Chapter 15

Randomized Algorithms III – Min Cut

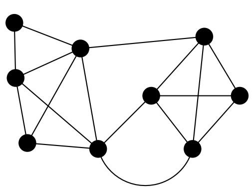
CS 573: Algorithms, Fall 2014
October 16, 2014

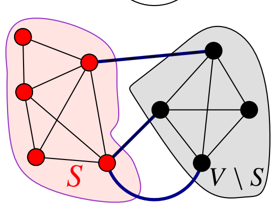
15.1 Min Cut

15.1.1 Problem Definition

15.2 Min cut

15.2.0.1 Min cut





$\mathsf{G} = (V, E)$: undirected graph, n vertices, m edges.
Interested in ***cuts*** in G.

Definition 15.2.1. ***cut*** in G: a partition of V: S and $V \setminus S$.
Edges of the cut:

$$(S, V \setminus S) = \left\{ uv \;\middle|\; u \in S, v \in V \setminus S, \text{ and } uv \in E \right\},$$

$|(S, V \setminus S)|$ is *size of the cut*

minimum cut / ***mincut***: cut in graph with min size.

15.2.0.2 Some definitions

(A) ***conditional probability*** of X given Y is $\mathbf{Pr}\Big[X = x \,|Y = y\Big] = \frac{\mathbf{Pr}\Big[(X=x)\cap(Y=y)\Big]}{\mathbf{Pr}\Big[Y=y\Big]}$.

$$\mathbf{Pr}\Big[(X = x) \cap (Y = y)\Big] = \mathbf{Pr}\Big[X = x \;\Big|\; Y = y\Big] \cdot \mathbf{Pr}[Y = y].$$

(B) X, Y events are ***independent***, if $\mathbf{Pr}\left[X = x \cap Y = y\right] = \mathbf{Pr}\left[X = x\right] \cdot \mathbf{Pr}\left[Y = y\right]$.
$\implies \mathbf{Pr}\left[X = x \,\middle|\, Y = y\right] = \mathbf{Pr}\left[X = x\right]$.

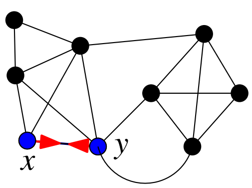
15.2.0.3 Some more probability

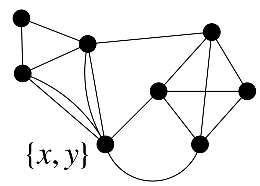
Lemma 15.2.2. *$\mathcal{E}_1, \ldots, \mathcal{E}_n$: n events (not necessarily independent). Then,*

$$\mathbf{Pr}\left[\cap_{i=1}^{n} \mathcal{E}_i\right] = \mathbf{Pr}\left[\mathcal{E}_1\right] * \mathbf{Pr}\left[\mathcal{E}_2 \,|\mathcal{E}_1\right] * \mathbf{Pr}\left[\mathcal{E}_3 \,\middle|\, \mathcal{E}_1 \cap \mathcal{E}_2\right] * \ldots$$
$$* \mathbf{Pr}\left[\mathcal{E}_n \,\middle|\, \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{n-1}\right].$$

15.3 The Algorithm

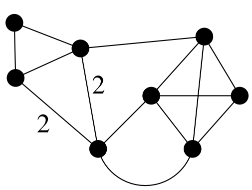
15.3.0.4 Edge contraction...

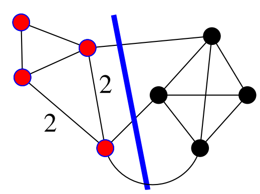
G:


G$/xy$:


(A) ***edge contraction***: $e = xy$ in G.
(B) ... merge x, y into a single vertex.
(C) ...remove self loops.
(D) ... parallel edges – ***multi-graph***.
(E) ... weights/ multiplicities on the edges.

