Approximation Algorithms using Linear Programming

Lecture 18 October 28, 2014

Part I

Weighted vertex cover

Weighted Vertex Cover problem

G = (V, E).

Each vertex $\mathbf{v} \in \mathbf{V}$: cost $\mathbf{c}_{\mathbf{v}}$.

Compute a vertex cover of minimum cost.

- vertex cover: subset of vertices **V** so each edge is covered.
- NP-Hard
- ...unweighted Vertex Cover problem.
- ... write as an integer program (IP):
- ⑤ \forall **v** ∈ **V**: x_v = 1 \iff **v** in the vertex cover.
- **⑤** \forall vu ∈ **E**: covered. $\implies x_{\mathsf{v}} \lor x_{\mathsf{u}}$ true. $\implies x_{\mathsf{v}} + x_{\mathsf{u}} \ge 1$.
- o minimize total cost: $\min \sum_{v \in V} x_v c_v$.

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- MP-Hard.
- 2 relax the integer program.
- allow x_v get values
 ∈ [0, 1].
- ① $x_{\mathsf{v}} \in \{0,1\}$ replaced by $0 \le x_{\mathsf{v}} \le 1$. The resulting LP is

min	$\sum_{v\inV}c_vx_v,$	
s.t.	$0 \leq x_{v}$	$\forall v \in V,$
	$x_{\!\scriptscriptstyleV} \leq 1$	$\forall v \in V,$
	$x_{\sf v}+x_{\sf u}\geq 1$	$\forall vu \in E$.

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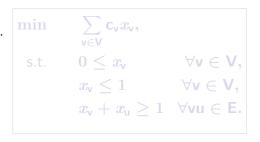
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- ② optimal value of LP solution is $\widehat{\alpha} = \sum_{\mathbf{v} \in \mathbf{V}} \mathbf{c}_{\mathbf{v}} \widehat{x}_{\mathbf{v}}$.
- ullet optimal integer solution: $x_{\mathbf{v}}^{I}$, $\forall \mathbf{v} \in \mathbf{V}$ and α^{I} .
- Any valid solution to IP is valid solution for LP!
- $\widehat{\alpha} \leq \alpha^I.$ Integral solution not better than LP.
- **6** Got fractional solution (i.e., values of $\widehat{x_{v}}$).
- Fractional solution is better than the optimal cost.
- Q: How to turn fractional solution into a (valid!) integer solution?
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- ② If $\widehat{x_{\mathsf{v}}} = 1$ then include in solution!
- If $\widehat{x_{\mathsf{v}}} = 0$ then do include in solution.
- ① if $\widehat{x_v} = 0.9 \implies \text{LP considers } \mathbf{v} \text{ as being } 0.9 \text{ useful.}$
- The LP puts its money where its belief is...
- \odot ... $\widehat{\alpha}$ value is a function of this "belief" generated by the LP
- 1 Big idea: Trust LP values as guidance to usefulness of vertices
- \odot Pick all vertices \geq threshold of usefulness according to LP.
- ullet Claim: S a valid vertex cover, and cost is low.
- ① Indeed, edge cover as: $\forall vu \in E$ have $\widehat{x}_v + \widehat{x}_u \geq 1$.
- $\begin{array}{c} @ \ \widehat{x_{\mathsf{v}}}, \, \widehat{x_{\mathsf{u}}} \in (0,1) \\ \Longrightarrow \ \widehat{x_{\mathsf{v}}} \geq 1/2 \ \mathsf{or} \ \widehat{x_{\mathsf{u}}} \ \end{array}$
 - \implies $\mathbf{v} \in \mathbf{S}$ or $\mathbf{u} \in \mathbf{S}$ (or both).
 - $\implies S$ covers all the edges of ${\sf G}.$

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since $\widehat{x_{\mathsf{v}}} \geq 1/2$ as $\mathsf{v} \in S$.

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The Weighted Vertex Cover problem can be 2-approximated by solving a single LP. Assuming computing the LP takes polynomial time, the resulting approximation algorithm takes polynomial time.

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- Weighted vertex cover is simple, but resulting approximation algorithm is non-trivial.
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Part II

Revisiting Set Cover

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- Purpose: See new technique for an approximation algorithm.
- ② Not better than greedy algorithm already seen $O(\log n)$ approximation.

