Chapter 11

Network Flow

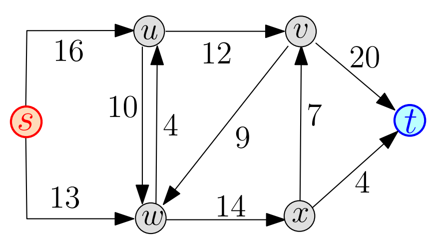
CS 573: Algorithms, Fall 2014
September 30, 2014

11.1 Network Flow

11.1.1 Network Flow

11.1.1.1 Network flow

(A) Transfer as much "merchandise" as possible from one point to another.
(B) Wireless network, transfer a large file from s to t.
(C) Limited capacities.

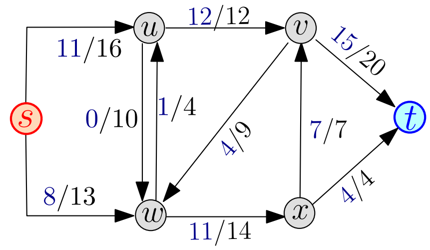


11.1.1.2 Network: Definition

(A) Given a network with capacities on each connection.
(B) Q: How much "flow" can transfer from source s to a sink t?
(C) The flow is ***splitable***.
(D) Network examples: water pipes moving water. Electricity network.
(E) Internet is packet base, so not quite splitable.

Definition 11.1.1. $\star$ $\mathsf{G} = (\mathsf{V}, \mathsf{E})$: a ***directed*** graph.
$\star$ $\forall\, (u \to v) \in \mathsf{E}(\mathsf{G})$: ***capacity*** $c(u, v) \geq 0$,
$\star$ $(u \to v) \notin G \implies c(u, v) = 0$.
$\star$ s: ***source*** vertex, t: target ***sink*** vertex.
$\star$ G, s, t and $c(\cdot)$: form ***flow network*** or ***network***.

11.1.1.3 Network Example



(A) All flow from the source ends up in the sink.
(B) Flow on edge: non-negative quantity $\leq$ capacity of edge.

11.1.1.4 Flow definition

Definition 11.1.2 (flow). ***flow*** in network is a function $f(\cdot,\cdot) : \mathsf{E}(\mathsf{G}) \to \mathbb{R}$:

(A) ***Bounded by capacity***:
$\forall (u \to v) \in \mathsf{E} \qquad f(u,v) \leq c(u,v).$

(B) ***Anti symmetry***:
$\forall u, v \qquad f(u,v) = -f(v,u).$

(C) Two special vertices: (i) the ***source*** s and the ***sink*** t.

(D) ***Conservation of flow*** (Kirchhoff's Current Law):
$\forall u \in \mathsf{V} \setminus \{s,t\} \qquad \sum_v f(u,v) = 0.$

flow/value of f: $|f| = \sum_{v \in V} f(s,v)$.

11.1.1.5 Problem: Max Flow

(A) Flow on edge can be negative (i.e., positive flow on edge in other direction).

Problem 11.1.3 (Maximum flow). Given a network G find the ***maximum flow*** in G. Namely, compute a legal flow f such that $|f|$ is maximized.

11.2 Some properties of flows and residual networks

11.2.0.6 Flow across sets of vertices

(A) $\forall X, Y \subseteq \mathsf{V}$, let $f(X,Y) = \sum_{x \in X, y \in Y} f(x,y)$.
$f(v,S) = f\Big(\{v\}, S\Big)$, where $v \in \mathsf{V}(\mathsf{G})$.

Observation 11.2.1. $|f| = f(s, \mathsf{V})$.

11.2.0.7 Basic properties of flows: (i)

Lemma 11.2.2. *For a flow f, the following properties holds:*
(i) $\forall u \in \mathsf{V}(\mathsf{G})$ we have $f(u,u) = 0$,

Proof: Holds since $(u \to u)$ it not an edge in G.
$(u \to u)$ capacity is zero,
Flow on $(u \to u)$ is zero. ∎

11.2.0.8 Basic properties of flows: (ii)

Lemma 11.2.3. *For a flow f, the following properties holds:*
(ii) $\forall X \subseteq \mathsf{V}$ *we have* $f(X, X) = 0$*,*

