. $\mathbf{x} = \mathbf{y}$ and $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$: $\langle \mathbf{x}, \mathbf{y} \rangle = 1$.
7. $\mathbf{x} = -\mathbf{y}$ and $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$: $\langle \mathbf{x}, \mathbf{y} \rangle = -1$.

Relaxing $-1, 1$...

Because 1 and -1 are just vectors.

1. max cut $w(S, \overline{S})$ as integer quadratic program:

$$(Q) \qquad \max \quad \frac{1}{2}\sum_{i<j} \omega_{ij}(1 - y_i y_j)$$
$$\text{subject to:} \quad y_i \in \{-1, 1\} \qquad \forall i \in \mathbf{V}.$$

2. Relaxed semi-definite programming version:

$$(P) \qquad \max \quad \gamma = \frac{1}{2}\sum_{i<j} \omega_{ij} \left(1 - \langle v_i, v_j \rangle\right)$$
$$\text{subject to:} \quad v_i \in \mathbb{S}^{(n)} \qquad \forall i \in V,$$

$\mathbb{S}^{(n)}$: n dimensional unit sphere in $\mathbb{R}^{n+1}$.

Discussion...

1. semi-definite programming: special case of convex programming.
2. Can be solved in polynomial time.
3. Solve within a factor of $(1+\varepsilon)$ of optimal, for any $\varepsilon > 0$, in polynomial time.
4. Intuition: vectors of one side of the cut, and vertices on the other sides, would have faraway vectors.

The approximation algorithm

For max cut

1. Given instance, compute Semi-definite program (P).
2. Compute optimal solution for (P).
3. $\vec{r}$: Pick random vector on the unit sphere $\mathbb{S}^{(n)}$.
4. induces hyperplane $h \quad \equiv \quad \langle \vec{r}, x \rangle = 0$
5. assign all vectors on one side of h to S, and rest to $\overline{S}$.

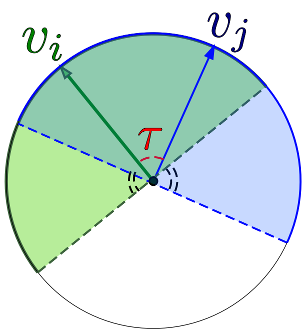
$$S = \left\{ v_i \,\middle|\, \langle v_i, \vec{r} \rangle \geq 0 \right\}.$$

Analysis...

Intuition: with good probability, vectors in the solution of (P) that have large angle between them would be separated by cut.

Lemma

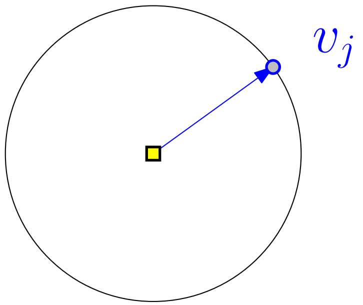
$$\Pr\left[\operatorname{sign}\Big(\langle v_i, \vec{r}\,\rangle\Big) \neq \operatorname{sign}(\langle v_j, \vec{r}\,\rangle)\right] = \frac{1}{\pi}\arccos\Big(\langle v_i, v_j\rangle\Big) = \frac{\tau}{\pi}.$$



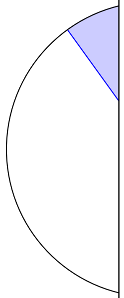
Proof...

1. Think v_i, v_j and $\vec{r}$ as being in the plane.
2. ... reasonable assumption!
   2.1 g: plane spanned by v_i and v_j.
   2.2 Only care about signs of $\langle v_i, \vec{r}\rangle$ and $\langle v_j, \vec{r}\rangle$
   2.3 can be decided by projecting $\vec{r}$ on g... and normalizing it to have length 1.
   2.4 Sphere is symmetric $\implies$ sampling $\vec{r}$ from $\mathbb{S}^{(n)}$ projecting it down to g, and then normalizing it $\equiv$ choosing uniformly a vector from the unit circle in g

Proof via figure...



$\tau = \arccos\big(\langle \boldsymbol{v_i}, \boldsymbol{v_j}\rangle\big)$



Proof...

1. Think $\boldsymbol{v_i}, \boldsymbol{v_j}$ and $\vec{\boldsymbol{r}}$ as being in the plane.
2. $\mathrm{sign}(\langle \boldsymbol{v_i}, \vec{\boldsymbol{r}}\,\rangle) \neq \mathrm{sign}(\langle \boldsymbol{v_j}, \vec{\boldsymbol{r}}\,\rangle)$ happens only if $\vec{\boldsymbol{r}}$ falls in the double wedge formed by the lines perpendicular to $\boldsymbol{v_i}$ and $\boldsymbol{v_j}$.
3. angle of double wedge $=$ angle $\boldsymbol{\tau}$ between $\boldsymbol{v_i}$ and $\boldsymbol{v_j}$.
4. $\boldsymbol{v_i}$ and $\boldsymbol{v_j}$ are unit vectors: $\langle \boldsymbol{v_i}, \boldsymbol{v_j}\rangle = \cos(\boldsymbol{\tau})$.
   $\boldsymbol{\tau} = \angle \boldsymbol{v_i}\boldsymbol{v_j}$.
5. Thus,

