Entropy, Randomness, and Information

Lecture 23

November 13, 2014

Part I

Entropy

Quote

"If only once - only once - no matter where, no matter before what audience - I could better the record of the great Rastelli and juggle with thirteen balls, instead of my usual twelve, I would feel that I had truly accomplished something for my country. But I am not getting any younger, and although I am still at the peak of my powers there are moments - why deny it? - when I begin to doubt - and there is a time limit on all of us."
–Romain Gary, The talent scout.

Entropy: Definition

Definition

The ***entropy*** in bits of a discrete random variable X is

$$\mathbb{H}(X) = -\sum_{x} \mathbf{Pr}\Big[X = x\Big] \lg \mathbf{Pr}\Big[X = x\Big].$$

Equivalently, $\mathbb{H}(X) = \mathbf{E}\Big[\lg \frac{1}{\mathbf{Pr}[X]}\Big]$.

Entropy intuition...

Intuition...

$\mathbb{H}(X)$ is the number of ***fair*** coin flips that one gets when getting the value of X.

Interpretation from last lecture...

Consider a (huge) string $S = s_1 s_2 \ldots s_n$ formed by picking characters independently according to X. Then

$$|S|\,\mathbb{H}(X) = n\mathbb{H}(X)$$

is the minimum number of bits one needs to store the string S.

Binary entropy

$$\mathbb{H}(X) = -\sum_x \Pr\Big[X = x\Big] \lg \Pr\Big[X = x\Big]$$

$\Longrightarrow$

Definition

The ***binary entropy*** function $\mathbb{H}(p)$ for a random binary variable that is 1 with probability p, is $\mathbb{H}(p) = -p \lg p - (1-p)\lg(1-p)$. We define $\mathbb{H}(0) = \mathbb{H}(1) = 0$.

Q: How many truly random bits are there when given the result of flipping a single coin with probability p for heads?

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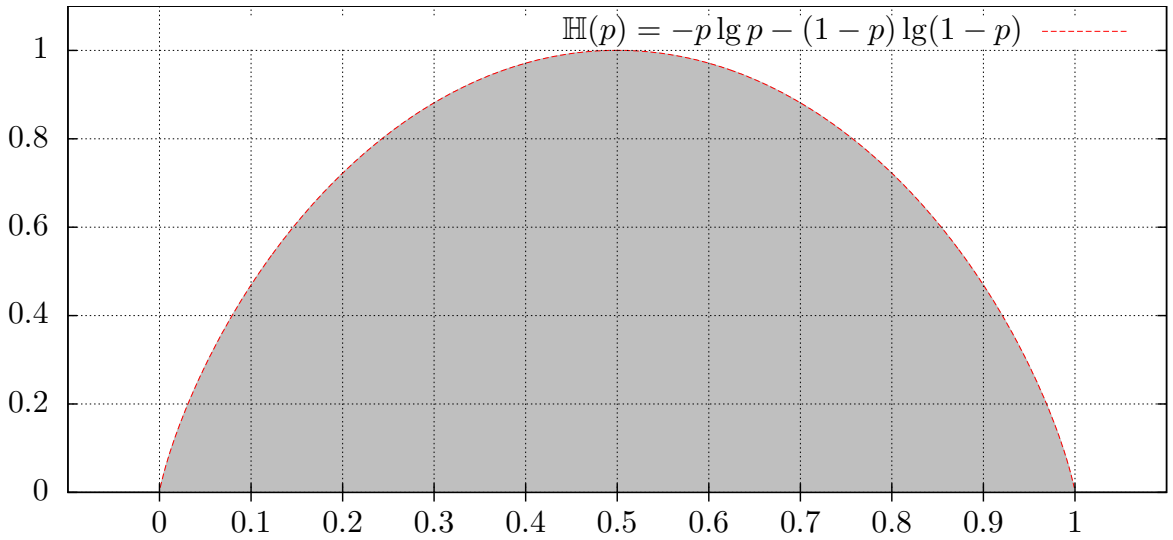

1. $\mathbb{H}(p)$ is a concave symmetric around $1/2$ on the interval $[0, 1]$.
2. maximum at $1/2$.
3. $\mathbb{H}(3/4) \approx 0.8113$ and $\mathbb{H}(7/8) \approx 0.5436$.
4. $\Longrightarrow$ coin that has $3/4$ probably to be heads have higher amount of "randomness" in it than a coin that has probability $7/8$ for heads.