Proof:

$$\begin{aligned} f(X, X) &= \sum_{\{u,v\} \subseteq X, u \neq v} (f(u,v) + f(v,u)) + \sum_{u \in X} f(u,u) \\ &= \sum_{\{u,v\} \subseteq X, u \neq v} (f(u,v) - f(u,v)) + \sum_{u \in X} 0 = 0, \end{aligned}$$

by the anti-symmetry property of flow. ■

11.2.0.9 Basic properties of flows: (iii)

Lemma 11.2.4. *For a flow f, the following properties holds:*
(iii) $\forall X, Y \subseteq \mathsf{V}$ *we have* $f(X, Y) = -f(Y, X)$*,*

Proof: By the anti-symmetry of flow, as

$$f(X, Y) = \sum_{x \in X, y \in Y} f(x, y) = -\sum_{x \in X, y \in Y} f(y, x) = -f(Y, X).$$

11.2.0.10 Basic properties of flows: (iv)

Lemma 11.2.5. *For a flow f, the following properties holds:*
(iv) $\forall X, Y, Z \subseteq \mathsf{V}$ *such that* $X \cap Y = \emptyset$ *we have that* $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ *and* $f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$*.*

Proof: Follows from definition. (Check!) ■

11.2.0.11 Basic properties of flows: (v)

Lemma 11.2.6. *For a flow f, the following properties holds:*
(v) $\forall u \in \mathsf{V} \setminus \{s, t\}$*, we have* $f(u, \mathsf{V}) = f(\mathsf{V}, u) = 0$*.*

Proof: This is a restatement of the conservation of flow property. ■

11.2.0.12 Basic properties of flows: summary

Lemma 11.2.7. *For a flow f, the following properties holds:*
(i) $\forall u \in \mathsf{V}(\mathsf{G})$ *we have* $f(u, u) = 0$*,*
(ii) $\forall X \subseteq \mathsf{V}$ *we have* $f(X, X) = 0$*,*
(iii) $\forall X, Y \subseteq \mathsf{V}$ *we have* $f(X, Y) = -f(Y, X)$*,*
(iv) $\forall X, Y, Z \subseteq \mathsf{V}$ *such that* $X \cap Y = \emptyset$ *we have that* $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ *and* $f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$*.*
(v) *For all* $u \in \mathsf{V} \setminus \{s, t\}$*, we have* $f(u, \mathsf{V}) = f(\mathsf{V}, u) = 0$*.*

11.2.0.13 All flow gets to the sink

Claim 11.2.8. $|f| = f(\mathsf{V}, t)$.

Proof:

$$
\begin{aligned}
|f| = f(s, \mathsf{V}) = f\Big(\mathsf{V} \setminus (\mathsf{V} \setminus \{s\}), V\Big) & \\
= f(\mathsf{V}, V) - f(V \setminus \{s\}, \mathsf{V}) & \\
= -f(V \setminus \{s\}, \mathsf{V}) & \qquad\qquad = f(\mathsf{V}, \mathsf{V} \setminus \{s\}) \\
= f(\mathsf{V}, t) + f(\mathsf{V}, \mathsf{V} \setminus \{s, t\}) & \\
= f(\mathsf{V}, t) + \sum_{u \in V \setminus \{s,t\}} f(\mathsf{V}, u) & \\
= f(\mathsf{V}, t) + \sum_{u \in V \setminus \{s,t\}} 0 & \\
= f(\mathsf{V}, t), &
\end{aligned}
$$

Since $f(\mathsf{V}, V) = 0$ by (i) and $f(\mathsf{V}, u) = 0$ by (iv).

11.2.0.14 Residual capacity

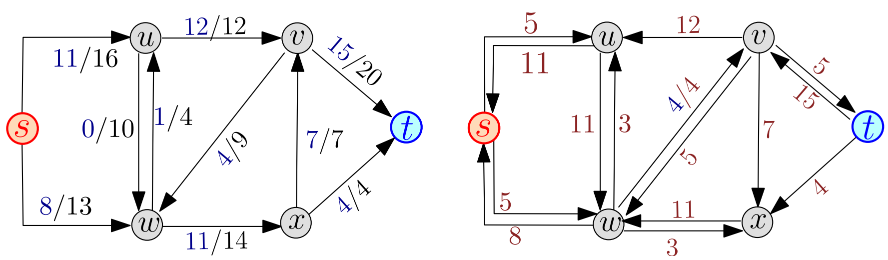
Definition 11.2.9. c: capacity, f: flow.

