

Linear Programming

Lecture 16

October 21, 2014

Part I

Linear Programming

Economic planning

Guns/nuclear-bombs/napkins/star-wars/professors/butter/mice problem

- 1 Penguinia: a country.
- 2 Ruler need to decide how to allocate resources.
- 3 Maximize benefit.
- 4 Budget allocation
 - (i) Nuclear bomb has a tremendous positive effect on security while being expensive.
 - (ii) Guns, on the other hand, have a weaker effect.
- 5 Penguinia need to prove a certain level of security:

$$x_{gun} + 1000 * x_{nuclear-bomb} \geq 1000,$$

where x_{guns} : # guns $x_{nuclear-bomb}$: # nuclear-bombs constructed.

- 6 $100 * x_{gun} + 1000000 * x_{nuclear-bomb} \leq x_{security}$

$x_{security}$: total amount spent on security.

100/1,000,000: price of producing a single gun/nuclear bomb.

Linear programming

An instance of *linear programming* (LP):

- ① x_1, \dots, x_n : variables.
- ② For $j = 1, \dots, m$: $a_{j1}x_1 + \dots + a_{jn}x_n \leq b_j$: linear inequality.
- ③ i.e., *constraint*.
- ④ Q: \exists s an assignment of values to x_1, \dots, x_n such that all inequalities are satisfied.
- ⑤ Many possible solutions... Want solution that maximizes some linear quantity.
- ⑥ *objective function*: linear inequality being maximized.

Linear programming – example

$$a_{11}x_1 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + \dots + a_{2n}x_n \leq b_2$$

...

$$a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m$$

$$\max c_1x_1 + \dots + c_nx_n.$$

History

- 1 1939: L. V. Kantorovich noticed the importance of certain type of Linear Programming problems for resource allocation.
- 2 1947: Dantzig invented the simplex method for solving LP problems for the US Air force planning problems.
- 3 1947: T. C. Koopmans showed LP provide the right model for the analysis of classical economic theories.
- 4 1975: Koopmans and Kantorovich got the Nobel prize of economics.
- 5 Kantorovich the only the Russian economist that got the Nobel prize

Network flow via linear programming

Input: $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with source \mathbf{s} and sink \mathbf{t} , and capacities $\mathbf{c}(\cdot)$ on the edges. Compute max flow in \mathbf{G} .

$$\forall (u \rightarrow v) \in E \quad \begin{aligned} 0 &\leq x_{u \rightarrow v} \\ x_{u \rightarrow v} &\leq c(u \rightarrow v) \end{aligned}$$

$$\forall v \in V \setminus \{\mathbf{s}, \mathbf{t}\} \quad \sum_{(u \rightarrow v) \in E} x_{u \rightarrow v} - \sum_{(v \rightarrow w) \in E} x_{v \rightarrow w} \leq 0$$

$$\sum_{(u \rightarrow v) \in E} x_{u \rightarrow v} - \sum_{(v \rightarrow w) \in E} x_{v \rightarrow w} \geq 0$$

maximizing

$$\sum_{(\mathbf{s} \rightarrow u) \in E} x_{\mathbf{s} \rightarrow u}$$

Part II

The Simplex Algorithm

Rewriting an LP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{subject to} & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m \end{array}$$

- 1 Rewrite: so every variable is non-negative.
- 2 Replace variable x_i by x'_i and x''_i , where new constraints are:
 $x_i = x'_i - x''_i$, $x'_i \geq 0$ and $x''_i \geq 0$.
- 3 Example: The (silly) LP $2x + y \geq 5$ rewritten:
 $2x' - 2x'' + y' - y'' \geq 5$,
 $x' \geq 0$, $y' \geq 0$,
 $x'' \geq 0$, and
 $y'' \geq 0$.

Rewriting an LP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{subject to} & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m \end{array}$$

- 1 Rewrite: so every variable is non-negative.
- 2 Replace variable x_i by x'_i and x''_i , where new constraints are:
 $x_i = x'_i - x''_i$, $x'_i \geq 0$ and $x''_i \geq 0$.
- 3 Example: The (silly) LP $2x + y \geq 5$ rewritten:
 $2x' - 2x'' + y' - y'' \geq 5$,
 $x' \geq 0, y' \geq 0$,
 $x'' \geq 0$, and
 $y'' \geq 0$.

Rewriting an LP

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{subject to} & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m \end{array}$$

- 1 Rewrite: so every variable is non-negative.
- 2 Replace variable x_i by x'_i and x''_i , where new constraints are:
 $x_i = x'_i - x''_i$, $x'_i \geq 0$ and $x''_i \geq 0$.
- 3 Example: The (silly) LP $2x + y \geq 5$ rewritten:
 $2x' - 2x'' + y' - y'' \geq 5$,
 $x' \geq 0$, $y' \geq 0$,
 $x'' \geq 0$, and
 $y'' \geq 0$.

Rewriting an LP into standard form

Lemma

Given an instance I of LP, one can rewrite it into an equivalent LP, such that all the variables must be non-negative. This takes linear time in the size of I .

An LP where all variables must be non-negative is in *standard form*

Rewriting an LP into standard form

Lemma

Given an instance I of LP, one can rewrite it into an equivalent LP, such that all the variables must be non-negative. This takes linear time in the size of I .

An LP where all variables must be non-negative is in ***standard form***

Standard form of LP

A linear program in standard form.

