Randomized Algorithms II – High Probability

Lecture 10

September 25, 2014

Part I

Movie...

Part II

Understanding the binomial distribution

Binomial distribution

X_n = numbers of heads when flipping a coin n times.

Claim

$\Pr\left[X_n = i\right] = \frac{\binom{n}{i}}{2^n}$.

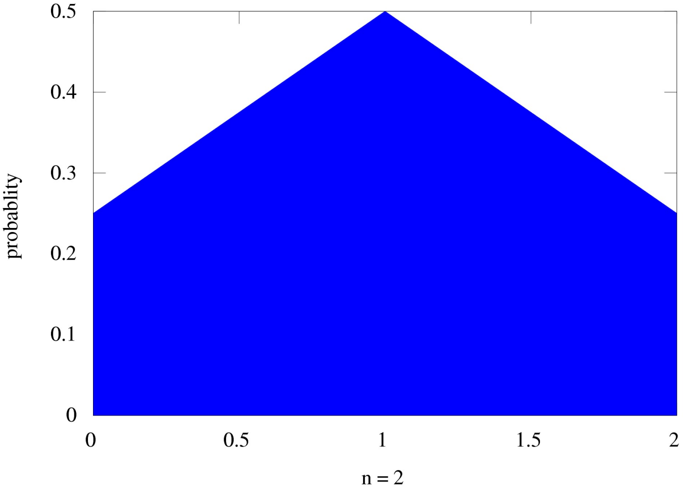
Where: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

Indeed, $\binom{n}{i}$ is the number of ways to choose i elements out of n elements (i.e., pick which i coin flip come up heads).

Each specific such possibility (say $0100010\ldots$) had probability $1/2^n$.

Massive randomness.. Is not that random.

Consider flipping a fair coin n times independently, head given 1, tail gives zero. How many heads? ...we get a binomial distribution.



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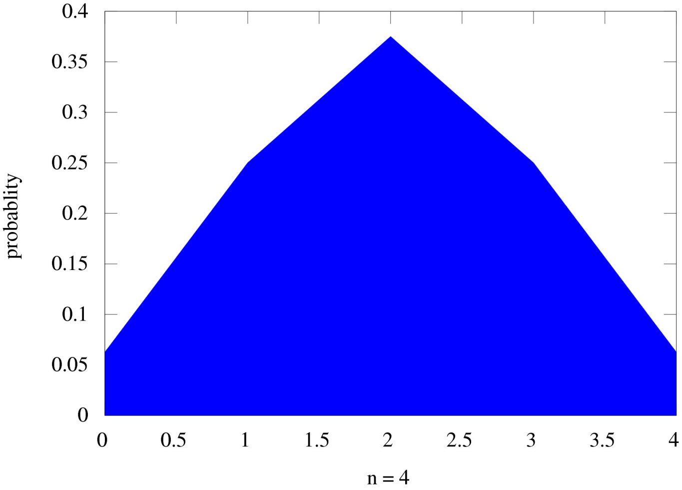

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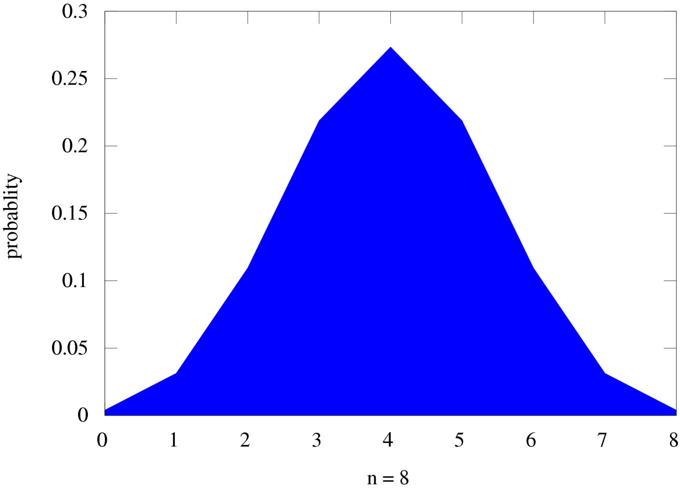