The ***residual capacity*** of an edge $(u \to v)$ is

$$c_f(u, v) = c(u, v) - f(u, v).$$

(A) residual capacity $c_f(u, v)$ on $(u \to v)$ = amount of unused capacity on $(u \to v)$.
(B) ... next construct graph with all edges not being fully used by f.

11.2.0.15 Residual graph



Graph

Residual graph

$f(u, w) = -f(w, u) = -1 \implies c_f(u, w) = 10 - (-1) = 11.$

11.2.0.16 Residual graph: Definition

Definition 11.2.10. Given f, $\mathsf{G} = (\mathsf{V}, E)$ and c, as above, the ***residual graph*** (or ***residual network***) of G and f is the graph $\mathsf{G}_f = (\mathsf{V}, \mathsf{E}_f)$ where

$$\mathsf{E}_f = \Big\{ (u, v) \in V \times \mathsf{V} \;\Big|\; c_f(u, v) > 0 \Big\}.$$

(A) $(u \to v) \in \mathsf{E}$: might induce two edges in E_f
(B) If $(u \to v) \in \mathsf{E}$, $f(u, v) < c(u, v)$ and $(v \to u) \notin \mathsf{E}(\mathsf{G})$

(C) $\implies c_f(u,v) = c(u,v) - f(u,v) > 0$
(D) ... and $(u \to v) \in \mathsf{E}_f$. Also,

$$c_f(v,u) = c(v,u) - f(v,u) = 0 - (-f(u,v)) = f(u,v),$$

since $c(v,u) = 0$ as $(v \to u)$ is not an edge of G.
(E) $\implies (v \to u) \in \mathsf{E}_f$.

11.2.0.17 Residual network properties

Since every edge of G induces at most two edges in G_f, it follows that G_f has at most twice the number of edges of G; formally, $|\mathsf{E}_f| \leq 2|\mathsf{E}|$.

Lemma 11.2.11. *Given a flow f defined over a network G, then the residual network G_f together with c_f form a flow network.*

Proof: One need to verify that $c_f(\cdot)$ is always a non-negative function, which is true by the definition of E_f. ■

11.2.0.18 Increasing the flow

Lemma 11.2.12. $\mathsf{G}(\mathsf{V}, E)$, *a flow f, and h a flow in G_f. G_f: residual network of f.*
Then $f + h$ is a flow in G and its capacity is $|f + h| = |f| + |h|$.

proof By definition: $(f+h)(u,v) = f(u,v)+h(u,v)$ and thus $(f+h)(X,Y) = f(X,Y)+h(X,Y)$. Verify legal...
(A) Anti symmetry: $(f + h)(u,v) = f(u,v) + h(u,v) = -f(v,u) - h(v,u) = -(f + h)(v,u)$.
(B) Bounded by capacity:

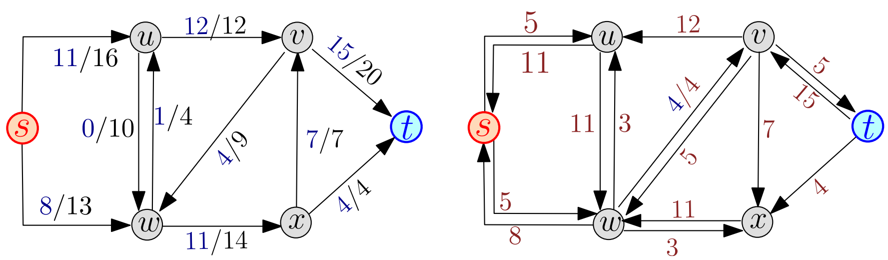
$$\begin{aligned}(f + h)(u,v) &\leq f(u,v) + h(u,v) \leq f(u,v) + c_f(u,v) \\ &= f(u,v) + (c(u,v) - f(u,v)) = c(u,v).\end{aligned}$$

