CS 573: Algorithms, Fall 2014

Network Flow

Lecture 11

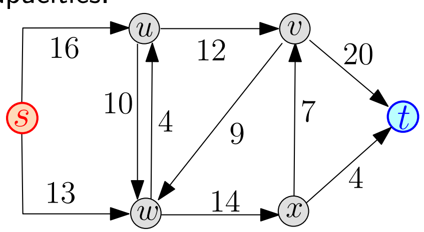
September 30, 2014

Part I

Network Flow

Network flow

1. Transfer as much "merchandise" as possible from one point to another.
2. Wireless network, transfer a large file from s to t.
3. Limited capacities.



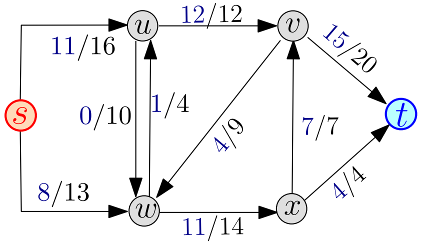
Network: Definition

1. Given a network with capacities on each connection.
2. Q: How much "flow" can transfer from source s to a sink t?
3. The flow is ***splitable***.
4. Network examples: water pipes moving water. Electricity network.
5. Internet is packet base, so not quite splitable.

Definition

- ⋆ $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: a ***directed*** graph.
- ⋆ $\forall (u \to v) \in \mathbf{E}(\mathbf{G})$: ***capacity*** $c(u, v) \geq 0$,
- ⋆ $(u \to v) \notin G \implies c(u, v) = 0$.
- ⋆ s: ***source*** vertex, t: target ***sink*** vertex.
- ⋆ $\mathbf{G}$, s, t and $c(\cdot)$: form ***flow network*** or ***network***.

Network Example



1. All flow from the source ends up in the sink.
2. Flow on edge: non-negative quantity $\leq$ capacity of edge.

Flow definition

Definition (flow)

flow in network is a function $f(\cdot,\cdot) : E(G) \to \mathbb{R}$:

(A) ***Bounded by capacity***:
$\forall (u \to v) \in E \quad f(u,v) \leq c(u,v)$.

(B) ***Anti symmetry***:
$\forall u, v \quad f(u,v) = -f(v,u)$.

(C) Two special vertices: (i) the ***source*** s and the ***sink*** t.

(D) ***Conservation of flow*** (Kirchhoff's Current Law):
$\forall u \in V \setminus \{s,t\} \quad \sum_{v} f(u,v) = 0$.

flow/value of f: $|f| = \sum_{v \in V} f(s,v)$.

Problem: Max Flow

1. Flow on edge can be negative (i.e., positive flow on edge in other direction).

Problem (Maximum flow)

Given a network $\mathbf{G}$ *find the* ***maximum flow*** *in* $\mathbf{G}$*. Namely, compute a legal flow* f *such that* $|f|$ *is maximized.*

Part II

Some properties of flows and residual networks

Flow across sets of vertices

1. $\forall \mathbf{X}, \mathbf{Y} \subseteq \mathbf{V}$, let $f(\mathbf{X}, \mathbf{Y}) = \sum_{x \in \mathbf{X}, y \in \mathbf{Y}} f(x, y)$.
   $f(v, S) = f\left(\{v\}, S\right)$, where $v \in \mathbf{V}(\mathbf{G})$.

Observation

$|f| = f(s, \mathbf{V})$.

Basic properties of flows: (i)

Lemma

For a flow f, the following properties holds:
(i) $\forall u \in \mathbf{V}(\mathbf{G})$ we have $f(u, u) = 0$,

Proof.

Holds since $(u \to u)$ it not an edge in $\mathbf{G}$.
$(u \to u)$ capacity is zero,
Flow on $(u \to u)$ is zero. □

Basic properties of flows: (ii)

Lemma

For a flow f, the following properties holds:
(ii) $\forall \mathbf{X} \subseteq \mathbf{V}$ we have $f(\mathbf{X}, \mathbf{X}) = 0$,

Proof.

$$f(X, X) = \sum_{\{u,v\} \subseteq X, u \neq v} \left(f(u, v) + f(v, u)\right) + \sum_{u \in X} f(u, u)$$
$$= \sum_{\{u,v\} \subseteq X, u \neq v} \left(f(u, v) - f(u, v)\right) + \sum_{u \in X} 0 = 0,$$

by the anti-symmetry property of flow. □

Basic properties of flows: (iii)

Lemma

For a flow f, the following properties holds:
(iii) $\forall \mathbf{X}, \mathbf{Y} \subseteq \mathbf{V}$ we have $f(\mathbf{X}, \mathbf{Y}) = -f(\mathbf{Y}, \mathbf{X})$,

Proof.