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m \\ & x_j \geq 0 \quad \text{for } j = 1, \dots, n. \end{aligned}$$

Standard form of LP

Because everything is clearer when you use matrices. Not.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(n-1)} & a_{2n} \\ \vdots & \dots & \dots & \dots & \vdots \\ a_{(m-1)1} & a_{(m-1)2} & \dots & a_{(m-1)(n-1)} & a_{(m-1)n} \\ a_{m1} & a_{m2} & \dots & a_{m(n-1)} & a_{mn} \end{pmatrix}$$

c , b and A :
prespecified.
 x is vector of
unknowns.
Solve LP for
 x .

LP in standard form.

(Matrix notation.)

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b. \\ & x \geq 0. \end{array}$$

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}$$

Slack Form

- 1 Next rewrite **LP** into **slack form**.
- 2 Every inequality becomes equality.
- 3 All variables must be positive.
- 4 See resulting form on the right.

$$\begin{array}{ll} \max & c^T x \\ \text{subject to} & Ax = b. \\ & x \geq 0. \end{array}$$

- 1 New **slack variables**. Rewrite inequality: $\sum_{i=1}^n a_i x_i \leq b$. As:

$$\begin{aligned} x_{n+1} &= b - \sum_{i=1}^n a_i x_i \\ x_{n+1} &\geq 0. \end{aligned}$$

- 2 Value of slack variable x_{n+1} encodes how far is the original inequality for holding with equality.

Slack Form

- 1 Next rewrite **LP** into **slack form**.
- 2 Every inequality becomes equality.
- 3 All variables must be positive.
- 4 See resulting form on the right.
- 1 New **slack variables**. Rewrite inequality: $\sum_{i=1}^n a_i x_i \leq b$. As:

$$\begin{array}{ll} \max & c^T x \\ \text{subject to} & Ax = b. \\ & x \geq 0. \end{array}$$

$$\begin{aligned} x_{n+1} &= b - \sum_{i=1}^n a_i x_i \\ x_{n+1} &\geq 0. \end{aligned}$$

- 2 Value of slack variable x_{n+1} encodes how far is the original inequality for holding with equality.

Slack Form

- 1 Next rewrite **LP** into **slack form**.
- 2 Every inequality becomes equality.
- 3 All variables must be positive.
- 4 See resulting form on the right.
- 1 New **slack variables**. Rewrite inequality: $\sum_{i=1}^n a_i x_i \leq b$. As:

$$\begin{array}{ll} \max & c^T x \\ \text{subject to} & Ax = b. \\ & x \geq 0. \end{array}$$

$$\begin{aligned} x_{n+1} &= b - \sum_{i=1}^n a_i x_i \\ x_{n+1} &\geq 0. \end{aligned}$$

- 2 Value of slack variable x_{n+1} encodes how far is the original inequality for holding with equality.

Slack Form

- 1 Next rewrite **LP** into **slack form**.
- 2 Every inequality becomes equality.
- 3 All variables must be positive.
- 4 See resulting form on the right.
- 1 New **slack variables**. Rewrite inequality: $\sum_{i=1}^n a_i x_i \leq b$. As:

$$\begin{array}{ll} \max & c^T x \\ \text{subject to} & Ax = b. \\ & x \geq 0. \end{array}$$

$$\begin{aligned} x_{n+1} &= b - \sum_{i=1}^n a_i x_i \\ x_{n+1} &\geq 0. \end{aligned}$$

- 2 Value of slack variable x_{n+1} encodes how far is the original inequality for holding with equality.

Slack form...

- ① LP now made of equalities of the form:

$$x_{n+1} = b - \sum_{i=1}^n a_i x_i$$

- ② Variables on left: *basic variables*.
③ Variables on right: *nonbasic variables*.
④ LP in this form is in *slack form*.

Linear program in slack form.

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

Slack form...

- ① LP now made of equalities of the form:

$$x_{n+1} = b - \sum_{i=1}^n a_i x_i$$

- ② Variables on left: **basic variables**.

- ③ Variables on right: *nonbasic variables*.

- ④ LP in this form is in *slack form*.

Linear program in slack form.

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

Slack form...

- ① LP now made of equalities of the form:

$$x_{n+1} = b - \sum_{i=1}^n a_i x_i$$

- ② Variables on left: **basic variables**.
③ Variables on right: **nonbasic variables**.
④ LP in this form is in **slack form**.

Linear program in slack form.

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

Slack form...

- ① LP now made of equalities of the form:

$$x_{n+1} = b - \sum_{i=1}^n a_i x_i$$

- ② Variables on left: **basic variables**.
③ Variables on right: **nonbasic variables**.
④ LP in this form is in **slack form**.

Linear program in slack form.

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

Slack form formally

Because everything is clearer when you use tuples. Not.

The slack form is defined by a tuple (N, B, A, b, c, v) .

B - Set of indices of basic variables

N - Set of indices of nonbasic variables

$n = |N|$ - number of original variables

b, c - two vectors of constants

$m = |B|$ - number of basic variables
(i.e., number of inequalities)

$A = \{a_{ij}\}$ - The matrix of coefficients

$N \cup B = \{1, \dots, n + m\}$

v - objective function constant.