11.2.0.19 Increasing the flow – proof continued

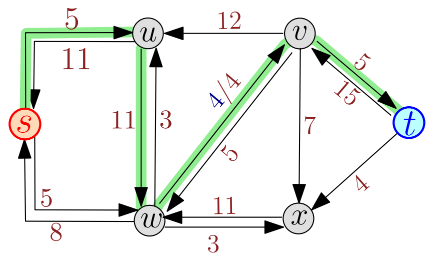
proof continued
(A) For $u \in V - s - t$ we have $(f + h)(u, \mathsf{V}) = f(u, \mathsf{V}) + h(u, \mathsf{V}) = 0 + 0 = 0$ and as such $f + h$ comply with the conservation of flow requirement.
(B) Total flow is

$$|f + h| = (f + h)(s, \mathsf{V}) = f(s, \mathsf{V}) + h(s, \mathsf{V}) = |f| + |h|.$$

11.2.0.20 Augmenting path



Graph   Residual graph



Definition 11.2.13. For G and a flow f, a path π in G_f between s and t is an ***augmenting path***.

11.2.0.21 More on augmenting paths

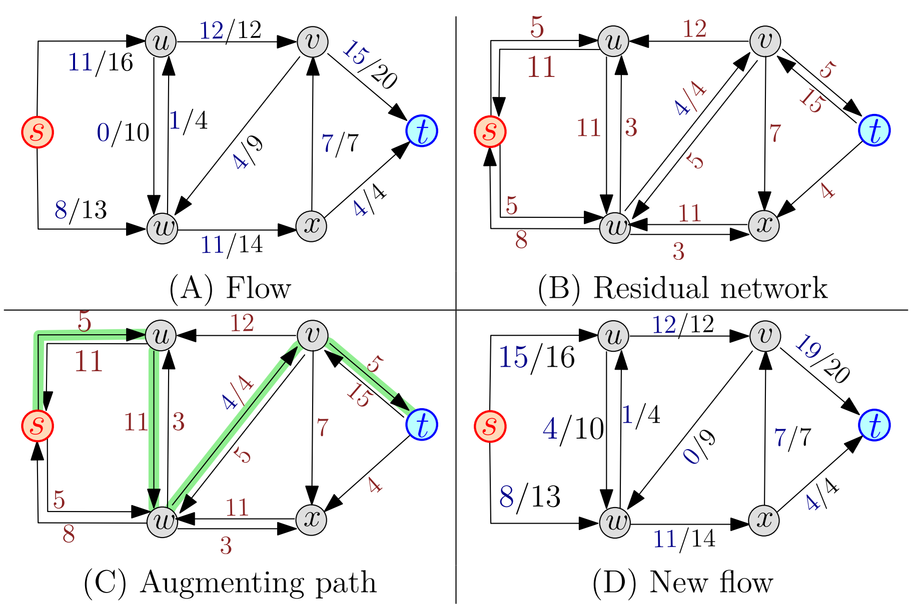
(A) π: augmenting path.
(B) All edges of π have positive capacity in G_f.
(C) ... otherwise not in E_f.
(D) f, π: can improve f by pushing positive flow along π.

11.2.0.22 Residual capacity

Definition 11.2.14. π: augmenting path of f.
$c_f(\pi)$: maximum amount of flow can push on π.
$c_f(\pi)$ is ***residual capacity*** of π.
Formally,

$$c_f(\pi) = \min_{(u \to v) \in \pi} c_f(u, v).$$

11.2.0.23 An example of an augmenting path



11.2.0.24 Flow along augmenting path

$$f_\pi(u, v) = \begin{cases} c_f(\pi) & \text{if } (u \to v) \text{ is in } \pi \\ -c_f(\pi) & \text{if } (v \to u) \text{ is in } \pi \\ 0 & \text{otherwise.} \end{cases}$$

11.2.0.25 Increase flow by augmenting flow

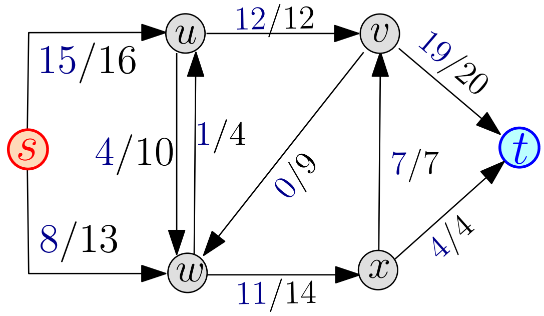
Lemma 11.2.15. *π: augmenting path. f_π is flow in G_f and $|f_\pi| = c_f(\pi) > 0$.*

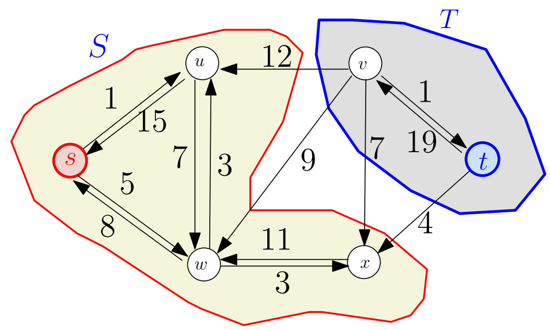
Get bigger flow...