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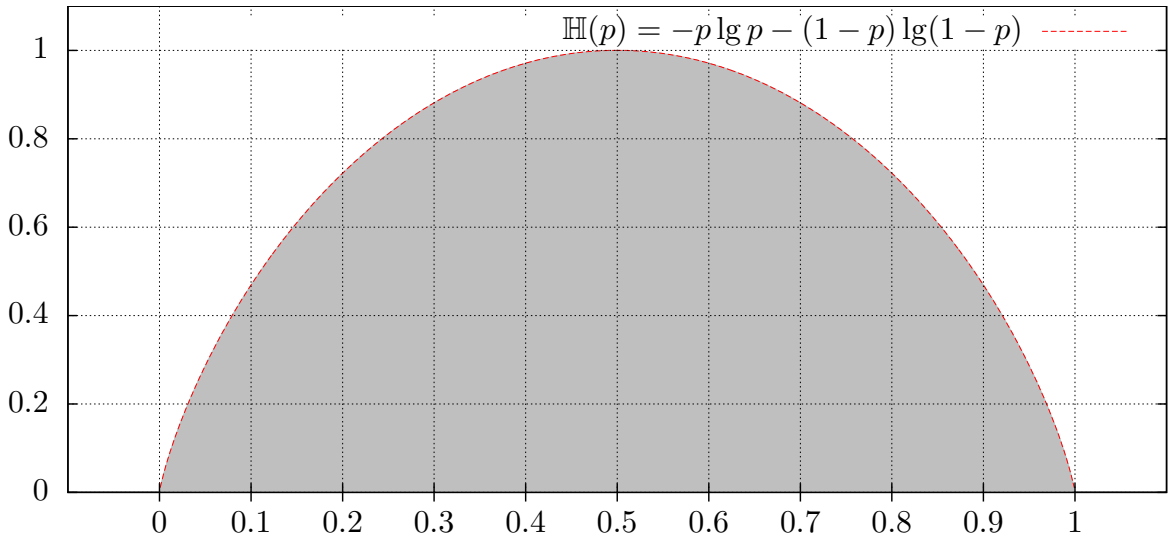

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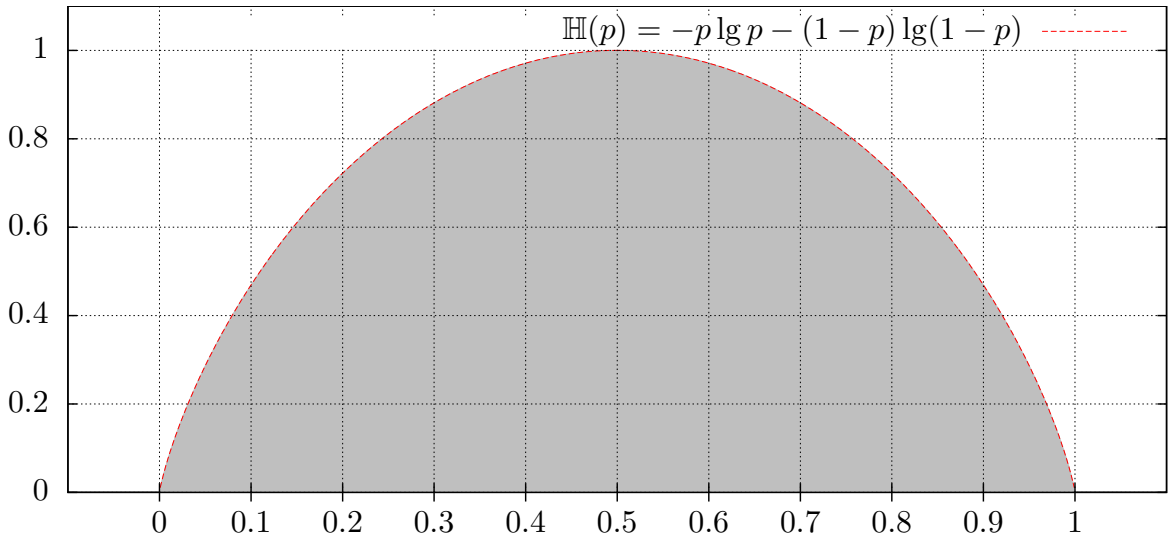