Set Cover

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Instance: (S, \mathcal{F})
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 $oldsymbol{S}$ - a set of $oldsymbol{n}$ elements

 ${\mathcal F}$ - a family of subsets of S, s.t. $\bigcup_{X\in{\mathcal F}}X=S$.

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- Resulting family of sets 9.
- **1** Z_S : indicator variable. 1 if $S \in \mathcal{G}$.
- ullet Cost of ullet is $\sum_{S\in\mathcal{F}} oldsymbol{Z}_S$, and the expected cost is

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- \odot In expectation, G is not too expensive

- **1** LP solution: $\forall U \in \mathcal{F}$, $\widehat{x_U}$, and $\widehat{\alpha}$.
- $\textbf{ Opt IP solution: } \forall \, U \in \mathfrak{F}, \, x_U^I \text{, and } \alpha^I.$
- Use LP solution to guide in rounding process.
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- **5** If $\widehat{x_U}$ close to **0** do not.
- **1** Idea: Pick $U \in \mathfrak{F}$: randomly choose U with **probability** $\widehat{x_U}$.
- \odot Resulting family of sets \mathfrak{G} .
- **3** Z_S : indicator variable. 1 if $S \in \mathcal{G}$.
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- \bullet In expectation, $\mathfrak G$ is not too expensive.
- lacktriangle Bigus problumos: $\mathfrak G$ might fail to cover some element $s\in S$.

Set Cover – Rounding continued

- **Solution**: Repeat rounding stage $m = 10 \lceil \lg n \rceil = O(\log n)$ times.
- n = |S|
- ${f 0}$ ${f \mathcal H}=\cup_i{f \mathcal G}_i.$ Return ${f \mathcal H}$ as the required cover.

Set Cover – Rounding continued

- **Solution**: Repeat rounding stage $m = 10 \lceil \lg n \rceil = O(\log n)$ times.
- **2** n = |S|.
- **3** \mathcal{G}_i : random cover computed in *i*th iteration.
- $\mathfrak{G} = \bigcup_i \mathfrak{G}_i$. Return \mathfrak{H} as the required cover.

1 For an element $s \in S$, we have that

$$\sum_{U \in \mathcal{F}, s \in U} \widehat{x_U} \ge 1,\tag{2}$$

② probability s not covered by \mathfrak{G}_i (ith iteration set). $\Prig[s \text{ not covered by } \mathfrak{G}_iig]$ $= \Prig[$ no $U \in \mathcal{F}, \text{ s.t. } s \in U \text{ picked into } \mathfrak{G}_iig]$

$$egin{aligned} &= \prod_{U \in \mathcal{F}, s \in U} \Pr \Big[oldsymbol{U} & ext{was not picked into } \mathcal{G}_i \Big] \ &= \prod_{U \in \mathcal{F}, s \in U} (1 - \widehat{x_U}) \leq \prod_{U \in \mathcal{F}, s \in U} \exp (-\widehat{x_U}) \ &= \exp \Big(- \sum_{U \in \mathcal{F}, s \in U} \widehat{x_U} \Big) \leq \exp (-1) \leq rac{1}{2} \end{aligned}$$

- ① probability s is not covered in all m iterations $\leq \left(\frac{1}{2}\right)^m < \frac{1}{n^{10}},$
- \bullet ...since $m = O(\log n)$.

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- $oldsymbol{\circ}$ probability one of n elements of S is not covered by ${\mathcal H}$ is $\leq n(1/n^{10})=1/n^9.$

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Cost of solution

- \cong Each iteration expected cost of cover \leq cost of optimal solution (i.e., α^I).

$$\mathsf{c}_{\mathfrak{H}} \leq \sum\limits_{i} \mathsf{c}_{B_i} \leq m lpha^I = \mathit{O}ig(lpha^I \log nig)$$
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Cost of solution

- \Longrightarrow Each iteration expected cost of cover \le cost of optimal solution (i.e., α^I).
- Expected cost of the solution is

$$\mathsf{c}_{\mathfrak{H}} \leq \sum_i \mathsf{c}_{B_i} \leq m lpha^I = \mathit{O}ig(lpha^I \log nig)$$
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The result

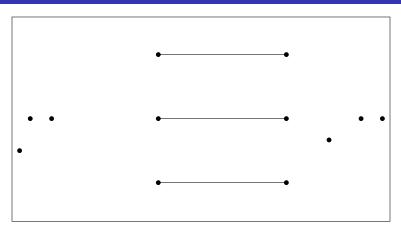
Theorem

By solving an LP one can get an $O(\log n)$ -approximation to set cover by a randomized algorithm. The algorithm succeeds with high probability.