15.3.0.5 Min cut in weighted graph





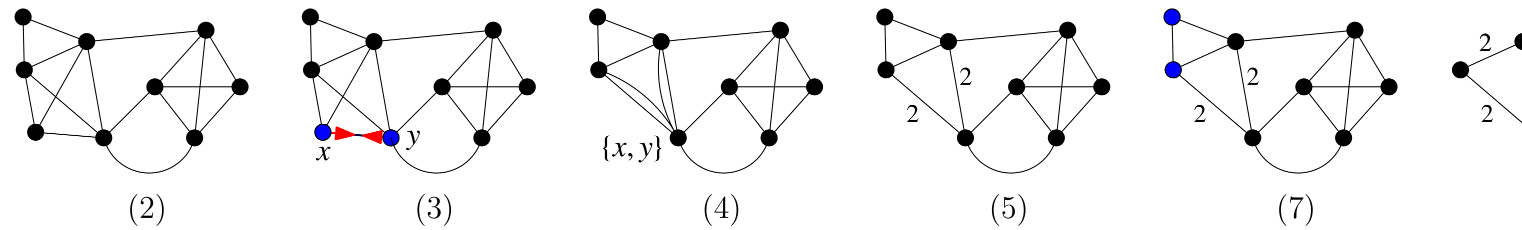
Edge contraction implemented in $O(n)$ time:
(A) Graph represented using adjacency lists.
(B) Merging the adjacency lists of the two vertices being contracted.
(C) Using hashing to do fix-ups.
(i.e., fix adjacency list of vertices connected to x, y.)
(D) Include edge weight in computing cut weight.

15.3.0.6 Cuts under contractions

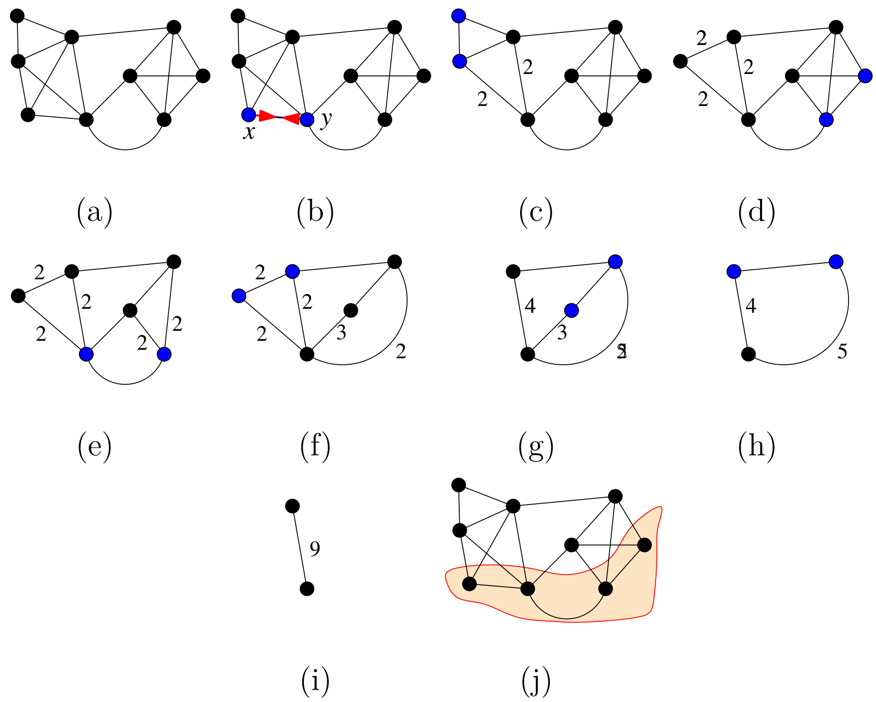
Observation 15.3.1. *(A) A cut in* G$/xy$ *is a valid cut in* G.
(B) There $\exists$ cuts in G *are not in* G$/xy$.
(C) The cut $S = \{x\}$ is not in G$/xy$.
(D) $\implies$ size mincut in G$/xy$ $\geq$ *mincut in* G.

