Union-Find

Lecture 21

November 6, 2014

Part I

Union Find

Requirements from the data-structure

1. Maintain a collection of sets.
2. **makeSet**(x) - creates a set that contains the single element x.
3. **find**(x) - returns the set that contains x.
4. **union**(A, B) - returns set = union of A and B. That is $A \cup B$.
   ... merges the two sets A and B and return the merged set.

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Amortized Analysis

1. Use data-structure as a black-box inside algorithm.
   ... Union-Find in Kruskal algorithm for computing MST.
2. Bounded worst case time per operation.
3. Care: *overall* running time spend in data-structure.
4. ***amortized running-time*** of operation
   = average time to perform an operation on data-structure.
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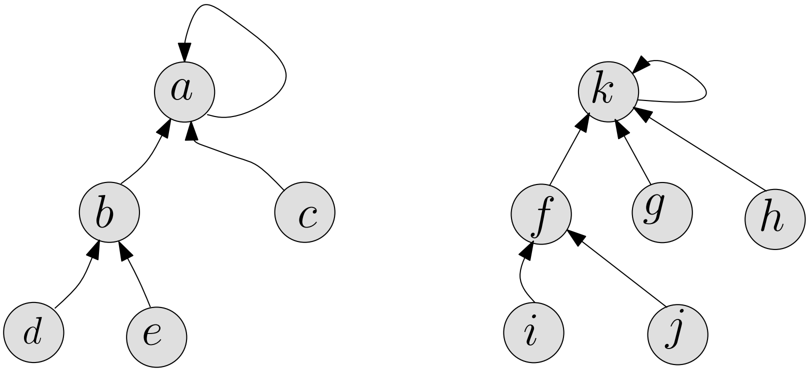

The Union-Find representation of the sets $A = \{a, b, c, d, e\}$ and $B = \{f, g, h, i, j, k\}$. The set A is uniquely identified by a pointer to the root of A, which is the node containing a.

Reversed Trees

!esrever ni retteb si gnihtyreve esuaceB

1. Reversed Trees:
   1. Initially: Every element is its own node.
   2. Node v: $\overline{\mathbf{p}}(v)$ pointer to its parent.
   3. Set uniquely identified by root node/element.
2.
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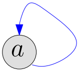
2. **makeSet**: Create a singleton pointing to itself: 

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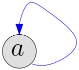
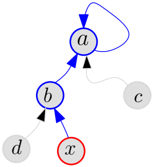
2. **makeSet**: Create a singleton pointing to itself: 

3. **find**(x):
   1. Start from node containing x, traverse up tree, till arriving to root.
   2. **find**(x):

      $x \rightarrow b \rightarrow a$
   3. a: returned as set.



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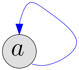
2. **makeSet**: Create a singleton pointing to itself: 

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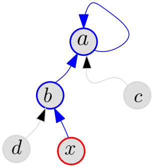

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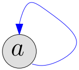
2. **makeSet**: Create a singleton pointing to itself: 

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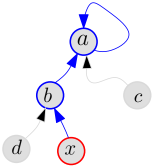

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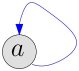
2. **makeSet**: Create a singleton pointing to itself: 

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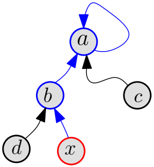
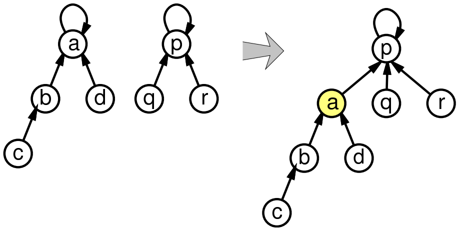


Union operation in reversed trees

Just hang them on each other.

union(a, p): Merge two sets.

1. Hanging the root of one tree, on the root of the other.
2. A destructive operation, and the two original sets no longer exist.



Pseudo-code of naive version...

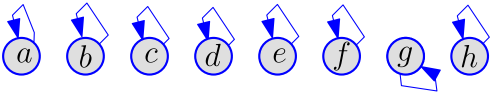
```
makeSet(x)
    p̄(x) ← x
```

```
find(x)
    if x = p̄(x) then
        return x
                return
find(p̄(x))
```

```
union( x, y )
    A ← find(x)
    B ← find(y)
    p̄(B) ← A
```

Example...

The long chain



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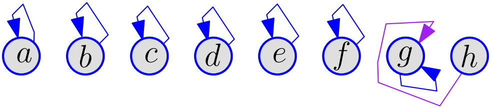

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union(g, h)

Example...