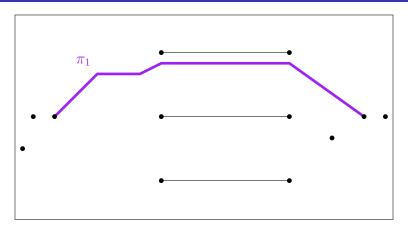
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Part III

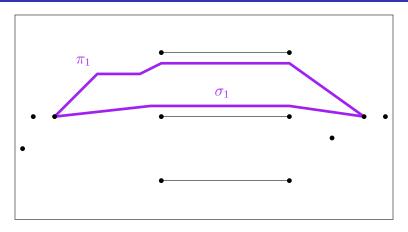
Minimizing congestion



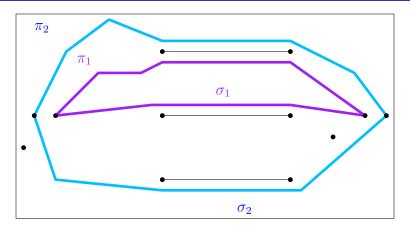
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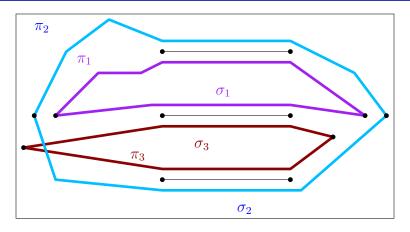


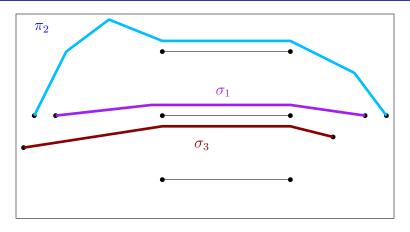
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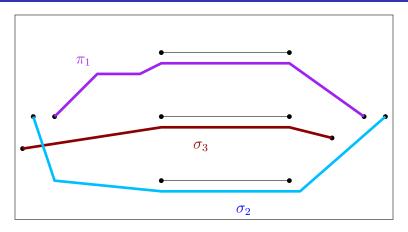


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- $oldsymbol{0}$ $oldsymbol{G}$: graph. $oldsymbol{n}$ vertices.
- ② π_i , σ_i paths with the same endpoints $\mathbf{v}_i, \mathbf{u}_i \in \mathbf{V}(\mathbf{G})$, for $i=1,\ldots,t$.
- 3 Rule I: Send one unit of flow from \mathbf{v}_i to \mathbf{u}_i .
- **1** Rule II: Choose whether to use π_i or σ_i .
- Target: No edge in G is being used too much.

Definition

Given a set X of paths in a graph G, the *congestion* of X is the maximum number of paths in X that use the same edge.

 \bullet IP \Longrightarrow LP:

min
$$w$$
 s.t. $x_i \geq 0$ $i=1,\ldots,t,$ $x_i \leq 1$ $i=1,\ldots,t,$ $\sum_{\mathbf{e} \in \pi_i} x_i + \sum_{\mathbf{e} \in \sigma_i} (1-x_i) \leq w$ $\forall \mathbf{e} \in E.$

- ② $\widehat{x_i}$: value of x_i in the optimal LP solution.
- $\widehat{\boldsymbol{w}}$: value of \boldsymbol{w} in LP solution.
- **1** Optimal congestion must be bigger than $\widehat{\boldsymbol{w}}$.
- **5** X_i : random variable one with probability $\widehat{x_i}$, and zero otherwise.
- **1** If $X_i = 1$ then use π to route from \mathbf{v}_i to \mathbf{u}_i .
- **1** Otherwise use σ_i .