Lemma 11.2.16. *Let f be a flow, and let π be an augmenting path for f. Then $f + f_\pi$ is a "better" flow. Namely, $|f + f_\pi| = |f| + |f_\pi| > |f|$.*

11.2.0.26 Flowing into the wall

(A) Namely, $f + f_\pi$ is flow with larger value than f.
(B) Can this flow be improved? Consider residual flow...





(C) s is disconnected from t in this residual network.
(D) unable to push more flow.
(E) Found local maximum!
(F) Is that a global maximum?
(G) Is this the maximum flow?

11.3 The Ford-Fulkerson method

11.3.0.27 The Ford-Fulkerson method

```
algFordFulkerson(G, c)
    begin
        f ← Zero flow on G
        while (G_f has augmenting
                        path p) do
            (* Recompute G_f for
               this check *)
            f ← f + f_p
        return f
    end
```

11.4 On maximum flows

11.4.0.28 Some definitions

Definition 11.4.1. (S, T): ***directed cut*** in flow network $\mathsf{G} = (\mathsf{V}, E)$.
A partition of V into S and $T = V \setminus S$, such that $s \in S$ and $t \in T$.

Definition 11.4.2. The net ***flow of f across a cut*** (S, T) is $f(S, T) = \sum_{s \in S, t \in T} f(s, t)$.

Definition 11.4.3. The ***capacity*** of (S, T) is $c(S, T) = \sum_{s \in S, t \in T} c(s, t)$.

Definition 11.4.4. The ***minimum cut*** is the cut in G with the minimum capacity.

11.4.0.29 Flow across cut is the whole flow

Lemma 11.4.5. G,f,s,t. (S, T): *cut of* G.
Then $f(S,T) = |f|$.

Proof:

$$\begin{aligned} f(S,T) &= f(S,\mathsf{V}) - f(S,S) = f(S,\mathsf{V}) \\ &= f(s,\mathsf{V}) + f(S-s,\mathsf{V}) = f(s,\mathsf{V}) \\ &= |f|, \end{aligned}$$

since $T = \mathsf{V} \setminus S$, and $f(S-s,\mathsf{V}) = \sum_{u \in S-s} f(u,\mathsf{V}) = 0$ (note that u can not be t as $t \in T$). ∎

11.4.0.30 Flow bounded by cut capacity

Claim 11.4.6. *The flow in a network is upper bounded by the capacity of any cut* (S,T) *in* G.

Proof: Consider a cut (S,T). We have $|f| = f(S,T) = \sum_{u \in S, v \in T} f(u,v) \le \sum_{u \in S, v \in T} c(u,v) = c(S,T)$. ∎

11.4.0.31 THE POINT

Key observation Maximum flow is bounded by the capacity of the minimum cut.
Surprisingly... Maximum flow is exactly the value of the minimum cut.

11.4.0.32 The Min-Cut Max-Flow Theorem

Theorem 11.4.7 (Max-flow min-cut theorem). *If f is a flow in a flow network* $\mathsf{G} = (\mathsf{V}, E)$ *with source s and sink t, then the following conditions are equivalent:*
(A) f is a maximum flow in G.
(B) The residual network G_f *contains no augmenting paths.*
(C) $|f| = c(S,T)$ *for some cut* (S,T) *of* G. *And* (S,T) *is a* minimum cut *in* G.