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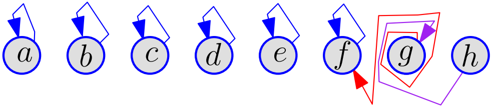

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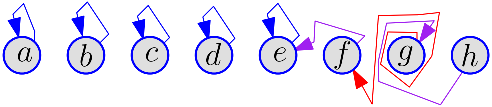

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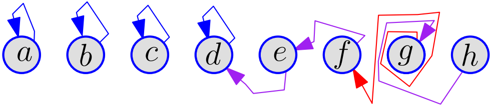

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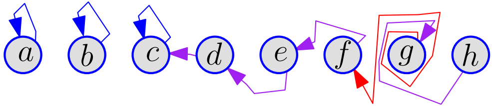

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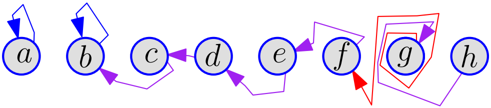

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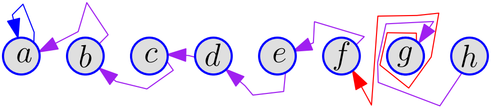

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Find is slow, hack it!

1. **find** might require $\Omega(n)$ time.
2. **Q**: How improve performance?
3. Two "hacks":

   (i) ***Union by rank***:
   Maintain in root of tree , a bound on its depth (***rank***).
   Rule: Hang the smaller tree on the larger tree in **union**.

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   During find, make all pointers on path point to root.

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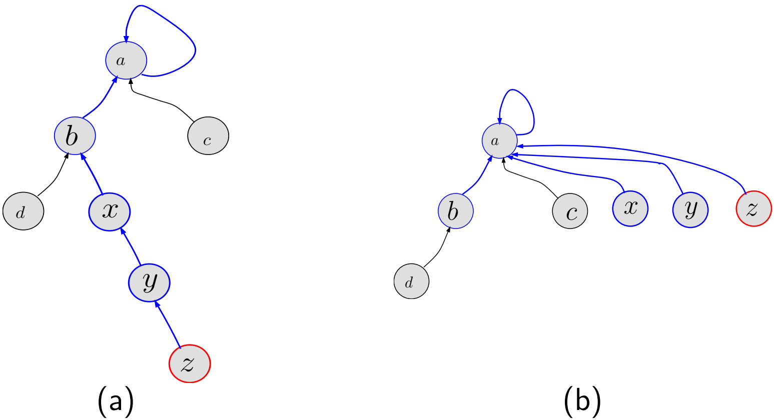
Path compression in action...



(a) The tree before performing **find**(z), and (b) The reversed tree after performing **find**(z) that uses path compression.

Pseudo-code of improved version...

```
makeSet(x)
    p̄(x) ← x
    rank(x) ← 0
```

```
find(x)
    if x ≠ p̄(x) then
        p̄(x) ← find(p̄(x))
    return p̄(x)
```

```
union(x, y)
    A ← find(x)
    B ← find(y)
    if rank(A) > rank(B) then
        p̄(B) ← A
    else
        p̄(A) ← B
        if rank(A) = rank(B) then
            rank(B) ← rank(B) + 1
```

Part II

Analyzing the Union-Find Data-Structure

Definition

Definition

v: Node **UnionFind** data-structure $\mathcal{D}$
v is ***leader*** $\iff$ v root of a (reversed) tree in $\mathcal{D}$.

"When you're not a leader, you're little people."

Lemma

Lemma

Once node v stop being a leader, can never become leader again.

Proof.

1. x stopped being leader because **union** operation hanged x on y.
2. From this point on...
3. x might change only its parent pointer (**find**).
4. x parent pointer will never become equal to x again.
5. x never a leader again.

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Once a node stop being a leader then its rank is fixed.

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1. rank of element changes only by **union** operation.
2. **union** operation changes rank only for... the "new" leader of the new set.
3. if an element is no longer a leader, than its rank is fixed.

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Ranks are strictly monotonically increasing

Lemma

Ranks are monotonically increasing in the reversed trees...
...along a path from node to root of the tree.

Proof...

1. Claim: $\forall u \to v$ in DS: $\mathrm{rank}(u) < \mathrm{rank}(v)$.
2. Proof by induction. Base: all singletons. Holds.
3. Assume claim holds at time t, before an operation.
4. If operation is **union**(A, B), and assume that we hanged $\mathrm{root}(A)$ on $\mathrm{root}(B)$.
   Must be that $\mathrm{rank}(\mathrm{root}(B))$ is now larger than $\mathrm{rank}(\mathrm{root}(A))$ (verify!).
   Claim true after operation!
5. If operation **find**: traverse path π, then all the nodes of π are made to point to the last node v of π.
   By induction, $\mathrm{rank}(v) >$ rank of all other nodes of π.
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Trees grow exponentially in size with rank

Lemma

When node gets rank $k \implies$ *at least* $\geq 2^k$ *elements in its subtree.*

Proof.