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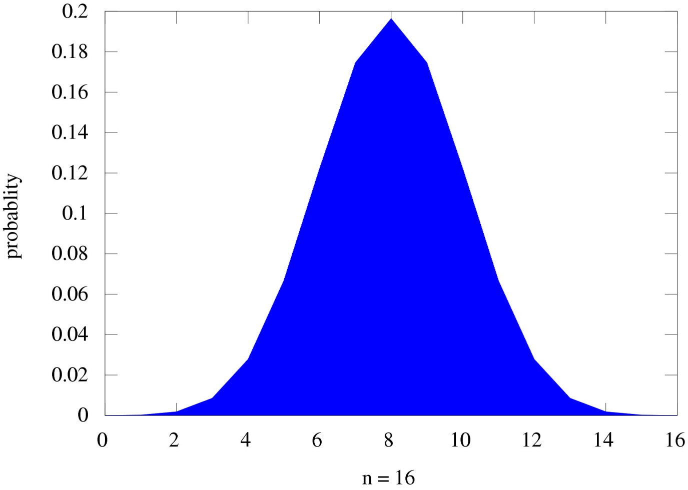

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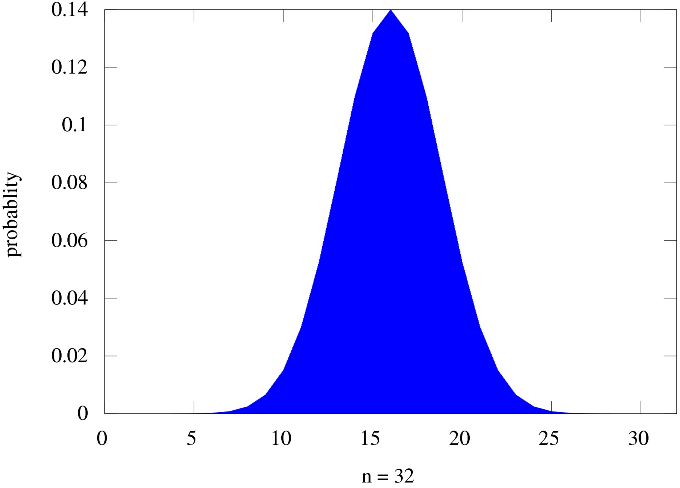

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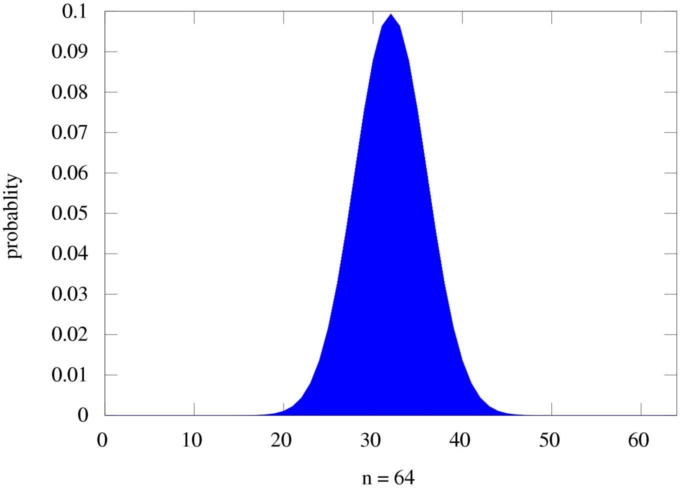

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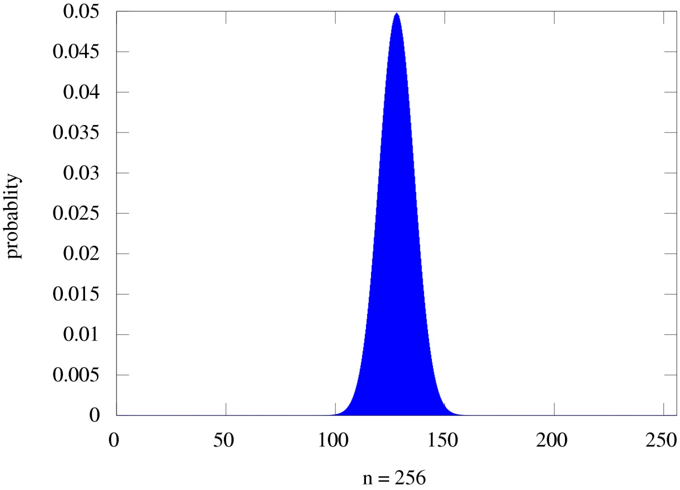