(A) **Idea**: Repeatedly perform edge contractions (benefits: shrink graph)...
(B) Every vertex in contracted graph is a connected component in the original graph.)

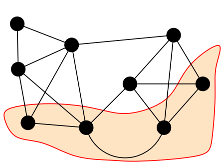
15.3.0.7 Contraction



15.3.0.8 Contraction - all together now



15.3.0.9 But...



(A) Not min cut!
(B) Contracted wrong edge somewhere...
(C) If never contract an edge in the cut...
(D) ...get min cut in the end!
(E) We might still get min cut even if we contract edge min cut. Why???

15.3.1 The resulting algorithm

15.3.1.1 The algorithm...

```
Algorithm MinCut(G)
        G_0 ← G
        i = 0
        while G_i has more than two vertices do
            e_i ← random edge from E(G_i)
            G_{i+1} ← G_i/e_i
            i ← i + 1
        Let (S, V \ S) be the cut in the original graph
                    corresponding to the single edge in G_i
        return (S, V \ S).
```

15.3.1.2 How to pick a random edge?

Lemma 15.3.2. *$X = \{x_1, \ldots, x_n\}$: elements, $\omega(x_i)$: integer positive weight.*

Pick randomly, in $O(n)$ time, an element $\in X$, with prob picking x_i being $\omega(x_i)/W$, where $W = \sum_{i=1}^{n} \omega(x_i)$.

Proof: Randomly choose $r \in [0, W]$.

Precompute $\beta_i = \sum_{k=1}^{i} \omega(x_k) = \beta_{i-1} + \omega(x_i)$.

Find first index i, $\beta_{i-1} < r \leq \beta_i$. Return x_i. ■

(A) Edges have weight...
(B) ...compute total weight of each vertex (adjacent edges).
(C) Pick randomly a vertex by weight.
(D) Pick random edge adjacent to this vertex.

15.3.2 Analysis

15.3.2.1 The probability of success

15.3.2.2 Lemma...

Lemma 15.3.3. G*: mincut of size k and n vertices, then* $|\mathsf{E}(\mathsf{G})| \geq \frac{kn}{2}$.

Proof: Each vertex degree is at least k, otherwise the vertex itself would form a minimum cut of size smaller than k. As such, there are at least $\sum_{v \in V} \mathrm{degree}(v)/2 \geq nk/2$ edges in the graph. ■

15.3.2.3 Lemma...

Lemma 15.3.4. *If we pick in random an edge e from a graph G, then with probability at most $\frac{2}{n}$ it belong to the minimum cut.*

Proof: There are at least $nk/2$ edges in the graph and exactly k edges in the minimum cut. Thus, the probability of picking an edge from the minimum cut is smaller then $k/(nk/2) = 2/n$. ■

15.3.2.4 Lemma

Lemma 15.3.5. **MinCut** *outputs the mincut with prob.* $\geq \dfrac{2}{n(n-1)}$.

Proof
(A) $\mathcal{E}_i$: event that e_i is not in the minimum cut of G_i.
(B) **MinCut** outputs mincut if all the events $\mathcal{E}_0, \ldots, \mathcal{E}_{n-3}$ happen.
(C) $\mathbf{Pr}\Big[\mathcal{E}_i \,\Big|\, \mathcal{E}_0 \cap \mathcal{E}_1 \cap \ldots \cap \mathcal{E}_{i-1}\Big] \geq 1 - \dfrac{2}{|V(G_i)|} = 1 - \dfrac{2}{n-i}$.

$$\implies \Delta = \mathbf{Pr}[\mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-3}] = \mathbf{Pr}[\mathcal{E}_0] \cdot \mathbf{Pr}\Big[\mathcal{E}_1 \,\Big|\, \mathcal{E}_0\Big] \cdot \mathbf{Pr}\Big[\mathcal{E}_2 \,\Big|\, \mathcal{E}_0 \cap \mathcal{E}_1\Big] \cdot \ldots \cdot \mathbf{Pr}\Big[\mathcal{E}_{n-3} \,\Big|\, \mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-4}\Big]$$