By the anti-symmetry of flow, as

$$f(X, Y) = \sum_{x \in X, y \in Y} f(x, y) = -\sum_{x \in X, y \in Y} f(y, x) = -f(Y, X).$$

□

Basic properties of flows: (iv)

Lemma

For a flow f, the following properties holds:

(iv) $\forall X, Y, Z \subseteq V$ *such that* $X \cap Y = \emptyset$ *we have that*
$f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ *and*
$f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$.

Proof.

Follows from definition. (Check!) □

Basic properties of flows: (v)

Lemma

For a flow f, the following properties holds:

(v) $\forall u \in V \setminus \{s, t\}$, *we have* $f(u, V) = f(V, u) = 0$.

Proof.

This is a restatement of the conservation of flow property. □

Basic properties of flows: summary

Lemma

For a flow f, the following properties holds:

(i) $\forall u \in V(G)$ *we have* $f(u, u) = 0$,

(ii) $\forall X \subseteq V$ *we have* $f(X, X) = 0$,

(iii) $\forall X, Y \subseteq V$ *we have* $f(X, Y) = -f(Y, X)$,

(iv) $\forall X, Y, Z \subseteq V$ *such that* $X \cap Y = \emptyset$ *we have that*
$f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ *and*
$f(Z, X \cup Y) = f(Z, X) + f(Z, Y)$.

(v) *For all* $u \in V \setminus \{s, t\}$, *we have*
$f(u, V) = f(V, u) = 0$.

All flow gets to the sink

Claim

$|f| = f(V, t)$.

Proof.

$$
\begin{aligned}
|f| &= f(s, V) = f\Big(V \setminus (V \setminus \{s\}), V\Big) \\
&= f(V, V) - f(V \setminus \{s\}, V) \\
&= -f(V \setminus \{s\}, V) \qquad\qquad = f(V, V \setminus \{s\}) \\
&= f(V, t) + f(V, V \setminus \{s, t\}) \\
&= f(V, t) + \sum_{u \in V \setminus \{s,t\}} f(V, u) \\
&= f(V, t) + \sum_{u \in V \setminus \{s,t\}} 0 \\
&= f(V, t),
\end{aligned}
$$

Since $f(V, V) = 0$ by (i) and $f(V, u) = 0$ by (iv). □

Residual capacity

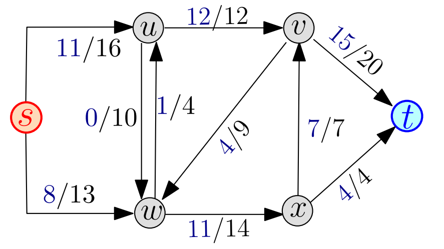
Definition

c: capacity, f: flow.
The ***residual capacity*** of an edge $(u \to v)$ is

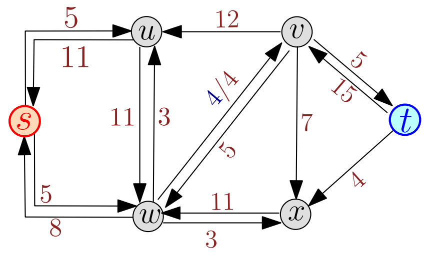
$$c_f(u, v) = c(u, v) - f(u, v).$$

1. residual capacity $c_f(u, v)$ on $(u \to v)$ = amount of unused capacity on $(u \to v)$.
2. ... next construct graph with all edges not being fully used by f.

Residual graph



Graph



Residual graph

$f(u, w) = -f(w, u) = -1 \implies$
$c_f(u, w) = 10 - (-1) = 11$.

Residual graph: Definition

Definition

Given f, $G = (V, E)$ and c, as above, the ***residual graph*** (or ***residual network***) of G and f is the graph $G_f = (V, E_f)$ where

$$E_f = \left\{ (u, v) \in V \times V \;\middle|\; c_f(u, v) > 0 \right\}.$$

1. $(u \to v) \in E$: might induce two edges in E_f
2. If $(u \to v) \in E$, $f(u, v) < c(u, v)$ and $(v \to u) \notin E(G)$
3. $\implies c_f(u, v) = c(u, v) - f(u, v) > 0$
4. ... and $(u \to v) \in E_f$. Also,

   $$c_f(v, u) = c(v, u) - f(v, u) = 0 - (-f(u, v)) = f(u, v),$$

   since $c(v, u) = 0$ as $(v \to u)$ is not an edge of G.
5. $\implies (v \to u) \in E_f$.