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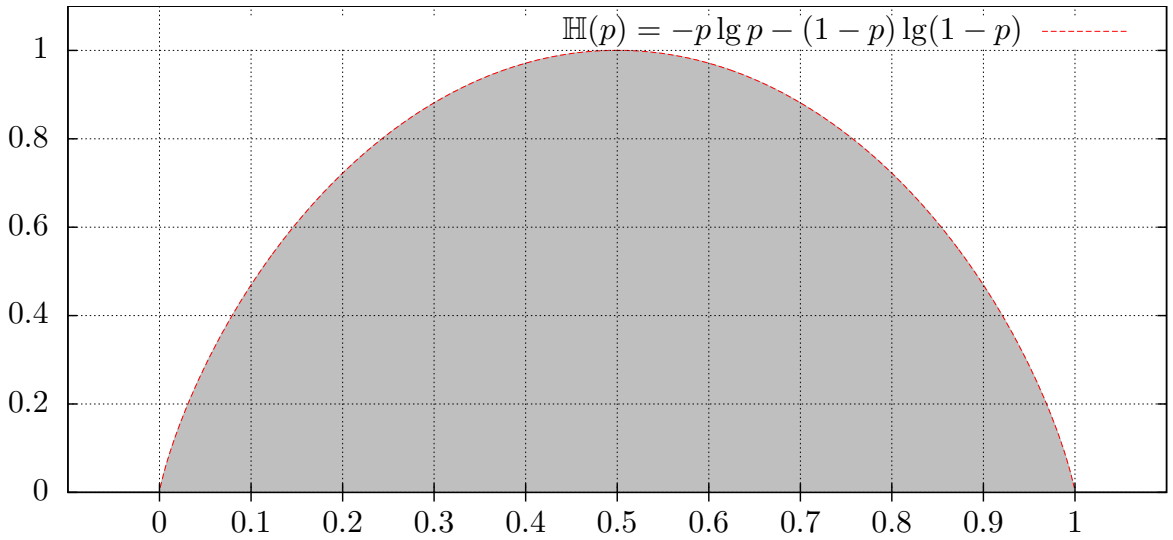

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3. $\mathbb{H}''(p) = \frac{p}{1-p} \cdot \left(-\frac{1}{p^2}\right) = -\frac{1}{p(1-p)}$.
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5. $\mathbb{H}'(1/2) = 0 \Longrightarrow \mathbb{H}(1/2) = 1$ **max** of binary entropy.
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Task at hand: Squeezing good random bits...

...out of bad random bits...

1. $b_1, \ldots, b_n$: result of n coin flips...
2. From a faulty coin!
3. p: probability for head.
4. We need fair bit coins!
5. Convert $b_1, \ldots, b_n \implies b'_1, \ldots, b'_m$.
6. **New bits must be truly random**: Probability for head is $1/2$.
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Intuitively...

Squeezing good random bits out of bad random bits...

Question...

Given the result of n coin flips: $b_1, \ldots, b_n$ from a faulty coin, with head with probability p, how many truly random bits can we extract?

If believe intuition about entropy, then this number should be $\approx n\mathbb{H}(p)$.

Back to Entropy

1. ***entropy*** of X is $\mathbb{H}(X) = -\sum_x \Pr[X = x] \lg \Pr[X = x]$.
2. Entropy of uniform variable..

Example

A random variable X that has probability $1/n$ to be i, for $i = 1, \ldots, n$, has entropy $\mathbb{H}(X) = -\sum_{i=1}^{n} \frac{1}{n} \lg \frac{1}{n} = \lg n$.

3. Entropy is oblivious to the exact values random variable can have.
4. $\Longrightarrow$ random variables over $-1, +1$ with equal probability has the same entropy (i.e., 1) as a fair coin.

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Lemma: Entropy additive for independent variables

Lemma

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two independent random variables, and let $\boldsymbol{Z}$ be the random variable $(\boldsymbol{X}, \boldsymbol{Y})$. Then $\mathbb{H}(\boldsymbol{Z}) = \mathbb{H}(\boldsymbol{X}) + \mathbb{H}(\boldsymbol{Y})$.

Proof

In the following, summation are over all possible values that the variables can have. By the independence of $\boldsymbol{X}$ and $\boldsymbol{Y}$ we have