Slack form formally

Final form

$$\begin{aligned} \text{Max} \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

Example

Consider the following LP which is in slack form.

$$\begin{aligned}\max \quad z &= 29 - \frac{1}{9}x_3 - \frac{1}{9}x_5 - \frac{2}{9}x_6 \\ x_1 &= 8 + \frac{1}{6}x_3 + \frac{1}{6}x_5 - \frac{1}{3}x_6 \\ x_2 &= 4 - \frac{8}{3}x_3 - \frac{2}{3}x_5 + \frac{1}{3}x_6 \\ x_4 &= 18 - \frac{1}{2}x_3 + \frac{1}{2}x_5\end{aligned}$$

Example

...translated into tuple form (N, B, A, b, c, v) .

$$B = \{1, 2, 4\}, N = \{3, 5, 6\}$$

$$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix} \quad c = \begin{pmatrix} c_3 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} -1/9 \\ -1/9 \\ -2/9 \end{pmatrix}$$

$$v = 29.$$

Note that indices depend on the sets N and B , and also that the entries in A are negation of what they appear in the slack form.

Another example...

$$\begin{aligned} \max \quad & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Transform into slack form...

$$\begin{aligned} \max \quad z = \quad & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

The Simplex algorithm by example

$$\begin{array}{ll} \max & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Next, we introduce slack variables, for example, rewriting $2x_1 + 3x_2 + x_3 \leq 5$ as the constraints: $w_1 \geq 0$ and $w_1 = 5 - 2x_1 - 3x_2 - x_3$. The resulting LP in slack form is

$$\begin{array}{ll} \max & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ \Rightarrow & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{array}$$

Example continued...

$$\begin{aligned}\max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0\end{aligned}$$

① $\implies w_1 = 5, w_2 = 11$ and $w_3 = 8$.

② Feasible!

③ Objection function value: $z = 0$.

④ Further improve the value of objective function (i.e., z). While keeping feasibility.

① w_1, w_2, w_3 : slack variables (Also currently basic variables).

② Consider the slack representation trivial solution...

all non-basic variables assigned zero:

$$x_1 = x_2 = x_3 = 0.$$

Example continued...

$$\begin{aligned}\max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0\end{aligned}$$

① $\implies w_1 = 5, w_2 = 11$ and $w_3 = 8$.

② Feasible!

③ Objection function value: $z = 0$.

④ Further improve the value of objective function (i.e., z). While keeping feasibility.

① w_1, w_2, w_3 : slack variables. (Also currently basic variables).

② Consider the slack representation trivial solution...
all non-basic variables assigned zero:
 $x_1 = x_2 = x_3 = 0$.

Example continued...

$$\begin{aligned}\max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0\end{aligned}$$

① $\implies w_1 = 5, w_2 = 11$ and $w_3 = 8$.

② Feasible!

③ Objection function value: $z = 0$.

④ Further improve the value of objective function (i.e., z). While keeping feasibility.

① w_1, w_2, w_3 : slack variables.
(Also currently basic variables).

② Consider the slack representation trivial solution...
all non-basic variables assigned zero:
 $x_1 = x_2 = x_3 = 0$.

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

① $\implies w_1 = 5, w_2 = 11$ and $w_3 = 8$.

② Feasible!

③ Objection function value: $z = 0$.

④ Further improve the value of objective function (i.e., z). While keeping feasibility.

① w_1, w_2, w_3 : slack variables.
(Also currently basic variables).

② Consider the slack representation trivial solution...
all non-basic variables assigned zero:
 $x_1 = x_2 = x_3 = 0$.

Example continued...

$$\begin{aligned}\max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0\end{aligned}$$

① $\implies w_1 = 5, w_2 = 11$ and $w_3 = 8$.

② Feasible!

③ Objection function value: $z = 0$.

④ Further improve the value of objective function (i.e., z). While keeping feasibility.

① w_1, w_2, w_3 : slack variables.
(Also currently basic variables).

② Consider the slack representation trivial solution...
all non-basic variables assigned zero:
 $x_1 = x_2 = x_3 = 0$.

Example continued...

$$\begin{aligned}\max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0\end{aligned}$$

① $\implies w_1 = 5, w_2 = 11$ and $w_3 = 8$.

② Feasible!

③ Objection function value: $z = 0$.

④ Further improve the value of objective function (i.e., z). While keeping feasibility.

① w_1, w_2, w_3 : slack variables.
(Also currently basic variables).

② Consider the slack representation trivial solution...
all non-basic variables assigned zero:
 $x_1 = x_2 = x_3 = 0$.

Example continued...

$$\begin{aligned}\max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0\end{aligned}$$

① $\implies w_1 = 5, w_2 = 11$ and $w_3 = 8$.

② Feasible!

③ Objection function value: $z = 0$.

④ Further improve the value of objective function (i.e., z). While keeping feasibility.

① w_1, w_2, w_3 : slack variables. (Also currently basic variables).

② Consider the slack representation trivial solution...
all non-basic variables assigned zero:
 $x_1 = x_2 = x_3 = 0$.

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

- ① $x_1 = x_2 = x_3 = 0$
 $\implies w_1 = 5, w_2 = 11$
and $w_3 = 8$.
- ② All w_i positive – change x_i a bit does not change feasibility.

- ① $z = 5x_1 + 4x_2 + 3x_3$: want to increase values of x_1 s... since z increases (since $5 > 0$).
- ② How much to increase x_1 ???
- ③ Careful! Might break feasibility.
- ④ Increase x_1 as much as possible without breaking feasibility!

Example continued...

$$\begin{aligned}\max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0\end{aligned}$$

- ① $x_1 = x_2 = x_3 = 0$
 $\implies w_1 = 5, w_2 = 11$
and $w_3 = 8$.
- ② All w_i positive – change x_i a bit does not change feasibility.

- ① $z = 5x_1 + 4x_2 + 3x_3$: want to increase values of x_1 s... since z increases (since $5 > 0$).
- ② How much to increase x_1 ???
- ③ Careful! Might break feasibility.
- ④ Increase x_1 as much as possible without breaking feasibility!