15.3.2.5 Proof continued...

As such, we have

$$\begin{aligned} \Delta &\geq \prod_{i=0}^{n-3}\left(1 - \frac{2}{n-i}\right) = \prod_{i=0}^{n-3} \frac{n-i-2}{n-i} \\ &= \frac{n-2}{n} * \frac{n-3}{n-1} * \frac{n-4}{n-2} \ldots \cdot \frac{2}{4} \cdot \frac{1}{3} \\ &= \frac{2}{n \cdot (n-1)}. \end{aligned}$$

15.3.2.6 Some math restated...

$$\begin{aligned} \alpha &= \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right)\left(1 - \frac{2}{n-2}\right) \cdots \left(1 - \frac{2}{4}\right)\left(1 - \frac{2}{3}\right) \\ &= \frac{n-2}{n} \cdot \frac{(n-1)-2}{n-1} \cdot \frac{(n-2)-2}{n-2} \cdots \frac{4-2}{4} \cdot \frac{3-2}{3} \\ &= \frac{\cancel{n-2}}{n} \cdot \frac{\cancel{n-3}}{n-1} \cdot \frac{\cancel{n-4}}{\cancel{n-2}} \cdot \frac{\cancel{n-5}}{\cancel{n-3}} \cdots \frac{\cancel{3}}{\cancel{5}} \cdot \frac{2}{\cancel{4}} \cdot \frac{1}{\cancel{3}} \\ &= \frac{}{n} \cdot \frac{}{n-1} \cdot \frac{}{} \cdot \frac{}{} \cdots \frac{}{} \cdot \frac{2}{} \cdot \frac{1}{} \\ &= \frac{2}{n(n-1)} \end{aligned}$$

15.3.2.7 Running time analysis.

15.3.2.8 Running time analysis...

Observation 15.3.6. **MinCut** *runs in* $O(n^2)$ *time.*

Observation 15.3.7. *The algorithm always outputs a cut, and the cut is not smaller than the minimum cut.*

Definition 15.3.8. ***Amplification***: running an experiment again and again till the things we want to happen, with good probability, do happen.

15.3.2.9 Getting a good probability

MinCutRep: algorithm runs **MinCut** $n(n-1)$ times and return the minimum cut computed.

Lemma 15.3.9. *probability* **MinCutRep** *fails to return the minimum cut is* < 0.14.

Proof: **MinCut** fails to output the mincut in each execution is at most $1 - \frac{2}{n(n-1)}$.

MinCutRep fails, only if all $n(n-1)$ executions of **MinCut** fail.

$\left(1 - \frac{2}{n(n-1)}\right)^{n(n-1)} \leq \exp\left(-\frac{2}{n(n-1)} \cdot n(n-1)\right) = \exp(-2) < 0.14$, since $1 - x \leq e^{-x}$ for $0 \leq x \leq 1$. ∎

15.3.2.10 Result

Theorem 15.3.10. *One can compute mincut in* $O(n^4)$ *time with constant probability to get a correct result. In* $O(n^4 \log n)$ *time the minimum cut is returned with high probability.*

15.4 A faster algorithm

15.4.0.11 Faster algorithm

Why **MinCutRep** needs so many executions?

Probability of failure in first ν iterations is

$$\begin{aligned}
\mathbf{Pr}\left[\mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{\nu-1}\right] &\geq \prod_{i=0}^{\nu-1}\left(1 - \frac{2}{n-i}\right) = \prod_{i=0}^{\nu-1} \frac{n-i-2}{n-i} \\
&= \frac{n-2}{n} * \frac{n-3}{n-1} * \frac{n-4}{n-2} \ldots \\
&= \frac{(n-\nu)(n-\nu-1)}{n \cdot (n-1)}.
\end{aligned}$$

$\implies \nu = n/2$: Prob of success $\approx 1/4$.

$\implies \nu = n - \sqrt{n}$: Prob of success $\approx 1/n$.