$$\begin{aligned}
\mathbb{H}(\boldsymbol{Z}) &= \sum_{x,y} \mathbf{Pr}\Big[(\boldsymbol{X},\boldsymbol{Y}) = (x,y)\Big] \lg \frac{1}{\mathbf{Pr}[(\boldsymbol{X},\boldsymbol{Y}) = (x,y)]} \\
&= \sum_{x,y} \mathbf{Pr}\Big[\boldsymbol{X} = x\Big] \mathbf{Pr}\Big[\boldsymbol{Y} = y\Big] \lg \frac{1}{\mathbf{Pr}[\boldsymbol{X} = x]\,\mathbf{Pr}[\boldsymbol{Y} = y]} \\
&= \sum_{x} \sum_{y} \mathbf{Pr}[\boldsymbol{X} = x]\,\mathbf{Pr}[\boldsymbol{Y} = y] \lg \frac{1}{\mathbf{Pr}[\boldsymbol{X} = x]} \\
&\quad + \sum_{y} \sum_{x} \mathbf{Pr}[\boldsymbol{X} = x]\,\mathbf{Pr}[\boldsymbol{Y} = y] \lg \frac{1}{\mathbf{Pr}[\boldsymbol{Y} = y]}
\end{aligned}$$

Proof continued

$$\begin{aligned}
\mathbb{H}(\boldsymbol{Z}) &= \sum_{x}\sum_{y} \Pr[\boldsymbol{X}=x]\Pr[\boldsymbol{Y}=y]\lg\frac{1}{\Pr[\boldsymbol{X}=x]} \\
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&= \mathbb{H}(\boldsymbol{X}) + \mathbb{H}(\boldsymbol{Y}).
\end{aligned}$$

■

Bounding the binomial coefficient using entropy

Lemma

$q \in [0, 1]$
nq *is integer in the range* $[0, n]$.
Then

$$\frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{nq} \leq 2^{n\mathbb{H}(q)}.$$

Proof

Holds if $q = 0$ or $q = 1$, so assume $0 < q < 1$. We have

$$\binom{n}{nq} q^{nq}(1-q)^{n-nq} \leq (q + (1-q))^n = 1.$$

We also have:
$q^{-nq}(1-q)^{-(1-q)n} = 2^{n(-q \lg q - (1-q)\lg(1-q))} = 2^{n\mathbb{H}(q)}$, we have

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Proof continued

Other direction...

1. $\mu(k) = \binom{n}{k} q^k (1-q)^{n-k}$
2. $\sum_{i=0}^{n} \binom{n}{i} q^i (1-q)^{n-i} = \sum_{i=0}^{n} \mu(i)$.
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5. sign of Δ_k = size of last term...
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   $= \mathbf{sign}\left(\frac{(k+1)(1-q)-(n-k)q}{(k+1)(1-q)}\right)$.

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Proof continued

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Generalization...

Corollary

We have:

(i) $q \in [0, 1/2] \Rightarrow \binom{n}{\lfloor nq \rfloor} \leq 2^{n\mathbb{H}(q)}$.

(ii) $q \in [1/2, 1]$ $\binom{n}{\lceil nq \rceil} \leq 2^{n\mathbb{H}(q)}$.

(iii) $q \in [1/2, 1] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lfloor nq \rfloor}$.

(iv) $q \in [0, 1/2] \Rightarrow \frac{2^{n\mathbb{H}(q)}}{n+1} \leq \binom{n}{\lceil nq \rceil}$.

Proof is straightforward but tedious.

What we have...

1. Proved that $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$.
2. Estimate is loose.
3. Sanity check...
   (I) A sequence of n bits generated by coin with probability q for head.
   (II) By Chernoff inequality... roughly nq heads in this sequence.
   (III) Generated sequence Y belongs to $\binom{n}{nq} \approx 2^{n\mathbb{H}(q)}$ possible sequences .
   (IV) ...of similar probability.
   (V) $\implies \mathbb{H}(Y) = n\mathbb{H}(q) \approx \lg \binom{n}{nq}$.

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Just one bit...

question

Given a coin C with:

p: Probability for head.

$q = 1 - p$: Probability for tail.

Q: How to get **one** true random bit, by flipping C.

Describe an algorithm!

Extracting randomness...

Entropy can be interpreted as the amount of unbiased random coin flips can be extracted from a random variable.

Definition

An extraction function **Ext** takes as input the value of a random variable X and outputs a sequence of bits y, such that $\Pr\left[\mathbf{Ext}(X) = y \;\middle|\; |y| = k\right] = \frac{1}{2^k}$, whenever $\Pr[|y| = k] > 0$, where $|y|$ denotes the length of y.

Extracting randomness...