11.4.0.33 Proof: (A) ⇒ (B):

Proof: (A) ⇒ (B): By contradiction. If there was an augmenting path p then $c_f(p) > 0$, and we can generate a new flow $f + f_p$, such that $|f + f_p| = |f| + c_f(p) > |f|$. A contradiction as f is a maximum flow. ∎

11.4.0.34 Proof: (B) ⇒ (C):

Proof: s and t are disconnected in G_f.
Set $S = \left\{ v \,\middle|\, \text{Exists a path between } s \text{ and } v \text{ in } \mathsf{G}_f \right\}$ $\quad T = \mathsf{V} \setminus S$.
Have: $s \in S$, $t \in T$, $\forall u \in S$ and $\forall v \in T$: $f(u,v) = c(u,v)$.
By contradiction: $\exists u \in S$, $v \in T$ s.t. $f(u,v) < c(u,v) \implies (u \to v) \in \mathsf{E}_f \implies v$ would be reachable from s in G_f. Contradiction.
$\implies |f| = f(S,T) = c(S,T)$.
(S,T) must be mincut. Otherwise $\exists (S',T')$: $c(S',T') < c(S,T) = f(S,T) = |f|$,
But... $|f| = f(S',T') \le c(S',T')$. A contradiction. ∎

11.4.0.35 Proof: (C) ⇒ (A):

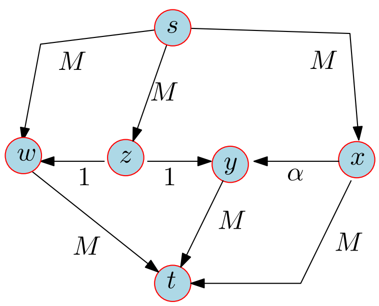
Proof: Well, for any cut (U, V), we know that $|f| \leq c(U, \mathsf{V})$. This implies that if $|f| = c(S, T)$ then the flow can not be any larger, and it is thus a maximum flow. ■

11.4.0.36 Implications

(A) The max-flow min-cut theorem $\implies$ if **algFordFulkerson** terminates, then computed max flow.
(B) Does not imply **algFordFulkerson** always terminates.
(C) **algFordFulkerson** might not terminate.

11.5 Non-termination of Ford-Fulkerson

11.5.0.37 Ford-Fulkerson runs in vain



(A) M: large positive integer.
(B) $\alpha = (\sqrt{5} - 1)/2 \approx 0.618$.
(C) $\alpha < 1$,
(D) $1 - \alpha < \alpha$.
(E) Maximum flow in this network is: $2M + 1$.

11.5.0.38 Some algebra...

For $\alpha = \dfrac{\sqrt{5}-1}{2}$:

$$
\begin{aligned}
\alpha^2 &= \left(\frac{\sqrt{5}-1}{2}\right)^2 = \frac{1}{4}\left(\sqrt{5}-1\right)^2 = \frac{1}{4}\left(5 - 2\sqrt{5} + 1\right) \\
&= 1 + \frac{1}{4}\left(2 - 2\sqrt{5}\right) \\
&= 1 + \frac{1}{2}\left(1 - \sqrt{5}\right) \\
&= 1 - \frac{\sqrt{5}-1}{2} \\
&= 1 - \alpha.
\end{aligned}
$$

11.5.0.39 Some algebra...

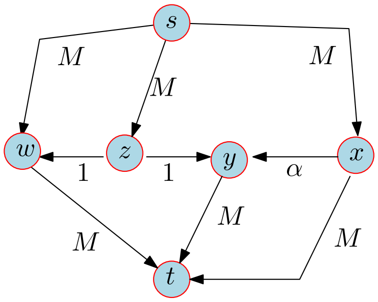
Claim 11.5.1. *Given:* $\alpha = (\sqrt{5} - 1)/2$ *and* $\alpha^2 = 1 - \alpha$.