Residual network properties

Since every edge of G induces at most two edges in G_f, it follows that G_f has at most twice the number of edges of G; formally, $|E_f| \leq 2\,|E|$.

Lemma

Given a flow f defined over a network G, then the residual network G_f together with c_f form a flow network.

Proof.

One need to verify that $c_f(\cdot)$ is always a non-negative function, which is true by the definition of E_f. □

Increasing the flow

Lemma

***G(V, E)**, a flow **f**, and **h** a flow in G_f. G_f: residual network of **f**.*

*Then $f + h$ is a flow in **G** and its capacity is $|f + h| = |f| + |h|$.*

proof

By definition: $(f + h)(u, v) = f(u, v) + h(u, v)$ and thus $(f + h)(X, Y) = f(X, Y) + h(X, Y)$. Verify legal...

1. Anti symmetry: $(f + h)(u, v) = f(u, v) + h(u, v) = -f(v, u) - h(v, u) = -(f + h)(v, u)$.
2. Bounded by capacity:

$$
\begin{aligned}
(f + h)(u, v) &\le f(u, v) + h(u, v) \le f(u, v) + c_f(u, v) \\
&= f(u, v) + (c(u, v) - f(u, v)) = c(u, v).
\end{aligned}
$$

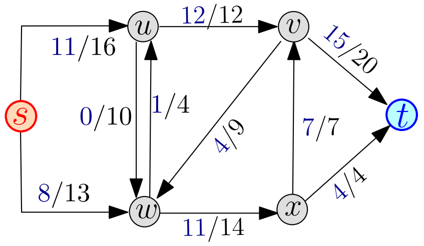
Increasing the flow – proof continued

proof continued

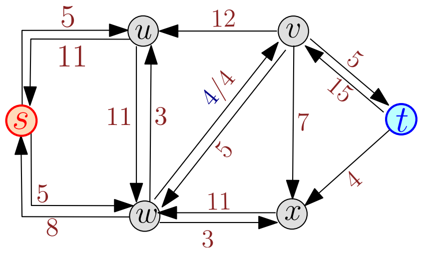
1. For $u \in V - s - t$ we have $(f + h)(u, V) = f(u, V) + h(u, V) = 0 + 0 = 0$ and as such $f + h$ comply with the conservation of flow requirement.
2. Total flow is

$$
|f + h| = (f + h)(s, V) = f(s, V) + h(s, V) = |f| + |h| .
$$

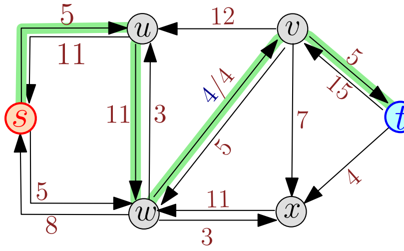
Augmenting path



Graph



Residual graph



Definition

For **G** and a flow f, a path π in G_f between s and t is an ***augmenting path***.

More on augmenting paths

1. π: augmenting path.
2. All edges of π have positive capacity in G_f.
3. ... otherwise not in E_f.
4. f, π: can improve f by pushing positive flow along π.