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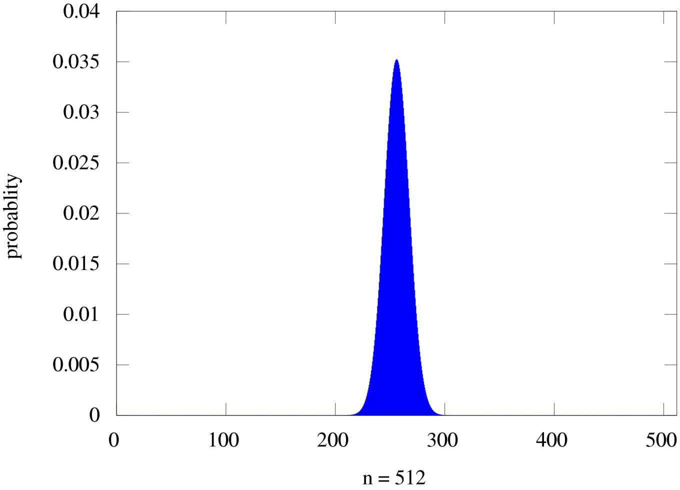

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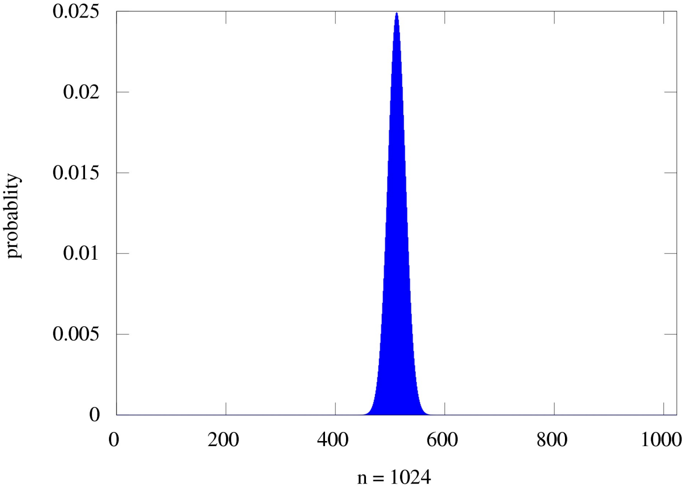

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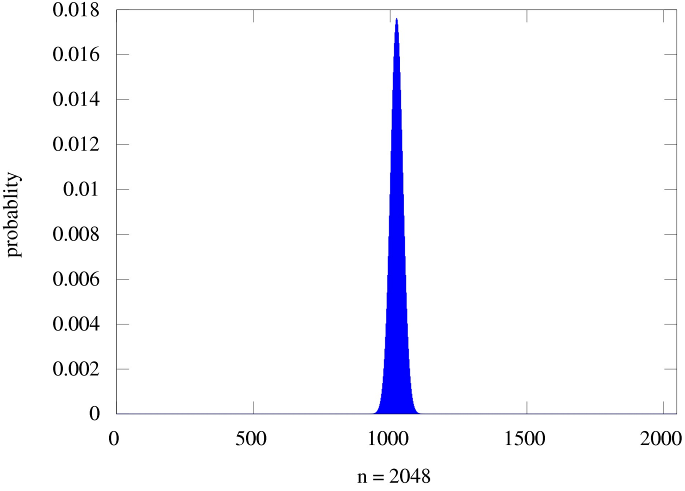

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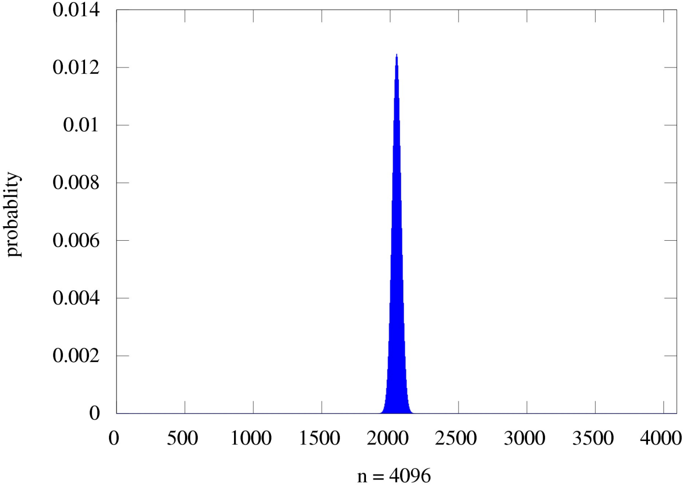