$$\begin{aligned}\mathbf{Pr}\Big[\mathrm{sign}(\langle \boldsymbol{v_i}, \vec{\boldsymbol{r}}\,\rangle) \neq \mathrm{sign}(\langle \boldsymbol{v_j}, \vec{\boldsymbol{r}}\,\rangle)\Big] &= \frac{2\tau}{2\pi} \\ &= \frac{1}{\pi}\cdot \arccos(\langle \boldsymbol{v_i}, \boldsymbol{v_j}\rangle)\,,\end{aligned}$$

as claimed. ■

Theorem

Theorem

Let $\boldsymbol{W}$ be the random variable which is the weight of the cut generated by the algorithm. We have

$$\mathbf{E}\Big[\boldsymbol{W}\Big] = \frac{1}{\pi}\sum_{i<j}\omega_{ij}\arccos\big(\langle \boldsymbol{v_i}, \boldsymbol{v_j}\rangle\big)\,.$$

Proof

1. $\boldsymbol{X_{ij}}$: indicator variable $= 1 \iff$ edge $\boldsymbol{ij}$ is in the cut.
2. $\mathbf{E}[\boldsymbol{X_{ij}}] = \mathbf{Pr}\Big[\mathrm{sign}(\langle \boldsymbol{v_i}, \vec{\boldsymbol{r}}\,\rangle) \neq \mathrm{sign}(\langle \boldsymbol{v_j}, \vec{\boldsymbol{r}}\,\rangle)\Big]$
   $= \frac{1}{\pi}\arccos\Big(\langle \boldsymbol{v_i}, \boldsymbol{v_j}\rangle\Big)$, by lemma.
3. $\boldsymbol{W} = \sum_{i<j}\omega_{ij}\boldsymbol{X_{ij}}$, and by linearity of expectation...

$$\mathbf{E}[\boldsymbol{W}] = \sum_{i<j}\omega_{ij}\,\mathbf{E}[\boldsymbol{X_{ij}}] = \frac{1}{\pi}\sum_{i<j}\omega_{ij}\arccos\big(\langle \boldsymbol{v_i}, \boldsymbol{v_j}\rangle\big)\,.$$

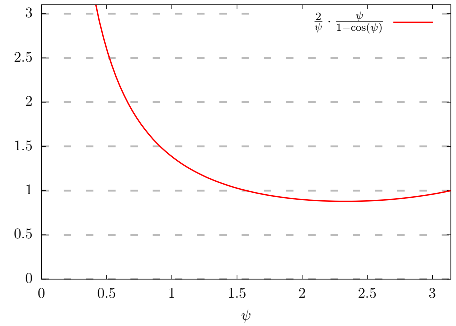
■

Lemma

Lemma

For $-1 \leq y \leq 1$*, we have* $\dfrac{\arccos(y)}{\pi} \geq \alpha \cdot \dfrac{1}{2}(1-y)$*,*

where $\alpha = \min\limits_{0 \leq \psi \leq \pi} \dfrac{2}{\pi} \dfrac{\psi}{1-\cos(\psi)}$.



Lemma restated + proof

Lemma

For $-1 \leq y \leq 1$*, we have* $\dfrac{\arccos(y)}{\pi} \geq \alpha \cdot \dfrac{1}{2}(1-y)$*,*

where $\alpha = \min\limits_{0 \leq \psi \leq \pi} \dfrac{2}{\pi} \dfrac{\psi}{1-\cos(\psi)}$.

Proof.

1. $y = \cos(\psi)$.
2. Inequality becomes: $\frac{\psi}{\pi} \geq \alpha \frac{1}{2}(1 - \cos\psi)$. Reorganizing,
3. $\implies \frac{2}{\pi} \frac{\psi}{1-\cos\psi} \geq \alpha$, holds by definition of α.

□

Lemma

Lemma

$\alpha > 0.87856$.

Proof.

Using simple calculus, one can see that α achieves its value for $\psi = 2.331122\ldots$, the nonzero root of $\cos\psi + \psi \sin\psi = 1$.

□

Result

Theorem

The above algorithm computes in expectation a cut with total weight $\alpha \cdot \mathrm{Opt} \geq 0.87856\mathrm{Opt}$*, where* Opt *is the weight of the maximal cut.*

Proof.