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

- 1 $x_1 = x_2 = x_3 = 0$
 $\implies w_1 = 5, w_2 = 11$
and $w_3 = 8$.
- 2 All w_i positive – change x_i a bit does not change feasibility.

- 1 $z = 5x_1 + 4x_2 + 3x_3$: want to increase values of x_1 s... since z increases (since $5 > 0$).
- 2 How much to increase x_1 ???
- 3 Careful! Might break feasibility.
- 4 Increase x_1 as much as possible without breaking feasibility!

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

- 1 $x_1 = x_2 = x_3 = 0$
 $\implies w_1 = 5, w_2 = 11$
and $w_3 = 8$.
- 2 All w_i positive – change x_i a bit does not change feasibility.

- 1 $z = 5x_1 + 4x_2 + 3x_3$: want to increase values of x_1 s... since z increases (since $5 > 0$).
- 2 How much to increase x_1 ???
- 3 Careful! Might break feasibility.
- 4 Increase x_1 as much as possible without breaking feasibility!

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

- 1 $x_1 = x_2 = x_3 = 0$
 $\implies w_1 = 5, w_2 = 11$
and $w_3 = 8$.
- 2 All w_i positive – change x_i a bit does not change feasibility.

- 1 $z = 5x_1 + 4x_2 + 3x_3$: want to increase values of x_1 s... since z increases (since $5 > 0$).
- 2 How much to increase x_1 ???
- 3 Careful! Might break feasibility.
- 4 Increase x_1 as much as possible without breaking feasibility!

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

- 1 $x_1 = x_2 = x_3 = 0$
 $\implies w_1 = 5, w_2 = 11$
and $w_3 = 8$.
- 2 All w_i positive – change x_i a bit does not change feasibility.

- 1 $z = 5x_1 + 4x_2 + 3x_3$: want to increase values of x_1 s... since z increases (since $5 > 0$).
- 2 How much to increase x_1 ???
- 3 Careful! Might break feasibility.
- 4 Increase x_1 as much as possible without breaking feasibility!

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

- 1 $x_1 = x_2 = x_3 = 0$
 $\implies w_1 = 5, w_2 = 11$
and $w_3 = 8$.
- 2 All w_i positive – change x_i a bit does not change feasibility.

- 1 $z = 5x_1 + 4x_2 + 3x_3$: want to increase values of x_1 s... since z increases (since $5 > 0$).
- 2 How much to increase x_1 ???
- 3 Careful! Might break feasibility.
- 4 Increase x_1 as much as possible without breaking feasibility!

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

① Set $x_2 = x_3 = 0$

$$\begin{aligned} w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\ &= 5 - 2x_1 \\ w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\ &= 11 - 4x_1 \\ w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \\ &= 8 - 3x_1. \end{aligned}$$

① Want to increase x_1 as much as possible, as long as:

$$\begin{aligned} w_1 &= 5 - 2x_1 \geq 0, \\ w_2 &= 11 - 4x_1 \geq 0, \\ \text{and } w_3 &= 8 - 3x_1 \geq 0. \end{aligned}$$

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

① Set $x_2 = x_3 = 0$

$$\begin{aligned} w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\ &= 5 - 2x_1 \end{aligned}$$

$$\begin{aligned} w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\ &= 11 - 4x_1 \end{aligned}$$

$$\begin{aligned} w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \\ &= 8 - 3x_1. \end{aligned}$$

① Want to increase x_1 as much as possible, as long as:

$$w_1 = 5 - 2x_1 \geq 0,$$

$$w_2 = 11 - 4x_1 \geq 0,$$

$$\text{and } w_3 = 8 - 3x_1 \geq 0.$$

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

① Set $x_2 = x_3 = 0$

$$\begin{aligned} w_1 &= 5 - 2x_1 - 3x_2 - x_3 \\ &= 5 - 2x_1 \end{aligned}$$

$$\begin{aligned} w_2 &= 11 - 4x_1 - x_2 - 2x_3 \\ &= 11 - 4x_1 \end{aligned}$$

$$\begin{aligned} w_3 &= 8 - 3x_1 - 4x_2 - 2x_3 \\ &= 8 - 3x_1. \end{aligned}$$

① Want to increase x_1 as much as possible, as long as:

$$w_1 = 5 - 2x_1 \geq 0,$$

$$w_2 = 11 - 4x_1 \geq 0,$$

$$\text{and } w_3 = 8 - 3x_1 \geq 0.$$

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

① Constraints:

$$w_1 = 5 - 2x_1 \geq 0,$$

$$w_2 = 11 - 4x_1 \geq 0,$$

$$\text{and } w_3 = 8 - 3x_1 \geq 0.$$

② $x_1 \leq 2.5$,
 $x_1 \leq 11/4 = 2.75$ and
 $x_1 \leq 8/3 = 2.66$

① Maximum we can increase x_1 is 2.5.

② $x_1 = 2.5, x_2 = 0, x_3 = 0, w_1 = 0, w_2 = 1, w_3 = 0.5$
 $\Rightarrow z = 5x_1 + 4x_2 + 3x_3 = 12.5.$

③ Improved target!

④ A nonbasic variable x_1 is now non-zero. One basic variable (w_1) became zero.

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

① Constraints:

$$w_1 = 5 - 2x_1 \geq 0,$$

$$w_2 = 11 - 4x_1 \geq 0,$$

$$\text{and } w_3 = 8 - 3x_1 \geq 0.$$

② $x_1 \leq 2.5$,
 $x_1 \leq 11/4 = 2.75$ and
 $x_1 \leq 8/3 = 2.66$

① Maximum we can increase x_1 is 2.5.