15.4.0.12 Faster algorithm...

Insight

(A) As the graph get smaller probability for bad choice increases.

(B) Currently do the amplification from the outside of the algorithm.

(C) Put amplification directly into the algorithm.

15.4.1 Contract...

15.4.1.1 Contract(G, t) shrinks G till it has only t vertices. FastCut computes the minimum cut using Contract.

```
Contract( G, t )
    while |(G)| > t do
        Pick a random edge
            e in G.
        G ← G/e
    return G
```

```
FastCut(G = (V, E))
    G -- multi-graph
begin
    n ← |V(G)|
    if n ≤ 6 then
        Compute minimum cut
        of G and return cut.
    t ← ⌈1 + n/√2⌉
    H_1 ← Contract(G, t)
    H_2 ← Contract(G, t)
    /* Contract is randomized!!! */
    X_1 ← FastCut(H_1),
    X_2 ← FastCut(H_2)
    return mincut of X_1 and X_2.
end
```

15.4.1.2 Lemma...

Lemma 15.4.1. *The running time of* **FastCut**(G) *is* $O(n^2 \log n)$*, where* $n = |V(G)|$.

Proof: Well, we perform two calls to **Contract**(G, t) which takes $O(n^2)$ time. And then we perform two recursive calls on the resulting graphs. We have:

$T(n) = O(n^2) + 2T\left(\frac{n}{\sqrt{2}}\right)$

The solution to this recurrence is $O(n^2 \log n)$ as one can easily (and should) verify. ■

15.4.1.3 Success at each step

Lemma 15.4.2. *Probability that mincut in contracted graph is original mincut is at least* $1/2$.

Proof: Plug in $\nu = n - t = n - \left\lceil 1 + n/\sqrt{2} \right\rceil$ into success probability:

$$\mathbf{Pr}\left[\mathcal{E}_0 \cap \ldots \cap \mathcal{E}_{n-t}\right] \geq \frac{t(t-1)}{n \cdot (n-1)} = \frac{\left\lceil 1 + n/\sqrt{2} \right\rceil \left(\left\lceil 1 + n/\sqrt{2} \right\rceil - 1 \right)}{n(n-1)} \geq \frac{1}{2}.$$

15.4.1.4 Probability of success...

Lemma 15.4.3. **FastCut** *finds the minimum cut with probability larger than* $\Omega\left(1/\log n\right)$.

See class notes for a formal proof. We provide a more elegant direct argument shortly.

15.4.1.5 Amplification

Lemma 15.4.4. *Running* **FastCut** *repeatedly* $c \cdot \log^2 n$ *times, guarantee that the algorithm outputs mincut with probability* $\geq 1 - 1/n^2$.

c is a constant large enough.

Proof: (A) **FastCut** succeeds with prob $\geq c'/\log n$, c' is a constant.
(B) ...fails with prob. $\leq 1 - c'/\log n$.
(C) ...fails in m reps with prob. $\leq (1 - c'/\log n)^m$. But then

$$(1 - c'/\log n)^m \leq \left(e^{-c'/\log n}\right)^m \leq e^{-mc'/\log n} \leq \tfrac{1}{n^2},$$

for $m = (2\log n)/c'$.

15.4.1.6 Theorem

Theorem 15.4.5. *One can compute the minimum cut in a graph* G *with* n *vertices in* $O(n^2 \log^3 n)$ *time. The algorithm succeeds with probability* $\geq 1 - 1/n^2$.

Proof: We do amplification on **FastCut** by running it $O(\log^2 n)$ times. The running time bound follows from lemma... ■

15.5 On coloring trees and min-cut

15.5.0.7 Trees and coloring edges...

(A) T_h be a complete binary tree of height h.
(B) Randomly color its edges by black and white.
(C) $\mathcal{E}_h$: there exists a black path from root T_h to one of its leafs.
(D) $\rho_h = \mathbf{Pr}[\mathcal{E}_h]$.
(E) $\rho_0 = 1$ and $\rho_1 = 3/4$ (see below).