1. Proof is by induction.
2. For $k = 0$: obvious since a singleton has a rank zero, and a single element in the set.
3. node u gets rank k only if the merged two roots u, v has rank $k - 1$.
4. By induction, u and v have $\geq 2^{k-1}$ nodes before merge.
5. merged tree has $\geq 2^{k-1} + 2^{k-1} = 2^k$ nodes.

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Find takes logarithmic time

Lemma

The time to perform a single **find** *operation when we perform union by rank and path compression is* $O(\log n)$ *time.*

Proof.

1. rank of leader v of reversed tree T, bounds depth of T.
2. By previous lemma: $\max$ rank $\leq \lg n$.
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1. $\log^*(n)$: number of times to take $\lg$ of number to get number smaller than two.
2. $\log^* 2 = 1$
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Theorem

For a sequence of m operations over n elements, the overall running time of the **UnionFind** *data-structure is $O((n + m) \log^* n)$.*

1. Intuitively: **UnionFind** data-structure takes constant time per operation...
   (unless n is larger than β which is unlikely).
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The tower function...

Definition

$\text{Tower}(b) = 2^{\text{Tower}(b-1)}$ and $\text{Tower}(0) = 1$.

$\text{Tower}(i)$: a tower of $2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$ of height i.
Observe that $\log^*(\text{Tower}(i)) = i$.

Definition

For $i \geq 0$, let $\text{Block}(i) = [\text{Tower}(i-1)+1, \text{Tower}(i)]$; that is

$$\text{Block}(i) = \left[z, 2^{z-1}\right] \qquad \text{for} \qquad z = \text{Tower}(i-1)+1.$$

Also $\text{Block}(0) = [0,1]$. As such,
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$\text{Tower}(i)$: a tower of $2^{2^{2^{\cdot^{\cdot^{2}}}}}$ of height i.
Observe that $\log^*(\text{Tower}(i)) = i$.

Definition

For $i \geq 0$, let $\text{Block}(i) = [\text{Tower}(i-1)+1, \text{Tower}(i)]$; that is

$$\text{Block}(i) = \left[z, 2^{z-1}\right] \qquad \text{for} \qquad z = \text{Tower}(i-1)+1.$$

Also $\text{Block}(0) = [0, 1]$. As such,

$\text{Block}(0) = \left[0, 1\right]$, $\text{Block}(1) = \left[2, 2\right]$, $\text{Block}(2) = \left[3, 4\right]$,

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Running time of find...

1. RT of **find**(x) proportional to length of the path from x to the root of its tree.
2. ...start from x and we visit the sequence: $x_1 = x$, $x_2 = \overline{\mathrm{p}}(x_1)$, $x_3 = \overline{\mathrm{p}}(x_2)$, ..., $x_i = \overline{\mathrm{p}}(x_{i-1})$, ..., $x_m = \overline{\mathrm{p}}(x_{m-1})$ = root of tree.
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Definition

A node x is ***in the ith block*** if $\mathrm{rank}(x) \in \mathrm{Block}(i)$.

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Blocks and jumping pointers

1. maximum rank of node v is $O(\log n)$.
2. # of blocks is $O(\log^* n)$, as $O(\log n) \in \mathrm{Block}(c \log^* n)$, ($c$: constant, say 2).
3. **find** (x): π path used.
4. partition π into each by rank.
5. Price of **find** length π.
6. node x: $\nu = \mathrm{index}_{\mathrm{B}}(x)$ index block containing $\mathrm{rank}(x)$.
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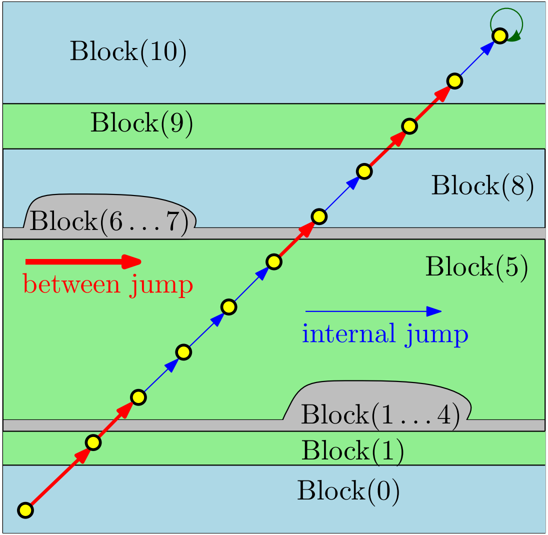
The path of find operation, and its pointers



The pointers between blocks...