② $x_1 = 2.5$, $x_2 = 0$, $x_3 = 0$, $w_1 = 0$, $w_2 = 1$, $w_3 = 0.5$
 $\Rightarrow z = 5x_1 + 4x_2 + 3x_3 = 12.5$.

③ Improved target!

④ A nonbasic variable x_1 is now non-zero. One basic variable (w_1) became zero.

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

① Constraints:

$$w_1 = 5 - 2x_1 \geq 0,$$

$$w_2 = 11 - 4x_1 \geq 0,$$

$$\text{and } w_3 = 8 - 3x_1 \geq 0.$$

② $x_1 \leq 2.5$,
 $x_1 \leq 11/4 = 2.75$ and
 $x_1 \leq 8/3 = 2.66$

① Maximum we can increase x_1 is 2.5.

② $x_1 = 2.5$, $x_2 = 0$, $x_3 = 0$, $w_1 = 0$, $w_2 = 1$, $w_3 = 0.5$
 $\Rightarrow z = 5x_1 + 4x_2 + 3x_3 = 12.5.$

③ Improved target!

④ A nonbasic variable x_1 is now non-zero. One basic variable (w_1) became zero.

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

① Constraints:

$$w_1 = 5 - 2x_1 \geq 0,$$

$$w_2 = 11 - 4x_1 \geq 0,$$

$$\text{and } w_3 = 8 - 3x_1 \geq 0.$$

② $x_1 \leq 2.5$,
 $x_1 \leq 11/4 = 2.75$ and
 $x_1 \leq 8/3 = 2.66$

① Maximum we can increase x_1 is 2.5.

② $x_1 = 2.5$, $x_2 = 0$, $x_3 = 0$, $w_1 = 0$, $w_2 = 1$, $w_3 = 0.5$
 $\Rightarrow z = 5x_1 + 4x_2 + 3x_3 = 12.5.$

③ Improved target!

④ A nonbasic variable x_1 is now non-zero. One basic variable (w_1) became zero.

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

① Constraints:

$$w_1 = 5 - 2x_1 \geq 0,$$

$$w_2 = 11 - 4x_1 \geq 0,$$

$$\text{and } w_3 = 8 - 3x_1 \geq 0.$$

② $x_1 \leq 2.5$,
 $x_1 \leq 11/4 = 2.75$ and

① Maximum we can increase x_1 is 2.5. $x_1 \leq 8/3 = 2.66$

② $x_1 = 2.5$, $x_2 = 0$, $x_3 = 0$, $w_1 = 0$, $w_2 = 1$, $w_3 = 0.5$
 $\Rightarrow z = 5x_1 + 4x_2 + 3x_3 = 12.5.$

③ Improved target!

④ A nonbasic variable x_1 is now non-zero. One basic variable (w_1) became zero.

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

① Constraints:

$$w_1 = 5 - 2x_1 \geq 0,$$

$$w_2 = 11 - 4x_1 \geq 0,$$

$$\text{and } w_3 = 8 - 3x_1 \geq 0.$$

② $x_1 \leq 2.5$,
 $x_1 \leq 11/4 = 2.75$ and
 $x_1 \leq 8/3 = 2.66$

① Maximum we can increase x_1 is 2.5.

② $x_1 = 2.5$, $x_2 = 0$, $x_3 = 0$, $w_1 = 0$, $w_2 = 1$, $w_3 = 0.5$
 $\Rightarrow z = 5x_1 + 4x_2 + 3x_3 = 12.5$.

③ Improved target!

④ A nonbasic variable x_1 is now non-zero. One basic variable (w_1) became zero.

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

- 1 $x_1 = 2.5, x_2 = 0, x_3 = 0, w_1 = 0, w_2 = 1, w_3 = 0.5$
- 2 A nonbasic variable x_1 is now non-zero. One basic variable (w_1) became zero.

- 1 Want to keep invariant: All non-basic variables in current solution are zero...
- 2 Idea: Exchange x_1 and w_1 !
- 3 Consider equality LP with w_1 and x_1 .
 $w_1 = 5 - 2x_1 - 3x_2 - x_3$.
- 4 Rewrite as: $x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3$.

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

- ① $x_1 = 2.5, x_2 = 0, x_3 = 0, w_1 = 0, w_2 = 1, w_3 = 0.5$
- ② A nonbasic variable x_1 is now non-zero. One basic variable (w_1) became zero.

- ① Want to keep invariant: All non-basic variables in current solution are zero...
- ② Idea: Exchange x_1 and w_1 !
- ③ Consider equality LP with w_1 and x_1 .
 $w_1 = 5 - 2x_1 - 3x_2 - x_3.$
- ④ Rewrite as: $x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3.$

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

- ① $x_1 = 2.5, x_2 = 0, x_3 = 0, w_1 = 0, w_2 = 1, w_3 = 0.5$
- ② A nonbasic variable x_1 is now non-zero. One basic variable (w_1) became zero.

- ① Want to keep invariant: All non-basic variables in current solution are zero...
- ② Idea: Exchange x_1 and w_1 !
- ③ Consider equality LP with w_1 and x_1 .
 $w_1 = 5 - 2x_1 - 3x_2 - x_3.$
- ④ Rewrite as: $x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3.$

Example continued...