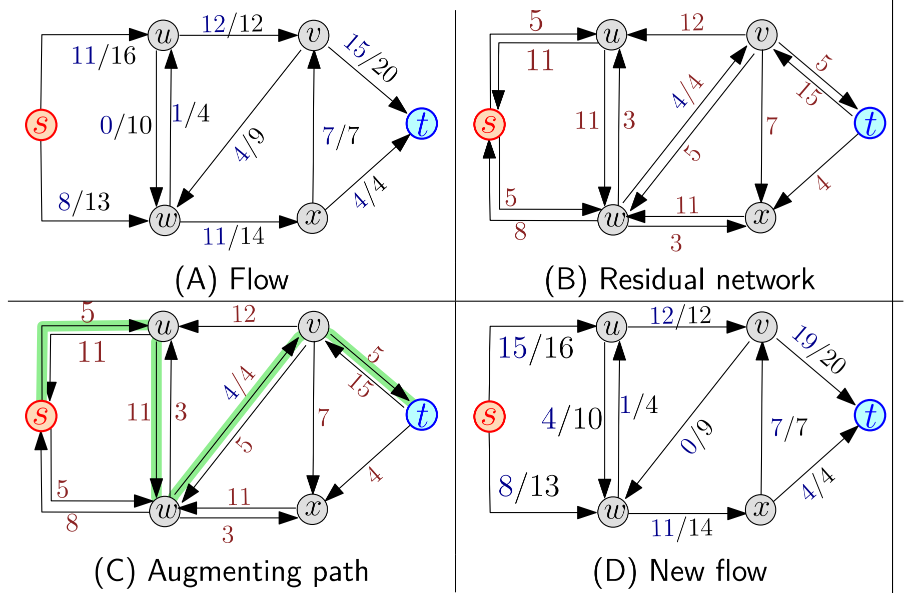
Residual capacity

Definition

π: augmenting path of f.
$c_f(\pi)$: maximum amount of flow can push on π.
$c_f(\pi)$ is ***residual capacity*** of π.
Formally,

$$c_f(\pi) = \min_{(u \to v) \in \pi} c_f(u, v).$$

An example of an augmenting path



Flow along augmenting path

$$f_\pi(u, v) = \begin{cases} c_f(\pi) & \text{if } (u \to v) \text{ is in } \pi \\ -c_f(\pi) & \text{if } (v \to u) \text{ is in } \pi \\ 0 & \text{otherwise.} \end{cases}$$

Increase flow by augmenting flow

Lemma

π: augmenting path. f_π is flow in G_f and $|f_\pi| = c_f(\pi) > 0$.

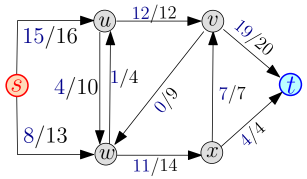
Get bigger flow...

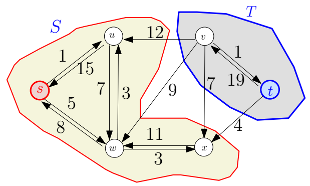
Lemma

Let f be a flow, and let π be an augmenting path for f. Then $f + f_\pi$ is a "better" flow. Namely,
$|f + f_\pi| = |f| + |f_\pi| > |f|$.

Flowing into the wall

1. Namely, $f + f_\pi$ is flow with larger value than f.
2. Can this flow be improved? Consider residual flow...





3. s is disconnected from t in this residual network.
4. unable to push more flow.
5. Found local maximum!
6. Is that a global maximum?
7. Is this the maximum flow?

The Ford-Fulkerson method

```
algFordFulkerson(G, c)
    begin
        f ← Zero flow on G
        while (G_f has augmenting
                        path p) do
            (* Recompute G_f for
               this check *)
            f ← f + f_p
        return f
    end
```


Part III

On maximum flows

Some definitions

Definition

(S, T): ***directed cut*** in flow network $G = (V, E)$.
A partition of V into S and $T = V \setminus S$, such that $s \in S$ and $t \in T$.

Definition

The net ***flow of f across a cut*** (S, T) is
$f(S, T) = \sum_{s \in S, t \in T} f(s, t)$.

Definition

The ***capacity*** of (S, T) is $c(S, T) = \sum_{s \in S, t \in T} c(s, t)$.

Definition

The ***minimum cut*** is the cut in G with the minimum capacity.

Flow across cut is the whole flow

Lemma

$\mathbf{G}, f, s, t$. (S, T): *cut of* $\mathbf{G}$.
Then $f(S, T) = |f|$.

Proof.

$$f(S, T) = f(S, \mathbf{V}) - f(S, S) = f(S, \mathbf{V})$$
$$= f(s, \mathbf{V}) + f(S - s, \mathbf{V}) = f(s, \mathbf{V})$$
$$= |f|,$$

since $T = \mathbf{V} \setminus S$, and $f(S - s, \mathbf{V}) = \sum_{u \in S - s} f(u, \mathbf{V}) = 0$ (note that u can not be t as $t \in T$). □

Flow bounded by cut capacity

Claim

The flow in a network is upper bounded by the capacity of any cut (S, T) *in* $\mathbf{G}$.

Proof.

