CS 573: Algorithms, Fall 2014

Approximation Algorithms using Linear Programming

Lecture 18

October 28, 2014

Part I

Weighted vertex cover

Weighted vertex cover

Weighted Vertex Cover problem

$\mathbf{G = (V, E)}$.

Each vertex $\mathbf{v} \in \mathbf{V}$: cost $\mathbf{c_v}$.

Compute a vertex cover of minimum cost.

1. vertex cover: subset of vertices $\mathbf{V}$ so each edge is covered.
2. **NP-Hard**
3. ...unweighted **Vertex Cover** problem.
4. ... write as an integer program (IP):
5. $\forall \mathbf{v} \in \mathbf{V}$: $x_v = 1 \iff \mathbf{v}$ in the vertex cover.
6. $\forall \mathbf{vu} \in \mathbf{E}$: covered. $\implies x_v \vee x_u$ **true**. $\implies x_v + x_u \geq 1$.
7. minimize total cost: $\min \sum_{v \in V} x_v c_v$.

Weighted vertex cover

State as IP $\implies$ Relax $\implies$ LP

$$
\begin{array}{llll}
\min & \sum_{v \in V} c_v x_v, & & \\
\text{such that} & x_v \in \{0,1\} & \forall v \in V & (1) \\
& x_v + x_u \geq 1 & \forall vu \in E. &
\end{array}
$$

1. ... **NP-Hard**.
2. relax the integer program.
3. allow x_v get values $\in [0,1]$.
4. $x_v \in \{0,1\}$ replaced by $0 \leq x_v \leq 1$. The resulting LP is

$$
\begin{array}{lll}
\min & \sum_{v \in V} c_v x_v, & \\
\text{s.t.} & 0 \leq x_v & \forall v \in V, \\
& x_v \leq 1 & \forall v \in V, \\
& x_v + x_u \geq 1 & \forall vu \in E.
\end{array}
$$

Weighted vertex cover – rounding the LP

1. Optimal solution to this LP: $\widehat{x_v}$ value of var X_v, $\forall v \in V$.
2. optimal value of LP solution is $\widehat{\alpha} = \sum_{v \in V} c_v \widehat{x_v}$.
3. optimal integer solution: x_v^I, $\forall v \in V$ and α^I.
4. **Any valid solution to IP is valid solution for LP!**
5. $\widehat{\alpha} \leq \alpha^I$.
   Integral solution not better than LP.
6. Got fractional solution (i.e., values of $\widehat{x_v}$).
7. Fractional solution is better than the optimal cost.
8. Q: How to turn fractional solution into a (valid!) integer solution?
9. Using ***rounding***.

How to round?

1. consider vertex v and fractional value $\widehat{x_v}$.
2. If $\widehat{x_v} = 1$ then include in solution!
3. If $\widehat{x_v} = 0$ then do notnot include in solution.
4. if $\widehat{x_v} = 0.9 \implies$ LP considers v as being 0.9 useful.
5. The LP puts its money where its belief is...
6. ...$\widehat{\alpha}$ value is a function of this "belief" generated by the LP.
7. Big idea: Trust LP values as guidance to usefulness of vertices.
8. Pick all vertices $\geq$ threshold of usefulness according to LP.
9. $S = \left\{ v \,\middle|\, \widehat{x_v} \geq 1/2 \right\}$.
10. **Claim**: S a valid vertex cover, and cost is low.
11. Indeed, edge cover as: $\forall vu \in E$ have $\widehat{x_v} + \widehat{x_u} \geq 1$.
12. $\widehat{x_v}, \widehat{x_u} \in (0, 1)$
    $\implies \widehat{x_v} \geq 1/2$ or $\widehat{x_u} \geq 1/2$.
    $\implies v \in S$ or $u \in S$ (or both).
    $\implies S$ covers all the edges of G.