$$\implies \forall i \qquad \alpha^i - \alpha^{i+1} = \alpha^{i+2}$$

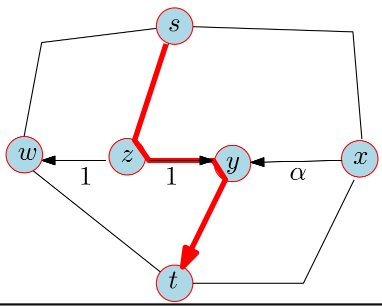
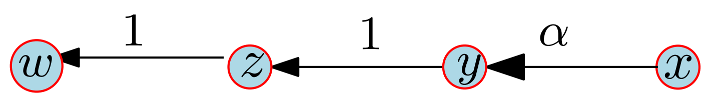
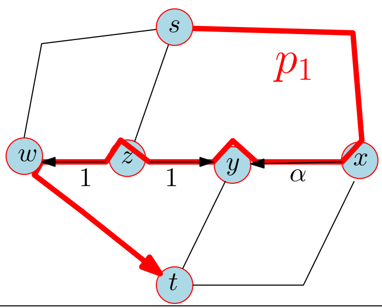
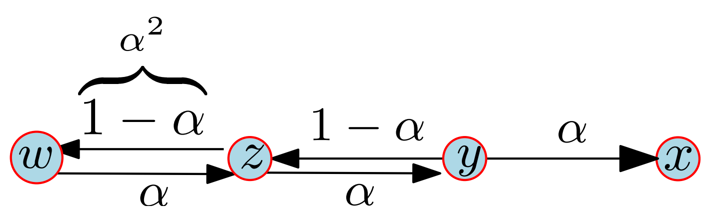
Proof:

$$\alpha^i - \alpha^{i+1} = \alpha^i(1 - \alpha) = \alpha^i \alpha^2 = \alpha^{i+2}.$$

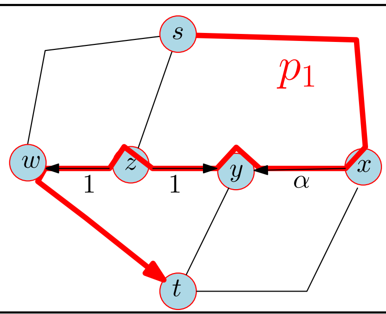
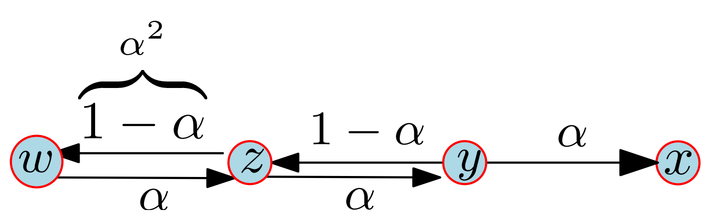
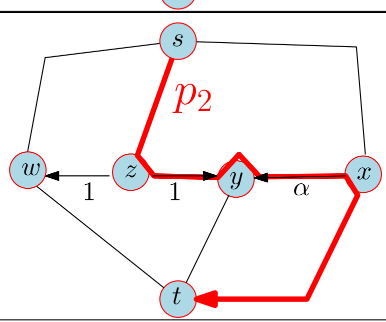
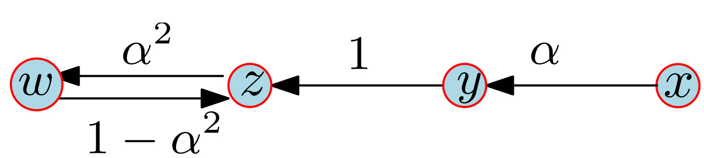
11.5.0.40 The network



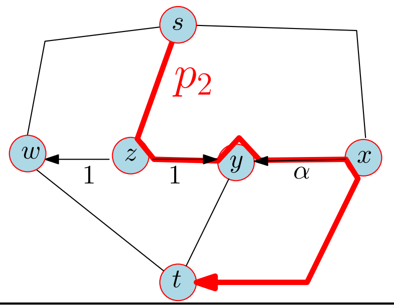
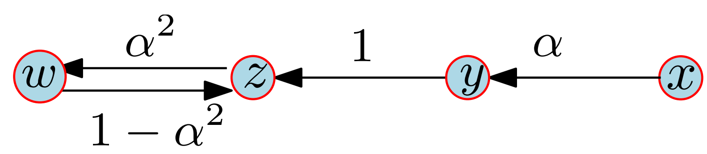
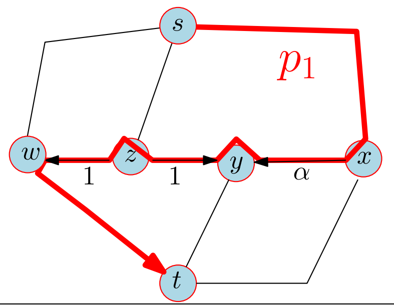
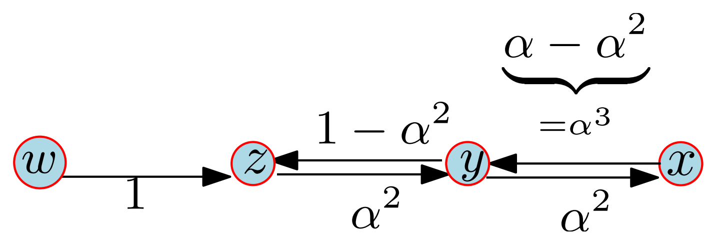
11.5.0.41 Let it flow...