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & w_1 = 5 - 2x_1 - 3x_2 - x_3 \\ & w_2 = 11 - 4x_1 - x_2 - 2x_3 \\ & w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\ & x_1, x_2, x_3, w_1, w_2, w_3 \geq 0 \end{aligned}$$

- ① $x_1 = 2.5, x_2 = 0, x_3 = 0, w_1 = 0, w_2 = 1, w_3 = 0.5$
- ② A nonbasic variable x_1 is now non-zero. One basic variable (w_1) became zero.

- ① Want to keep invariant: All non-basic variables in current solution are zero...
- ② Idea: Exchange x_1 and w_1 !
- ③ Consider equality LP with w_1 and x_1 .
 $w_1 = 5 - 2x_1 - 3x_2 - x_3.$
- ④ Rewrite as: $x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3.$

Example continued...

Substituting $x_1 = 5 - 2x_2 - x_3$, the new LP

$$\begin{aligned}\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\ x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.\end{aligned}$$

- 1 nonbasic variables: $\{w_1, x_2, x_3\}$
basic variables: $\{x_1, w_2, w_3\}$.
- 2 Trivial solution: all nonbasic variables = 0 is feasible.
- 3 $w_1 = x_2 = x_3 = 0$. Value: $z = 12.5$.

Example continued...

Substituting $x_1 = 5 - 2x_2 - x_3$, the new LP

$$\begin{aligned}\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\ x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.\end{aligned}$$

- 1 nonbasic variables: $\{w_1, x_2, x_3\}$
basic variables: $\{x_1, w_2, w_3\}$.
- 2 Trivial solution: all nonbasic variables $= 0$ is feasible.
- 3 $w_1 = x_2 = x_3 = 0$. Value: $z = 12.5$.

Example continued...

Substituting $x_1 = 5 - 2x_2 - x_3$, the new LP

$$\begin{aligned}\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\ x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.\end{aligned}$$

- 1 nonbasic variables: $\{w_1, x_2, x_3\}$
basic variables: $\{x_1, w_2, w_3\}$.
- 2 Trivial solution: all nonbasic variables = 0 is feasible.
- 3 $w_1 = x_2 = x_3 = 0$. Value: $z = 12.5$.

Example continued...

① Rewriting step done is called **pivoting**.

② pivoted on x_1 .

③ Continue pivoting till reach optimal solution.

$$\max \quad z = 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3$$

$$x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$w_3 = 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.$$

④ Can not pivot on w_1 , since if w_1 increase, then z decreases.
Bad.

⑤ Can not pivot on x_2 (coefficient in objective function is -3.5).

⑥ Can only pivot on x_3 since its coefficient ub objective 0.5 .
Positive number.

Example continued...

① Rewriting step done is called **pivoting**.

② pivoted on x_1 .

③ Continue pivoting till reach optimal solution.

$$\max \quad z = 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3$$

$$x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$w_3 = 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.$$

④ Can not pivot on w_1 , since if w_1 increase, then z decreases.
Bad.

⑤ Can not pivot on x_2 (coefficient in objective function is -3.5).

⑥ Can only pivot on x_3 since its coefficient ub objective 0.5 .
Positive number.

Example continued...

- 1 Rewriting step done is called **pivoting**.
- 2 pivoted on x_1 .
- 3 Continue pivoting till reach optimal solution.

$$\begin{aligned}\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\ x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.\end{aligned}$$

- 4 Can not pivot on w_1 , since if w_1 increase, then z decreases. Bad.
- 5 Can not pivot on x_2 (coefficient in objective function is -3.5).
- 6 Can only pivot on x_3 since its coefficient ub objective 0.5 . Positive number.

Example continued...

① Rewriting step done is called **pivoting**.

② pivoted on x_1 .

③ Continue pivoting till reach optimal solution.

$$\max \quad z = 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3$$

$$x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$w_3 = 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.$$

④ Can not pivot on w_1 , since if w_1 increase, then z decreases.
Bad.

⑤ Can not pivot on x_2 (coefficient in objective function is -3.5).

⑥ Can only pivot on x_3 since its coefficient ub objective 0.5 .
Positive number.

Example continued...

① Rewriting step done is called **pivoting**.

② pivoted on x_1 .

③ Continue pivoting till reach optimal solution.

$$\max \quad z = 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3$$

$$x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$w_3 = 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.$$

④ Can not pivot on w_1 , since if w_1 increase, then z decreases.
Bad.

⑤ Can not pivot on x_2 (coefficient in objective function is -3.5).

⑥ Can only pivot on x_3 since its coefficient ub objective 0.5 .
Positive number.

Example continued...

① Rewriting step done is called **pivoting**.

② pivoted on x_1 .

③ Continue pivoting till reach optimal solution.

$$\max \quad z = 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3$$

$$x_1 = 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3$$

$$w_2 = 1 + 2w_1 + 5x_2$$

$$w_3 = 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.$$

④ Can not pivot on w_1 , since if w_1 increase, then z decreases.
Bad.

⑤ Can not pivot on x_2 (coefficient in objective function is -3.5).

⑥ Can only pivot on x_3 since its coefficient ub objective 0.5 .
Positive number.

Example continued...

- 1 Rewriting step done is called **pivoting**.
- 2 pivoted on x_1 .
- 3 Continue pivoting till reach optimal solution.

$$\begin{aligned}\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\ x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.\end{aligned}$$

- 4 Can not pivot on w_1 , since if w_1 increase, then z decreases.
Bad.
- 5 Can not pivot on x_2 (coefficient in objective function is -3.5).
- 6 Can only pivot on x_3 since its coefficient ub objective 0.5 .
Positive number.