Cost of solution

Cost of S:

$$c_S = \sum_{v \in S} c_v = \sum_{v \in S} 1 \cdot c_v \leq \sum_{v \in S} 2\widehat{x_v} \cdot c_v \leq 2 \sum_{v \in V} \widehat{x_v} c_v = 2\widehat{\alpha} \leq 2\alpha^I,$$

since $\widehat{x_v} \geq 1/2$ as $v \in S$.
α^I is cost of the optimal solution $\implies$

Theorem

*The **Weighted Vertex Cover** problem can be 2-approximated by solving a single LP. Assuming computing the LP takes polynomial time, the resulting approximation algorithm takes polynomial time.*

The lessons we can take away

Or not - boring, boring, boring.

1. Weighted vertex cover is simple, but resulting approximation algorithm is non-trivial.
2. Not aware of any other 2-approximation algorithm does not use LP. (For the weighted case!)
3. Solving a ***relaxation*** of an optimization problem into a LP provides us with insight.
4. But... have to be creative in the rounding.

Part II

Revisiting Set Cover

Revisiting **Set Cover**

1. Purpose: See new technique for an approximation algorithm.
2. Not better than greedy algorithm already seen $O(\log n)$ approximation.

Set Cover

Instance: $(S, \mathcal{F})$
S - a set of n elements
$\mathcal{F}$ - a family of subsets of S, s.t. $\cup_{X \in \mathcal{F}} X = S$.
Question: The set $\mathcal{X} \subseteq F$ such that $\mathcal{X}$ contains as few sets as possible, and $\mathcal{X}$ covers S.

Set Cover – IP & LP

$$\begin{aligned} \min \quad & \alpha = \sum_{U \in \mathfrak{F}} x_U, & \\ \text{s.t.} \quad & x_U \in \{0, 1\} & \forall U \in \mathfrak{F}, \\ & \sum_{U \in \mathfrak{F}, s \in U} x_U \geq 1 & \forall s \in S. \end{aligned}$$

Next, we relax this IP into the following LP.

$$\begin{aligned} \min \quad & \alpha = \sum_{U \in \mathfrak{F}} x_U, & \\ & 0 \leq x_U \leq 1 & \forall U \in \mathfrak{F}, \\ & \sum_{U \in \mathfrak{F}, s \in U} x_U \geq 1 & \forall s \in S. \end{aligned}$$

Set Cover – IP & LP

1. LP solution: $\forall U \in \mathfrak{F}$, $\widehat{x_U}$, and $\widehat{\alpha}$.
2. Opt IP solution: $\forall U \in \mathfrak{F}$, x^I_U, and α^I.
3. Use LP solution to guide in rounding process.
4. If $\widehat{x_U}$ is close to 1 then pick U to cover.
5. If $\widehat{x_U}$ close to 0 do not.
6. Idea: Pick $U \in \mathfrak{F}$: randomly choose U with ***probability*** $\widehat{x_U}$.
7. Resulting family of sets $\mathcal{G}$.
8. Z_S: indicator variable. 1 if $S \in \mathcal{G}$.
9. Cost of $\mathcal{G}$ is $\sum_{S \in \mathfrak{F}} Z_S$, and the expected cost is $\mathbf{E}\left[\text{cost of } \mathcal{G}\right] = \mathbf{E}[\sum_{S \in \mathfrak{F}} Z_S] = \sum_{S \in \mathfrak{F}} \mathbf{E}\left[Z_S\right] = \sum_{S \in \mathfrak{F}} \mathbf{Pr}\left[S \in \mathcal{G}\right] = \sum_{S \in \mathfrak{F}} \widehat{x_S} = \widehat{\alpha} \leq \alpha^I.$
10. In expectation, $\mathcal{G}$ is not too expensive.
11. Bigus problumos: $\mathcal{G}$ might fail to cover some element $s \in S$.

Set Cover – Rounding continued

1. **Solution**: Repeat rounding stage $m = 10 \lceil \lg n \rceil = O(\log n)$ times.
2. $n = |S|$.
3. $\mathcal{G}_i$: random cover computed in ith iteration.
4. $\mathcal{H} = \cup_i \mathcal{G}_i$. Return $\mathcal{H}$ as the required cover.