1. During a **find** operation...
2. π: path traversed.
3. Ranks of the nodes visited in π monotone increasing.
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Jumping pointers

Definition

π: path traversed by **find**.
$x \in \pi$, $\overline{\mathrm{p}}(x)$ is in a different block, is a ***jump between blocks***.
jump inside a block is an ***internal jump*** (i.e., x and $\overline{\mathrm{p}}(x)$ are in same block).

Not too many jumps between blocks

Lemma

During a single **find**(x) *operation, the number of jumps between blocks along the search path is* $O(\log^* n)$.

Proof.

1. $\pi = x_1, \ldots, x_m$: path followed by **find**.
2. $x_i = \overline{\mathrm{p}}(x_{i-})$, for all i.
3. $0 \leq \mathrm{index_B}(x_1) \leq \mathrm{index_B}(x_2) \leq \ldots \leq \mathrm{index_B}(x_m)$.
4. $\mathrm{index_B}(x_m) = O(\log^* n)$.
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Benefits of an internal jump

1. x and $\overline{\mathrm{p}}(x)$ are in same block.
2. $\mathbf{index_B}(x) = \mathbf{index_B}(\overline{\mathrm{p}}(x))$.
3. **find** passes through x.
4. $r_{\mathrm{bef}} = \mathrm{rank}(\overline{\mathrm{p}}(x))$ before **find** operation.
5. $r_{\mathrm{aft}} = \mathrm{rank}(\overline{\mathrm{p}}(x))$ after **find** operation.
6. By path compression: $r_{\mathrm{aft}} > r_{\mathrm{bef}}$.
7. $\Longrightarrow$ parent pointer x jumped forward...
8. ...and new parent has higher rank.
9. Charge internal block jumps to this "progress".

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2. $\mathbf{index_B}(x) = \mathbf{index_B}(\overline{\mathrm{p}}(x))$.
3. **find** passes through x.
4. $r_{\mathrm{bef}} = \mathbf{rank}(\overline{\mathrm{p}}(x))$ before **find** operation.
5. $r_{\mathrm{aft}} = \mathbf{rank}(\overline{\mathrm{p}}(x))$ after **find** operation.
6. By path compression: $r_{\mathrm{aft}} > r_{\mathrm{bef}}$.
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Changing parents...

Your parent can be promoted only a few times before leaving block

Lemma

At most $|\mathrm{Block}(i)| \leq \mathrm{Tower}(i)$ **find** *operations can pass through an element* x*, which is in the* i*th block (i.e.,* $\mathrm{index_B}(x) = i$*) before* $\overline{\mathrm{p}}(x)$ *is no longer in the* i*th block. That is* $\mathrm{index_B}(\overline{\mathrm{p}}(x)) > i$*.*

Proof.

1. parent of x incr rank every-time internal jump goes through x.
2. At most $|\mathrm{Block}(i)|$ different values in the ith block.
3. $\mathrm{Block}(i) = [\mathrm{Tower}(i-1)+1, \mathrm{Tower}(i)]$
4. Claim follows, as: $|\mathrm{Block}(i)| \leq \mathrm{Tower}(i)$.

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Few elements are in the bigger blocks

Lemma

At most $n/\mathrm{Tower}(i)$ *nodes are assigned ranks in the* i*th block throughout the algorithm execution.*

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By lemma, the number of elements with rank in the ith block

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Total number of internal jumps is $\mathbf{O(n)}$

Lemma

The number of internal jumps performed, inside the ith block, during the lifetime of the union-find data-structure is $O(n)$.

Proof.

1. x in ith block, have at most $|\mathrm{Block}(i)|$ internal jumps...
2. ... all jumps through x are between blocks, by lemma...
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Lemma

The number of internal jumps performed by the Union-Find data-structure overall is $O(n \log^* n)$.

Proof.

1. Every internal jump associated with block it is in.
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3. There are $O(\log^* n)$ blocks.
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Result...

Lemma

The overall time spent on m **find** *operations, throughout the lifetime of a union-find data-structure defined over n elements, is* $O((m+n)\log^* n)$ *.*

Theorem

If we perform a sequence of m operations over n elements, the overall running time of the Union-Find data-structure is $O((n+m)\log^* n)$.

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Idea: Define a sequence of functions $f_i(x) = f_{i-1}^{(x)}(0)$

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Inverse Ackerman function:
$\alpha(n) = A^{-1}(n) = \min i$ s.t. $g_i(n) \leq i$.

Union-Find: Tarjan result

Theorem (?)

If we perform a sequence of m operations over n elements, the overall running time of the Union-Find data-structure is $O((n + m)\alpha(n))$.

(The above is not quite correct, but close enough.)

Notes

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