Consider a cut (S, T). We have $|f| = f(S, T) = \sum_{u \in S, v \in T} f(u, v) \leq \sum_{u \in S, v \in T} c(u, v) = c(S, T)$. □

THE POINT

Key observation

Maximum flow is bounded by the capacity of the minimum cut.

Surprisingly...

Maximum flow is exactly the value of the minimum cut.

The Min-Cut Max-Flow Theorem

Theorem (Max-flow min-cut theorem)

If f *is a flow in a flow network* $\mathbf{G} = (\mathbf{V}, E)$ *with source* s *and sink* t*, then the following conditions are equivalent:*

- *(A)* f *is a maximum flow in* $\mathbf{G}$.
- *(B) The residual network* $\mathbf{G}_f$ *contains no augmenting paths.*
- *(C)* $|f| = c(S, T)$ *for some cut* (S, T) *of* $\mathbf{G}$. *And* (S, T) *is a minimum cut in* $\mathbf{G}$.

Proof: (A) $\Rightarrow$ (B):

Proof.

(A) $\Rightarrow$ (B): By contradiction. If there was an augmenting path p then $c_f(p) > 0$, and we can generate a new flow $f + f_p$, such that $|f + f_p| = |f| + c_f(p) > |f|$. A contradiction as f is a maximum flow. □

Proof: (B) $\Rightarrow$ (C):

Proof.

s and t are disconnected in $\mathbf{G}_f$.

Set $S = \left\{ v \,\middle|\, \text{Exists a path between } s \text{ and } v \text{ in } \mathbf{G}_f \right\}$ $\quad T = \mathbf{V} \setminus S$.

Have: $s \in S$, $t \in T$, $\forall u \in S$ and $\forall v \in T$: $f(u, v) = c(u, v)$.

By contradiction: $\exists u \in S$, $v \in T$ s.t. $f(u, v) < c(u, v)$ $\implies (u \rightarrow v) \in \mathbf{E}_f \implies v$ would be reachable from s in $\mathbf{G}_f$. Contradiction.

$\implies |f| = f(S, T) = c(S, T)$.

(S, T) must be mincut. Otherwise $\exists(S', T')$:

$c(S', T') < c(S, T) = f(S, T) = |f|$,

But... $|f| = f(S', T') \leq c(S', T')$. A contradiction. □

Proof: (C) $\Rightarrow$ (A):

Proof.

Well, for any cut $(U, \mathbf{V})$, we know that $|f| \leq c(U, \mathbf{V})$. This implies that if $|f| = c(S, T)$ then the flow can not be any larger, and it is thus a maximum flow. □

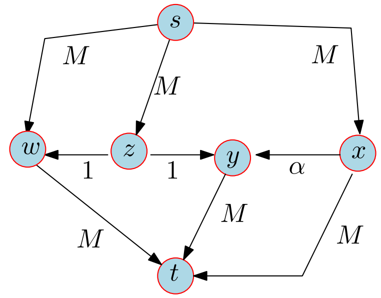
Implications

1. The max-flow min-cut theorem $\implies$ if **algFordFulkerson** terminates, then computed max flow.
2. Does not imply **algFordFulkerson** always terminates.
3. **algFordFulkerson** might not terminate.

Part IV

Non-termination of Ford-Fulkerson

Ford-Fulkerson runs in vain



1. M: large positive integer.
2. $\alpha = (\sqrt{5} - 1)/2 \approx 0.618$.
3. $\alpha < 1$,
4. $1 - \alpha < \alpha$.
5. Maximum flow in this network is: $2M + 1$.

Some algebra...

For $\alpha = \dfrac{\sqrt{5} - 1}{2}$:

$$\alpha^2 = \left(\frac{\sqrt{5}-1}{2}\right)^2 = \frac{1}{4}\left(\sqrt{5}-1\right)^2 = \frac{1}{4}\left(5 - 2\sqrt{5} + 1\right)$$
$$= 1 + \frac{1}{4}\left(2 - 2\sqrt{5}\right)$$
$$= 1 + \frac{1}{2}\left(1 - \sqrt{5}\right)$$
$$= 1 - \frac{\sqrt{5}-1}{2}$$
$$= 1 - \alpha.$$

Some algebra...

Claim

Given: $\alpha = (\sqrt{5} - 1)/2$ *and* $\alpha^2 = 1 - \alpha$.