The set $\mathcal{H}$ covers S

1. For an element $s \in S$, we have that
$$\sum_{U \in \mathcal{F}, s \in U} \widehat{x_U} \geq 1, \qquad (2)$$
2. probability s not covered by $\mathcal{G}_i$ (ith iteration set).
$\mathbf{Pr}\Big[s \text{ not covered by } \mathcal{G}_i\Big]$
$= \mathbf{Pr}\Big[\text{ no } U \in \mathcal{F}, \text{ s.t. } s \in U \text{ picked into } \mathcal{G}_i\Big]$
$= \prod_{U \in \mathcal{F}, s \in U} \mathbf{Pr}\Big[U \text{ was not picked into } \mathcal{G}_i\Big]$
$= \prod_{U \in \mathcal{F}, s \in U} (1 - \widehat{x_U}) \leq \prod_{U \in \mathcal{F}, s \in U} \exp(-\widehat{x_U})$
$= \exp\Big(-\sum_{U \in \mathcal{F}, s \in U} \widehat{x_U}\Big) \leq \exp(-1) \leq \frac{1}{2}, \leq \frac{1}{2}$
3. probability s is not covered in all m iterations $\leq \left(\frac{1}{2}\right)^m < \frac{1}{n^{10}}$,
4. ...since $m = O(\log n)$.
5. probability one of n elements of S is not covered by $\mathcal{H}$ is $\leq n(1/n^{10}) = 1/n^9$.

Cost of solution

1. Have: $\mathbf{E}\Big[\text{cost of } \mathcal{G}_i\Big] \leq \alpha^I$.
2. $\Longrightarrow$ Each iteration expected cost of cover $\leq$ cost of optimal solution (i.e., α^I).
3. Expected cost of the solution is

$$c_{\mathcal{H}} \leq \sum_i c_{B_i} \leq m\alpha^I = O\Big(\alpha^I \log n\Big).$$