15.5.0.8 Bounding ρ_h

(A) u root of T_h: children u_l and u_r.
(B) ρ_{h-1}: Probability for black path $u_l \rightsquigarrow$ children
(C) Prob of black path from u through u_1 is:

$\mathbf{Pr}\Big[u u_l \text{ is black}\Big] \cdot \rho_{h-1} = \rho_{h-1}/2$

(D) Prob. no black path through u_l is $1 - \rho_{h-1}/2$.
(E) Prob no black path is: $(1 - \rho_{h-1}/2)^2$
(F) We have

$$\rho_h = 1 - \left(1 - \frac{\rho_{h-1}}{2}\right)^2 = \frac{\rho_{h-1}}{2}\left(2 - \frac{\rho_{h-1}}{2}\right) = \rho_{h-1} - \frac{\rho_{h-1}^2}{4}.$$

15.5.0.9 Lemma...

Lemma 15.5.1. *We have that* $\rho_h \geq 1/(h+1)$.

Proof: (A) By induction. For $h = 1$: $\rho_1 = 3/4 \geq 1/(1+1)$.
(B) $\rho_h = \rho_{h-1} - \frac{\rho_{h-1}^2}{4} = f(\rho_{h-1})$, for $f(x) = x - x^2/4$.

(C) $f'(x) = 1 - x/2. \implies f'(x) > 0$ for $x \in [0, 1]$.
(D) $f(x)$ is increasing in the range $[0, 1]$
(E) By induction: $\rho_h = f(\rho_{h-1}) \geq f\left(\frac{1}{(h-1)+1}\right) = \frac{1}{h} - \frac{1}{4h^2}$.
(F) $\frac{1}{h} - \frac{1}{4h^2} \geq \frac{1}{h+1} \quad \Leftrightarrow \quad 4h(h+1) - (h+1) \geq 4h^2 \quad \Leftrightarrow \quad 4h^2 + 4h - h - 1 \geq 4h^2 \quad \Leftrightarrow \quad 3h \geq 1,$

15.5.0.10 Back to FastCut...

(A) Recursion tree for **FastCut** corresponds to such a coloring.
(B) Every call performs two recursive calls.
(C) Contraction in recursion succeeds with prob 1/2.
Draw recursion edge in black if successful.
(D) algorithm succeeds $\iff$ there black path from root of recursion tree to leaf.
(E) Since depth of tree $H \leq 2 + \log_{\sqrt{2}} n$.
(F) by above... probability of success is $\geq 1/(h+1) \geq 1/(3 + \log_{\sqrt{2}} n)$.

15.5.0.11 Galton-Watson processes

(A) Start with a single node.
(B) Each node has two children.
(C) Each child survives with probability half (independently).
(D) If a child survives then it is going to have two children, and so on.
(E) A single node give a rise to a random tree.
(F) Q: Probability that the original node has descendants h generations in the future.
(G) Prove this probability is at least $1/(h+1)$.

15.5.0.12 Galton-Watson process

(A) Victorians worried: aristocratic surnames were disappearing.
(B) Family names passed on only through the male children.
(C) Family with no male children had its family name disappear.
(D) # male children of a person is an independent random variable $X \in \{0, 1, 2, \ldots\}$.
(E) Starting with a single person, its family (as far as male children are concerned) is a random tree with the degree of a node being distributed according to X.
(F) .. A family disappears if $\mathbf{E}[X] \leq 1$, and it has a constant probability of surviving if $\mathbf{E}[X] > 1$.

15.5.0.13 Galton-Watson process

(A) ... Infant mortality is dramatically down. No longer a problem.
(B) Countries with family names that were introduced long time ago...
(C) ...have very few surnames.
Koreans have 250 surnames, and three surnames form 45% of the population).
(D) Countries introduced surnames recently have more surnames.
Dutch have surnames only for the last 200 years, and there are $68,000$ different family names).