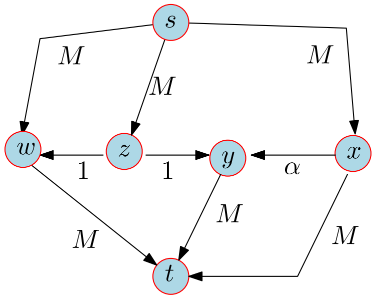
$$\implies \forall i \qquad \alpha^i - \alpha^{i+1} = \alpha^{i+2}$$

Proof.

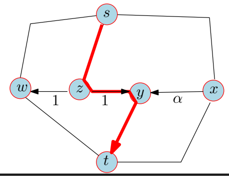
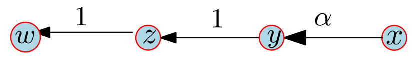
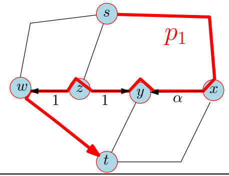
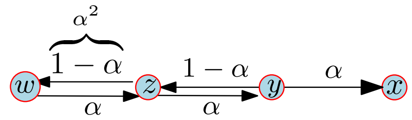
$$\alpha^i - \alpha^{i+1} = \alpha^i(1-\alpha) = \alpha^i\alpha^2 = \alpha^{i+2}.$$

□

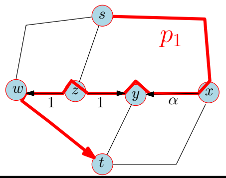
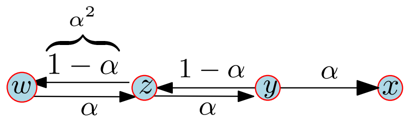
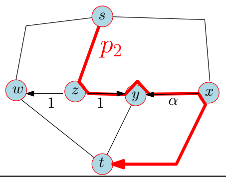
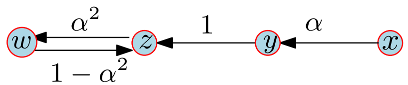
The network



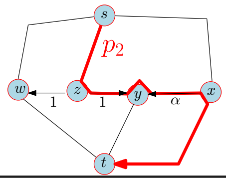
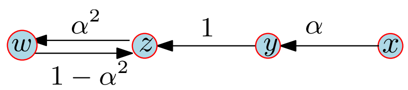
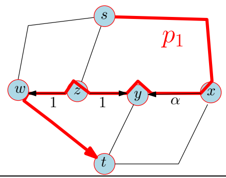
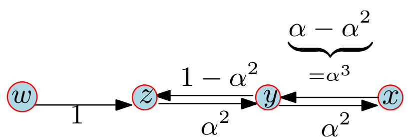
Let it flow...

#	Augment. path π	c_π	New residual network
0.		$\mathbf{1}$	
1.		$\boldsymbol{\alpha}$	

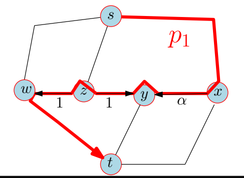
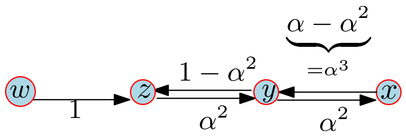
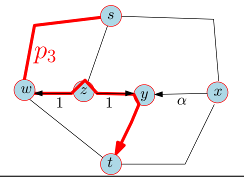
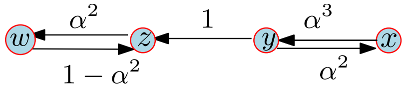
Let it flow II

#	Augment. path π	c_π	New residual network
1.		$\boldsymbol{\alpha}$	
2.		$\boldsymbol{\alpha}$	

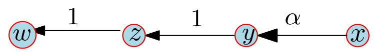
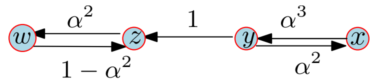
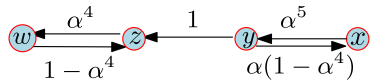
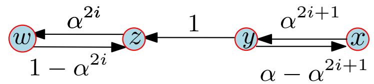
Let it flow II

2.		$\boldsymbol{\alpha}^2$	
3.		$\boldsymbol{\alpha}^2$	

Let it flow III

3.		α^2	
4.		α^2	

Let it flow III

moves	Residual network after
0	
moves $\mathbf{0, (1, 2, 3, 4)}$	
moves $\mathbf{0, (1, 2, 3, 4)^2}$	
$\mathbf{0.(1, 2, 3, 4)^i}$	

Namely, the algorithm never terminates.