Consider the optimal solution to (P), and lets its value be $\gamma \geq \mathrm{Opt}$. By lemma:

$$
\begin{aligned}
\mathbf{E}[W] &= \frac{1}{\pi} \sum_{i<j} \omega_{ij} \arccos(\langle v_i, v_j \rangle) \\
&\geq \sum_{i<j} \omega_{ij} \alpha \frac{1}{2}(1 - \langle v_i, v_j \rangle) = \alpha\gamma \geq \alpha \cdot \mathrm{Opt}. \quad \blacksquare
\end{aligned}
$$

□

SDP: Semi-definite programming

1. $x_{ij} = \langle v_i, v_j \rangle$.
2. M: $n \times n$ matrix with x_{ij} as entries.
3. $x_{ii} = 1$, for $i = 1, \ldots, n$.
4. V: matrix having vectors $v_1, \ldots, v_n$ as its columns.
5. $M = V^T V$.
6. $\forall v \in \mathbb{R}^n$: $v^T M v = v^T A^T A v = (Av)^T (Av) \geq 0$.
7. M is ***positive semidefinite*** (PSD).
8. Fact: Any PSD matrix P can be written as $P = B^T B$.
9. Furthermore, given such a matrix P of size $n \times n$, we can compute B such that $P = B^T B$ in $O(n^3)$ time.
10. Known as ***Cholesky decomposition***.

SDP: Semi-definite programming

1. If PSD $P = B^T B$ has a diagonal of 1
2. $\implies$ B has columns which are unit vectors.
3. If solve SDP (P), get back semi-definite matrix...
4. ... recover the vectors realizing the solution (i.e., compute B)
5. Now, do the rounding.
6. SDP (P) can be restated as

$$(SD) \qquad \max \quad \frac{1}{2}\sum_{i<j} \omega_{ij}(1 - x_{ij})$$
$$\text{subject to:} \quad x_{ii} = 1 \quad \text{for } i = 1, \ldots, n$$
$$\Big(x_{ij}\Big)_{i=1,\ldots,n, j=1,\ldots,n} \quad \text{is a PSD matrix.}$$

SDP: Semi-definite programming

1. SDP is

$$(SD) \qquad \max \quad \frac{1}{2}\sum_{i<j} \omega_{ij}(1 - x_{ij})$$
$$\text{subject to:} \quad x_{ii} = 1 \quad \text{for } i = 1, \ldots, n$$
$$\Big(x_{ij}\Big)_{i=1,\ldots,n, j=1,\ldots,n} \quad \text{is a PSD matrix.}$$

2. find optimal value of a linear function...
3. ... over a set which is the intersection of:
    - 3.1 linear constraints, and
    - 3.2 set of positive semi-definite matrices.

Lemma

Lemma
Let $\mathcal{U}$ be the set of $n \times n$ positive semidefinite matrices. The set $\mathcal{U}$ is convex.

Proof.
Consider $A, B \in \mathcal{U}$, and observe that for any $t \in [0, 1]$, and vector $v \in \mathbb{R}^n$, we have:

$$v^T\Big(tA + (1-t)B\Big)v = v^T\Big(tAv + (1-t)Bv\Big)$$
$$= tv^T Av + (1-t)v^T Bv \geq 0 + 0 \geq 0,$$

since A and B are positive semidefinite. □

More on positive semidefinite matrices

1. PSD matrices corresponds to ellipsoids.
2. $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = \mathbf{1}$: the set of vectors solve this equation is an ellipsoid.
3. Eigenvalues of a PSD are all non-negative real numbers.
4. Given matrix: can in polynomial time decide if it is PSD.
5. ... by computing the eigenvalues of the matrix.
6. $\implies$ SDP: optimize a linear function over a convex domain.
7. SDP can be solved using interior point method, or the ellipsoid method.
8. See **Boyd and Vandenberghe [2004], Grötschel et al. [1993]** for more details.
9. Membership oracle: ability to decide in polynomial time, given a solution, whether its feasible or not.

Bibliographical Notes

1. Approx. algorithm presented by Goemans and Williamson **Goemans and Williamson [1995]**.
2. **Håstad [2001]** showed that MAX CUT can not be approximated within a factor of $\mathbf{16/17 \approx 0.941176}$.
3. **Khot et al. [2004]** showed a hardness result that matches the constant of Goemans and Williamson (i.e., one can not approximate it better than α, unless **P = NP**).

Bibliographical Notes

1. Relies on two conjectures: “Unique Games Conjecture” and “Majority is Stablest”.
2. “Majority is Stablest” conjecture was proved by **Mossel et al. [2005]**.
3. Not clear if the “Unique Games Conjecture” is true, see the discussion in **Khot et al. [2004]**.
4. Goemans and Williamson work spurred wide research on using SDP for approximation algorithms.

S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge, 2004. URL `http://www.stanford.edu/~boyd/cvxbook/`.

M. X. Goemans and D. P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. Assoc. Comput. Mach.*, 42(6):1115–1145, November 1995.

M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*, volume 2 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin Heidelberg, 2nd edition, 1993.

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