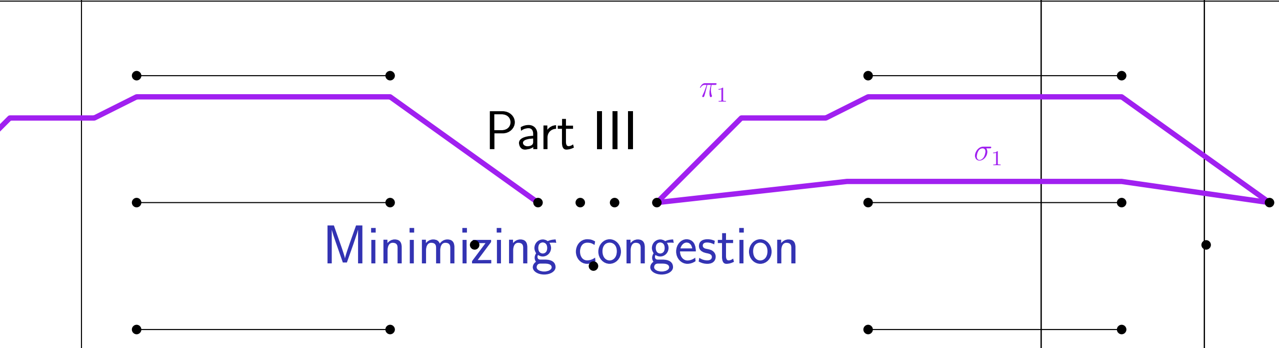
The result

Theorem

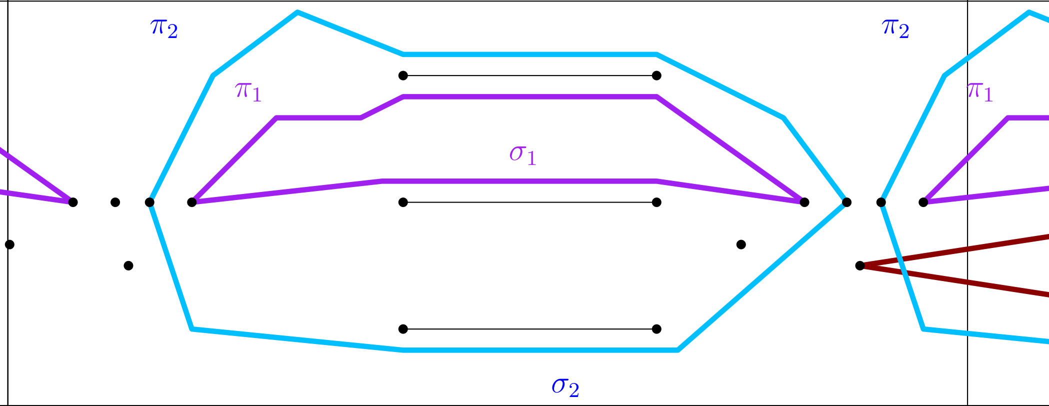
By solving an LP one can get an $O(\log n)$-approximation to set cover by a randomized algorithm. The algorithm succeeds with high probability.

Part III

Minimizing congestion



Minimizing congestion by example



Minimizing congestion

1. $\mathbf{G}$: graph. n vertices.
2. π_i, σ_i paths with the same endpoints $\mathbf{v}_i, \mathbf{u}_i \in \mathbf{V(G)}$, for $i = 1, \ldots, t$.
3. Rule I: Send one unit of flow from $\mathbf{v}_i$ to $\mathbf{u}_i$.
4. Rule II: Choose whether to use π_i or σ_i.
5. Target: No edge in $\mathbf{G}$ is being used too much.

Definition

Given a set X of paths in a graph $\mathbf{G}$, the ***congestion*** of X is the maximum number of paths in X that use the same edge.

Minimizing congestion

1. IP $\implies$ LP:

$$
\begin{array}{lll}
\min & w & \\
\text{s.t.} & x_i \geq 0 & i = 1, \ldots, t, \\
 & x_i \leq 1 & i = 1, \ldots, t, \\
 & \sum_{e \in \pi_i} x_i + \sum_{e \in \sigma_i} (1 - x_i) \leq w & \forall e \in E.
\end{array}
$$

2. $\widehat{x}_i$: value of x_i in the optimal LP solution.
3. $\widehat{w}$: value of w in LP solution.
4. Optimal congestion must be bigger than $\widehat{w}$.
5. X_i: random variable one with probability $\widehat{x}_i$, and zero otherwise.
6. If $X_i = 1$ then use π to route from $\mathbf{v}_i$ to $\mathbf{u}_i$.
7. Otherwise use σ_i.

Minimizing congestion

1. Congestion of $\mathbf{e}$ is $Y_e = \sum_{e\in\pi_i} X_i + \sum_{e\in\sigma_i}(1 - X_i)$.
2. And in expectation

$$\alpha_e = \mathbf{E}\left[Y_e\right] = \mathbf{E}\left[\sum_{e\in\pi_i} X_i + \sum_{e\in\sigma_i}(1 - X_i)\right]$$
$$= \sum_{e\in\pi_i} \mathbf{E}\left[X_i\right] + \sum_{e\in\sigma_i} \mathbf{E}\left[(1 - X_i)\right]$$
$$= \sum_{e\in\pi_i} \widehat{x}_i + \sum_{e\in\sigma_i} (1 - \widehat{x}_i) \le \widehat{w}.$$

3. $\widehat{w}$: Fractional congestion (from LP solution).

Minimizing congestion - continued

1. $Y_e = \sum_{e\in\pi_i} X_i + \sum_{e\in\sigma_i}(1 - X_i)$.
2. Y_e is just a sum of independent $0/1$ random variables!
3. Chernoff inequality tells us sum can not be too far from expectation!