Example continued...

$$\begin{aligned}\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\ x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.\end{aligned}$$

- 1 Can only pivot on x_3 ...
- 2 x_1 can only be increased to **1** before $w_3 = 0$.
- 3 Rewriting the equality for w_3 in LP:
 $w_3 = 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3$,
- 4 ...for x_3 : $x_3 = 1 + 3w_1 + x_2 - 2w_3$.
- 5 Substituting into LP, we get the following LP.

$$\begin{aligned}\max \quad z &= 13 - w_1 - 3x_2 - w_3 \\ s.t. \quad x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ x_3 &= 1 + 3w_1 + x_2 - 2w_3\end{aligned}$$

Example continued...

$$\begin{aligned}\max \quad z &= 12.5 - 2.5w_1 - 3.5x_2 + 0.5x_3 \\ x_1 &= 2.5 - 0.5w_1 - 1.5x_2 - 0.5x_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ w_3 &= 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3.\end{aligned}$$

- 1 Can only pivot on x_3 ...
- 2 x_1 can only be increased to 1 before $w_3 = 0$.
- 3 Rewriting the equality for w_3 in LP:
 $w_3 = 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3$,
- 4 ...for x_3 : $x_3 = 1 + 3w_1 + x_2 - 2w_3$.
- 5 Substituting into LP, we get the following LP.

$$\begin{aligned}\max \quad z &= 13 - w_1 - 3x_2 - w_3 \\ s.t. \quad x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ x_3 &= 1 + 3w_1 + x_2 - 2w_3\end{aligned}$$

Example continued – can this be further improved?

$$\begin{aligned} \max \quad z &= 13 - w_1 - 3x_2 - w_3 \\ \text{s.t.} \quad x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ x_3 &= 1 + 3w_1 + x_2 - 2w_3 \end{aligned}$$

- 1 NO!
- 2 All coefficients in objective negative (or zero).
- 3 trivial solution (all nonbasic variables zero) is maximal.

Example continued – can this be further improved?

$$\begin{aligned} \max \quad z &= 13 - w_1 - 3x_2 - w_3 \\ \text{s.t.} \quad x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ x_3 &= 1 + 3w_1 + x_2 - 2w_3 \end{aligned}$$

- 1 NO!
- 2 All coefficients in objective negative (or zero).
- 3 trivial solution (all nonbasic variables zero) is maximal.

Example continued – can this be further improved?

$$\begin{aligned}\max \quad z &= 13 - w_1 - 3x_2 - w_3 \\ \text{s.t.} \quad x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ x_3 &= 1 + 3w_1 + x_2 - 2w_3\end{aligned}$$

- 1 NO!
- 2 All coefficients in objective negative (or zero).
- 3 trivial solution (all nonbasic variables zero) is maximal.

Example continued – can this be further improved?

$$\begin{aligned} \max \quad z &= 13 - w_1 - 3x_2 - w_3 \\ \text{s.t.} \quad x_1 &= 2 - 2w_1 - 2x_2 + w_3 \\ w_2 &= 1 + 2w_1 + 5x_2 \\ x_3 &= 1 + 3w_1 + x_2 - 2w_3 \end{aligned}$$

- 1 NO!
- 2 All coefficients in objective negative (or zero).
- 3 trivial solution (all nonbasic variables zero) is maximal.

Pivoting changes nothing

Observation

Every pivoting step just rewrites the **LP** into EQUIVALENT **LP**.
When **LP** objective can no longer be improved because of rewrite, it implies that the original **LP** objective function can not be increased any further.

Simplex algorithm – summary

- ① This was an informal description of the simplex algorithm.
- ② At each step pivot on a nonbasic variable that improves objective function.
- ③ Till reach optimal solution.
- ④ Problem: Assumed that the starting (trivial) solution (all zero nonbasic vars) is feasible.

Starting somewhere...

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

- 1 **L**: Transformed **LP** to slack form.
- 2 **Simplex** starts from feasible solution and walks around till reaches opt.
- 3 **L** might not be feasible at all.
- 4 Example on left, trivial sol is not feasible, if $\exists b_i < 0$.

Idea: Add a variable x_0 , and minimize it!

$$\begin{aligned} \min \quad & x_0 \\ \text{s.t.} \quad & x_i = x_0 + b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

Starting somewhere...

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

- 1 **L**: Transformed **LP** to slack form.
- 2 **Simplex** starts from feasible solution and walks around till reaches opt.
- 3 **L** might not be feasible at all.
- 4 Example on left, trivial sol is not feasible, if $\exists b_i < 0$.

Idea: Add a variable x_0 , and minimize it!

$$\begin{aligned} \min \quad & x_0 \\ \text{s.t.} \quad & x_i = x_0 + b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

Starting somewhere...

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

- 1 **L**: Transformed **LP** to slack form.
- 2 **Simplex** starts from feasible solution and walks around till reaches opt.
- 3 **L** might not be feasible at all.
- 4 Example on left, trivial sol is not feasible, if $\exists b_i < 0$.

Idea: Add a variable x_0 , and minimize it!

$$\begin{aligned} \min \quad & x_0 \\ \text{s.t.} \quad & x_i = x_0 + b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

Starting somewhere...

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

- 1 **L**: Transformed **LP** to slack form.
- 2 **Simplex** starts from feasible solution and walks around till reaches opt.
- 3 **L** might not be feasible at all.
- 4 Example on left, trivial sol is not feasible, if $\exists b_i < 0$.