1. X: uniform random integer variable out of $0, \ldots, 7$.
2. $\mathsf{Ext}(X)$: binary representation of x.
3. Def. subtle: all extracted seqs of same len have same probability.
4. Another example of extraction scheme:
   1. X: uniform random integer variable $0, \ldots, 11$.
   2. $\mathsf{Ext}(x)$: output the binary representation for x if $0 \leq x \leq 7$.
   3. If x is between 8 and 11?
   4. Idea... Output binary representation of $x - 8$ as a two bit number.
5. A valid extractor...
   $\Pr\left[\mathsf{Ext}(X) = 00 \;\middle|\; |\mathsf{Ext}(X)| = 2\right] = \frac{1}{4}$,

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Technical lemma

The following is obvious, but we provide a proof anyway.

Lemma

Let x/y be a faction, such that $x/y < 1$. Then, for any i, we have $x/y < (x+i)/(y+i)$.

Proof.

We need to prove that $x(y+i) - (x+i)y < 0$. The left size is equal to $i(x-y)$, but since $y > x$ (as $x/y < 1$), this quantity is negative, as required. □

A uniform variable extractor...

Theorem

1. X: *random variable chosen uniformly at random from* $\{0, \ldots, m-1\}$.
2. *Then there is an extraction function for* X*:*
   1. *outputs on average at least*

$$\lfloor \lg m \rfloor - 1 = \lfloor \mathbb{H}(X) \rfloor - 1$$

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Proof

1. m: A sum of unique powers of 2, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.
2. Example:
3. decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of 2.
4. If x is in block 2^k, output its relative location in the block in binary representation.
5. Example: $x = 10$:
   then falls into block 2^2...
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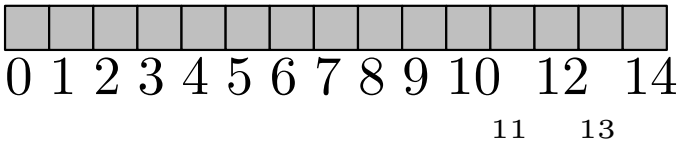

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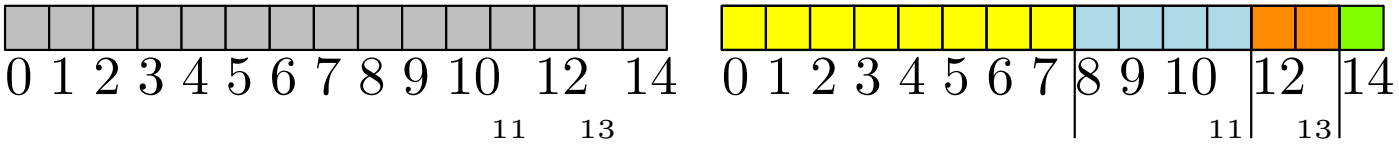

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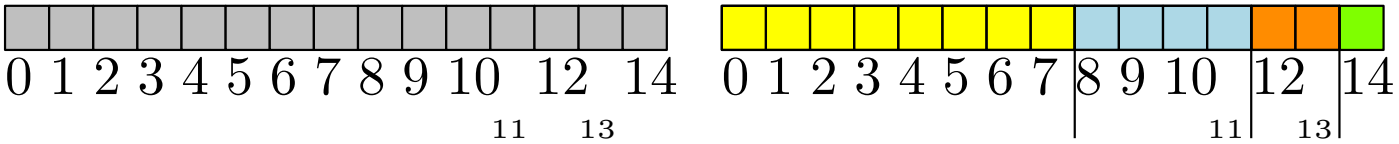

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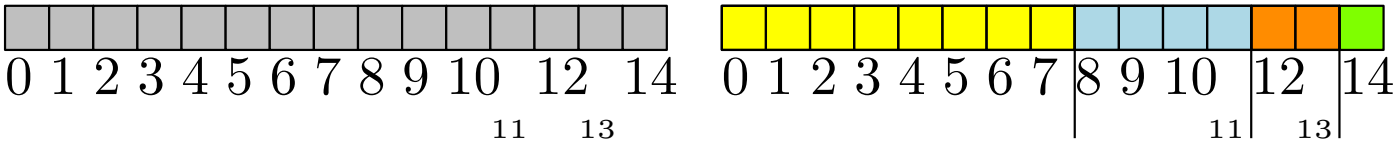

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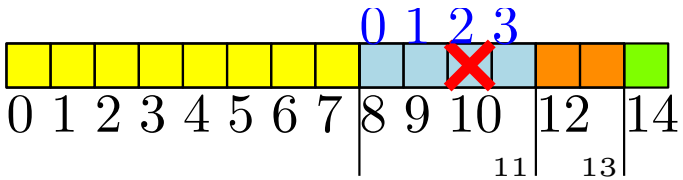