#	Augment. path π	c_π	New residual network
0.		1	
1.		α	

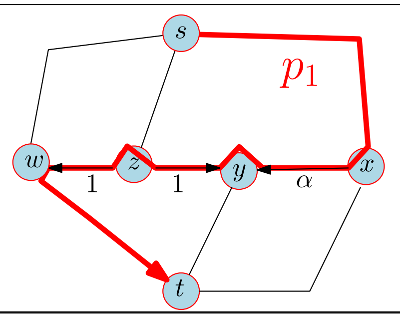
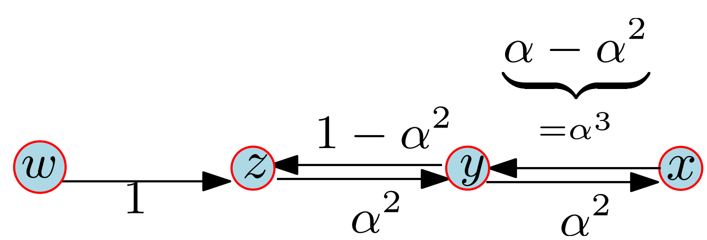
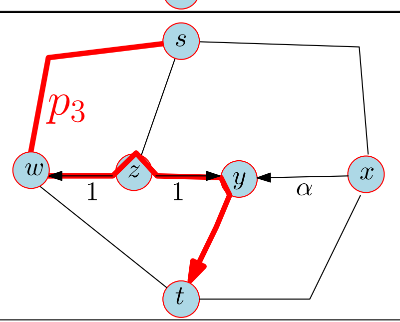
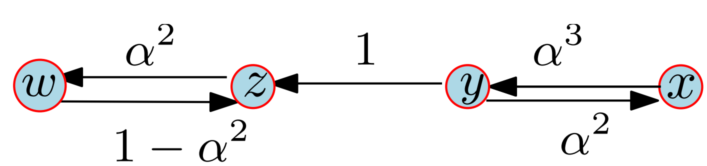
11.5.0.42 Let it flow II

#	Augment. path π	c_π	New residual network
1.		α	
2.		α	

11.5.0.43 Let it flow II

2.		α^2	
3.		α^2	

11.5.0.44 Let it flow III

3.		α^2	
4.		α^2	

11.5.0.45 Let it flow III

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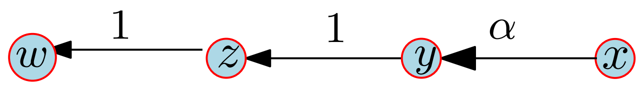
<tr><th>moves</th><th>Residual network after</th></tr>
<tr><td>0</td><td>
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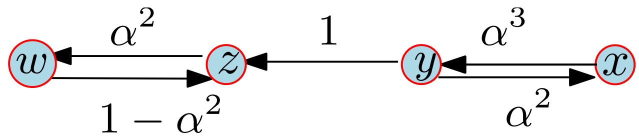
<tr><td>moves $0, (1, 2, 3, 4)$</td><td>
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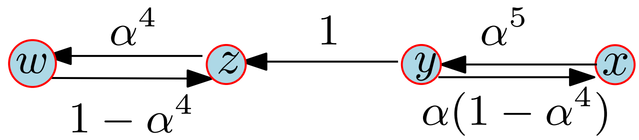
<tr><td>moves $0, (1, 2, 3, 4)^2$</td><td>
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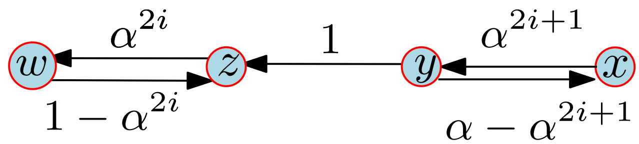
<tr><td>$0.(1, 2, 3, 4)^i$</td><td>
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Namely, the algorithm never terminates.