Minimizing congestion - continued

1. By Chernoff inequality:

$$\Pr\left[Y_e \ge (1+\delta)\alpha_e\right] \le \exp\left(-\frac{\alpha_e \delta^2}{4}\right) \le \exp\left(-\frac{\widehat{w}\delta^2}{4}\right).$$

2. Let $\delta = \sqrt{\frac{400}{\widehat{w}} \ln t}$. We have that

$$\Pr\left[Y_e \ge (1+\delta)\alpha_e\right] \le \exp\left(-\frac{\delta^2 \widehat{w}}{4}\right) \le \frac{1}{t^{100}},$$

3. If $t \ge n^{1/50} \implies \forall$ edges in graph congestion $\le (1+\delta)\widehat{w}$.
4. t: Number of pairs, n: Number of vertices in $\mathbf{G}$.

Minimizing congestion - continued

1. Got: For $\delta = \sqrt{\frac{400}{\widehat{w}} \ln t}$. We have

$$\Pr\left[Y_e \ge (1+\delta)\alpha_e\right] \le \exp\left(-\frac{\delta^2 \widehat{w}}{4}\right) \le \frac{1}{t^{100}},$$

2. Play with the numbers. If $t = n$, and $\widehat{w} \ge \sqrt{n}$. Then, the solution has congestion larger than the optimal solution by a factor of

$$1 + \delta = 1 + \sqrt{\frac{20}{\widehat{w}} \ln t} \le 1 + \frac{\sqrt{20 \ln n}}{n^{1/4}},$$

which is of course extremely close to 1, if n is sufficiently large.

Minimizing congestion: result

Theorem

1. $\mathbf{G}$: *Graph* n *vertices.*
2. $(s_1, t_1), \ldots, (s_t, t_t)$: *pairs o vertices*
3. π_i, σ_i: *two different paths connecting* s_i *to* t_i
4. $\widehat{w}$: *Fractional congestion at least* $n^{1/2}$.
5. opt: *Congestion of optimal solution.*
6. $\implies$ *In polynomial time (*LP *solving time) choose paths*
   6.1 *congestion* $\forall$ *edges:* $\leq (1+\delta)\mathrm{opt}$
   6.2 $\delta = \sqrt{\frac{20}{\widehat{w}} \ln t}$.

When the congestion is low

1. Assume $\widehat{w}$ is a constant.
2. Can get a better bound by using the Chernoff inequality in its more general form.
3. set $\delta = c \ln t / \ln \ln t$, where c is a constant. For $\mu = \alpha_e$, we have that

$$\Pr\Big[Y_e \geq (1+\delta)\mu\Big] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu = \exp\Big(\mu\big(\delta - (1+\delta)\ln(1+\delta)\big)\Big) = \exp\Big(-\mu c' \ln t\Big) \leq \frac{1}{t^{O(1)}},$$

where c' is a constant that depends on c and grows if c grows.

When the congestion is low

1. Just proved that...
2. if the optimal congestion is $O(1)$, then...
3. algorithm outputs a solution with congestion $O(\log t / \log \log t)$, and this holds with high probability.

Part IV

Reminder about Chernoff inequality

Chernoff inequality

Problem

Let $X_1, \ldots X_n$ be n independent Bernoulli trials, where

$$\Pr\left[X_i = 1\right] = p_i, \qquad \Pr\left[X_i = 0\right] = 1 - p_i,$$
$$Y = \sum_i X_i, \qquad \text{and} \qquad \mu = \mathbf{E}\left[Y\right].$$

We are interested in bounding the probability that $Y \geq (1+\delta)\mu$.

Chernoff inequality

Theorem (Chernoff inequality)

For any $\delta > 0$,

$$\Pr\left[Y > (1+\delta)\mu\right] < \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu.$$

Or in a more simplified form, for any $\delta \leq 2e - 1$,

$$\Pr\left[Y > (1+\delta)\mu\right] < \exp\left(-\mu\delta^2/4\right),$$

and

$$\Pr\left[Y > (1+\delta)\mu\right] < 2^{-\mu(1+\delta)},$$

for $\delta \geq 2e - 1$.

More Chernoff...

Theorem

Under the same assumptions as the theorem above, we have

$$\Pr\left[Y < (1-\delta)\mu\right] \leq \exp\left(-\mu\frac{\delta^2}{2}\right).$$