- **①** Congestion of **e** is $Y_{\mathsf{e}} = \sum_{\mathsf{e} \in \pi_i} X_i + \sum_{\mathsf{e} \in \sigma_i} (1 X_i)$.
- And in expectation

$$egin{aligned} lpha_{ ext{e}} &= ext{E}ig[Y_{ ext{e}}ig] = ext{E}igg[\sum_{ ext{e} \in \pi_i} X_i + \sum_{ ext{e} \in \sigma_i} (1 - X_i)igg] \ &= \sum_{ ext{e} \in \pi_i} ext{E}ig[X_iig] + \sum_{ ext{e} \in \sigma_i} ext{E}ig[(1 - X_i)ig] \ &= \sum_{ ext{e} \in \pi_i} \widehat{x_i} + \sum_{ ext{e} \in \sigma_i} (1 - \widehat{x_i}) \leq \widehat{w}. \end{aligned}$$

 $\widehat{\boldsymbol{w}}$: Fractional congestion (from LP solution).

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- lacksquare Congestion of f e is $Y_{f e} = \sum_{{f e} \in \pi_i} X_i + \sum_{{f e} \in \sigma_i} (1-X_i)$.
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- ② $Y_{\rm e}$ is just a sum of independent 0/1 random variables!
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By Chernoff inequality:

$$\Pr\Bigl[Y_{\mathsf{e}} \geq (1+\delta)\alpha_{\mathsf{e}}\Bigr] \leq \exp\Bigl(-\frac{\alpha_{\mathsf{e}}\delta^2}{4}\Bigr) \leq \exp\Bigl(-\frac{\widehat{w}\delta^2}{4}\Bigr)\,.$$

② Let $\delta = \sqrt{rac{400}{\widehat{w}}} \ln t$. We have that

$$ext{Pr}ig[Y_{ ext{e}} \geq (1+\delta)lpha_{ ext{e}}ig] \leq ext{exp}igg(-rac{\delta^2\widehat{w}}{4}igg) \leq rac{1}{t^{100}},$$

- ① t: Number of pairs, n: Number of vertices in G.

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- ① If $t \geq n^{1/50} \implies \forall$ edges in graph congestion $\leq (1+\delta)\widehat{w}$.
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② Play with the numbers. If t=n, and $\widehat{w} \geq \sqrt{n}$. Then, the solution has congestion larger than the optimal solution by a factor of

$$1 + \delta = 1 + \sqrt{rac{20}{\widehat{w}} \ln t} \le 1 + rac{\sqrt{20 \ln n}}{n^{1/4}},$$

which is of course extremely close to 1, if n is sufficiently large

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- **1 G**: Graph **n** vertices.
- $(s_1,t_1),\ldots,(s_t,t_t)$: pairs o vertices
- $oldsymbol{\circ}$ $oldsymbol{\pi}_i, oldsymbol{\sigma}_i$: two different paths connecting s_i to t_i
- \widehat{w} : Fractional congestion at least $n^{1/2}$.
- opt: Congestion of optimal solution
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- Assume $\widehat{\boldsymbol{w}}$ is a constant.
- Can get a better bound by using the Chernoff inequality in its more general form.
- ③ set $\delta = c \ln t / \ln \ln t$, where c is a constant. For $\mu = \alpha_{\rm e}$, we have that

$$egin{split} \Prig[Y_{\mathsf{e}} & \geq (1+\delta)\muig] & \leq \left(rac{e^{\delta}}{(1+\delta)^{1+\delta}}
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where c' is a constant that depends on c and grows if c grows.

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Part IV

Reminder about Chernoff inequality

Chernoff inequality

Problem

Let $X_1, \ldots X_n$ be n independent Bernoulli trials, where

$$ext{Pr}ig[X_i=1ig]=p_i, \qquad ext{Pr}ig[X_i=0ig]=1-p_i, \ Y=\sum_i X_i, \qquad ext{and} \qquad \mu= ext{E}ig[Yig]\,.$$

We are interested in bounding the probability that $Y \geq (1+\delta)\mu$.

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Chernoff inequality

Theorem (Chernoff inequality)

For any $\delta > 0$,

$$ext{Pr}ig[Y>(1+\delta)\muig]$$

Or in a more simplified form, for any $\delta \leq 2e-1$,

$$\Pr\!\left[\,Y>(1+\delta)\mu\right]<\exp\!\left(-\mu\delta^2/4\right),$$

and

$$\Prig[Y>(1+\delta)\muig]<2^{-\mu(1+\delta)},$$

for
$$\delta \geq 2e-1$$
.

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More Chernoff...

Theorem

Under the same assumptions as the theorem above, we have

$$ext{Pr}ig[Y < (1-\delta)\muig] \leq \expigg(-\murac{\delta^2}{2}igg)$$
 .

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