Idea: Add a variable x_0 , and minimize it!

$$\begin{aligned} \min \quad & x_0 \\ \text{s.t.} \quad & x_i = x_0 + b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

Starting somewhere...

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

- 1 **L**: Transformed **LP** to slack form.
- 2 **Simplex** starts from feasible solution and walks around till reaches opt.
- 3 **L** might not be feasible at all.
- 4 Example on left, trivial sol is not feasible, if $\exists b_i < 0$.

Idea: Add a variable x_0 , and minimize it!

$$\begin{aligned} \min \quad & x_0 \\ \text{s.t.} \quad & x_i = x_0 + b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

Starting somewhere...

$$\begin{aligned} \max \quad & z = v + \sum_{j \in N} c_j x_j, \\ \text{s.t.} \quad & x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

- 1 **L**: Transformed **LP** to slack form.
- 2 **Simplex** starts from feasible solution and walks around till reaches opt.
- 3 **L** might not be feasible at all.
- 4 Example on left, trivial sol is not feasible, if $\exists b_i < 0$.

Idea: Add a variable x_0 , and minimize it!

$$\begin{aligned} \min \quad & x_0 \\ \text{s.t.} \quad & x_i = x_0 + b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B, \\ & x_i \geq 0, \quad \forall i = 1, \dots, n + m. \end{aligned}$$

Finding a feasible solution...

- 1 $L' = \mathbf{Feasible}(L)$ (see previous slide).
- 2 Add new variable x_0 and make it large enough.
- 3 $x_0 = \max(-\min_i b_i, 0)$, $\forall i > 0, x_i = 0$: feasible!
- 4 $\mathbf{LPStartSolution}(L')$: Solution of $\mathbf{Simplex}$ to L' .
- 5 If x_0 is solution then L feasible, and we found a valid basic solution.
- 6 If $x_0 > 0$ then LP not feasible. Done.

Finding a feasible solution...

- ① $L' = \mathbf{Feasible}(L)$ (see previous slide).
- ② Add new variable x_0 and make it large enough.
- ③ $x_0 = \max(-\min_i b_i, 0)$, $\forall i > 0, x_i = 0$: feasible!
- ④ $\mathbf{LPStartSolution}(L')$: Solution of $\mathbf{Simplex}$ to L' .
- ⑤ If x_0 is solution then L feasible, and we found a valid basic solution.
- ⑥ If $x_0 > 0$ then LP not feasible. Done.

Finding a feasible solution...

- ① $L' = \text{Feasible}(L)$ (see previous slide).
- ② Add new variable x_0 and make it large enough.
- ③ $x_0 = \max(-\min_i b_i, 0)$, $\forall i > 0, x_i = 0$: feasible!
- ④ **LPStartSolution**(L'): Solution of **Simplex** to L' .
- ⑤ If x_0 is solution then L feasible, and we found a valid basic solution.
- ⑥ If $x_0 > 0$ then LP not feasible. Done.

Finding a feasible solution...

- ① $L' = \text{Feasible}(L)$ (see previous slide).
- ② Add new variable x_0 and make it large enough.
- ③ $x_0 = \max(-\min_i b_i, 0)$, $\forall i > 0, x_i = 0$: feasible!
- ④ $\text{LPStartSolution}(L')$: Solution of **Simplex** to L' .
- ⑤ If x_0 is solution then L feasible, and we found a valid basic solution.
- ⑥ If $x_0 > 0$ then LP not feasible. Done.

Finding a feasible solution...

- ① $L' = \text{Feasible}(L)$ (see previous slide).
- ② Add new variable x_0 and make it large enough.
- ③ $x_0 = \max(-\min_i b_i, 0)$, $\forall i > 0, x_i = 0$: feasible!
- ④ $\text{LPStartSolution}(L')$: Solution of **Simplex** to L' .
- ⑤ If x_0 is solution then L feasible, and we found a valid basic solution.
- ⑥ If $x_0 > 0$ then **LP** not feasible. Done.

Lemma...

Lemma

LP L is feasible \iff optimal objective value of LP L' is zero.

Proof.

A feasible solution to L is immediately an optimal solution to L' with $x_0 = 0$, and vice versa. Namely, given a solution to L' with $x_0 = 0$ we can transform it to a feasible solution to L by removing x_0 . \square

Technicalities, technicalities everywhere

- ① Starting solution for L' , generated by **LPStartSolution**(L)..
- ② .. not legal in slack form as non-basic variable x_0 assigned non-zero value.
- ③ Trick: Immediately pivoting on x_0 when running **Simplex**(L').
- ④ First try to decrease x_0 as much as possible.

Technicalities, technicalities everywhere

- ① Starting solution for L' , generated by **LPStartSolution**(L)..
- ② .. not legal in slack form as non-basic variable x_0 assigned non-zero value.
- ③ Trick: Immediately pivoting on x_0 when running **Simplex**(L').
- ④ First try to decrease x_0 as much as possible.

Technicalities, technicalities everywhere

- ① Starting solution for L' , generated by **LPStartSolution**(L)..
- ② .. not legal in slack form as non-basic variable x_0 assigned non-zero value.
- ③ Trick: Immediately pivoting on x_0 when running **Simplex**(L').
- ④ First try to decrease x_0 as much as possible.

Technicalities, technicalities everywhere

- ① Starting solution for L' , generated by **LPStartSolution**(L)..
- ② .. not legal in slack form as non-basic variable x_0 assigned non-zero value.
- ③ Trick: Immediately pivoting on x_0 when running **Simplex**(L').
- ④ First try to decrease x_0 as much as possible.

Notes