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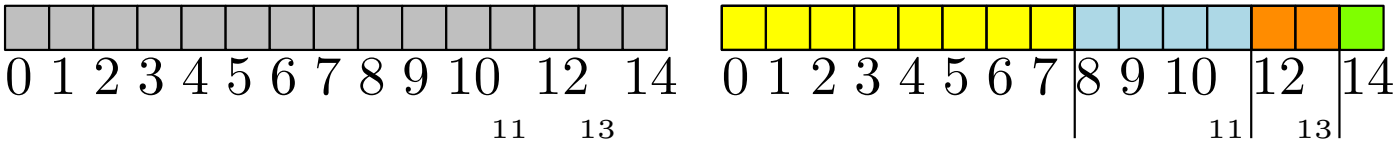
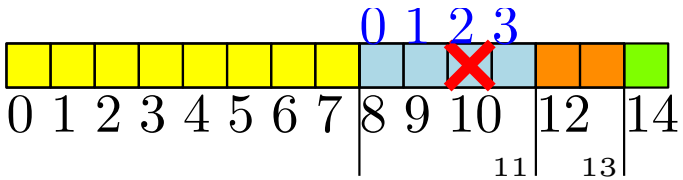


3. decomposed $\{0, \ldots, m-1\}$ into disjoint union of blocks sizes are powers of 2.
4. If x is in block 2^k, output its relative location in the block in binary representation.
5. Example: $x = 10$:
   then falls into block 2^2...
   x relative location is 2. Output 2 written using two bits,
   Output: "10".



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1. m: A sum of unique powers of 2, namely $m = \sum_i a_i 2^i$, where $a_i \in \{0, 1\}$.
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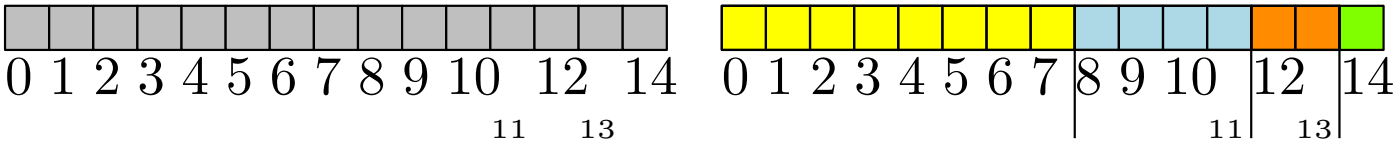

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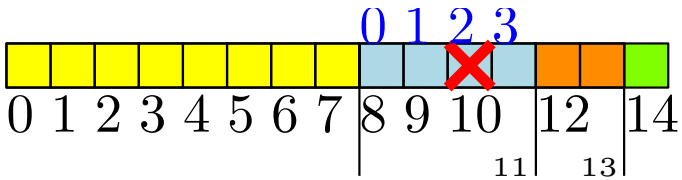

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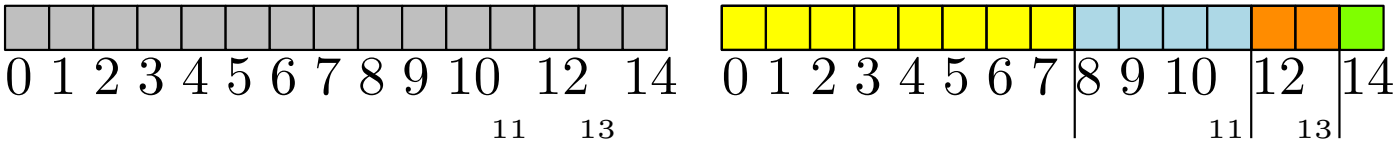

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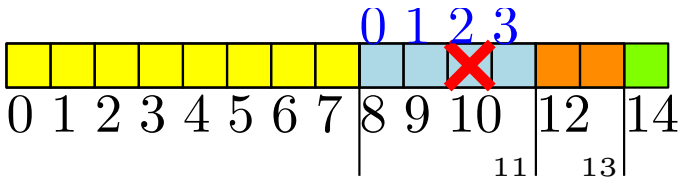

Proof continued

1. Valid extractor...
2. Theorem holds if m is a power of two. Only one block.
3. m not a power of 2...
4. X falls in block of size 2^k: then output k complete random bits..
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5. Let $2^k < m < 2^{k+1}$ biggest block.
6. $u = \left\lfloor \lg(m - 2^k) \right\rfloor < k$.
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8. Largest two blocks...
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2. By induction (assumed holds for all numbers smaller than m):

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since $u - k - 1 \leq 0$ as $k > u$.

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