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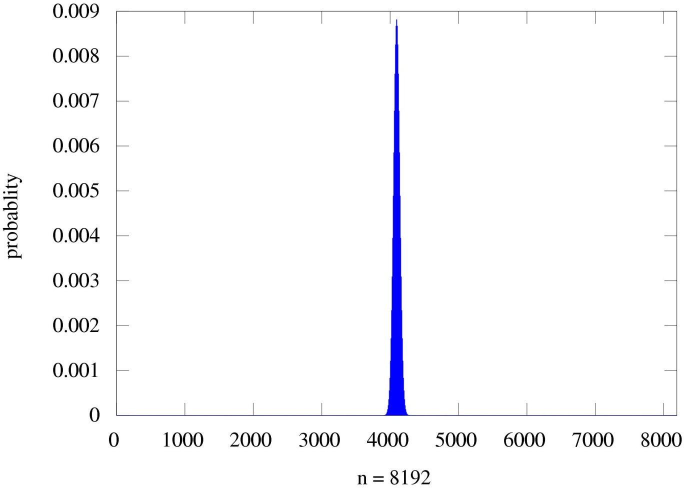

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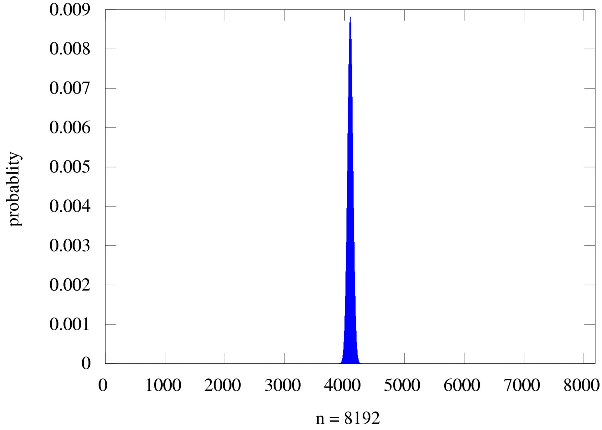


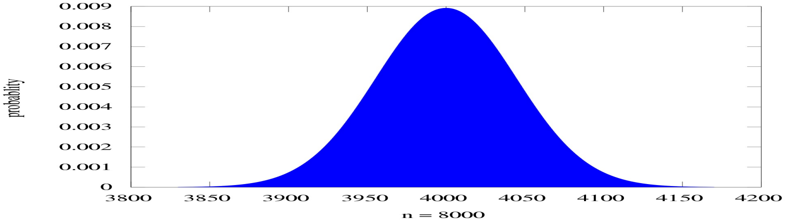
This is known as ***concentration of mass***.
This is a very special case of the ***law of large numbers***.

Side note...

Law of large numbers (weakest form)...

Informal statement of law of large numbers

For n large enough, the middle portion of the binomial distribution looks like (converges to) the normal/Gaussian distribution.



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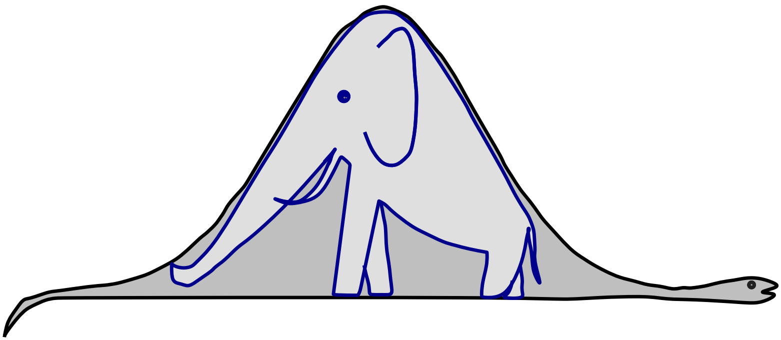
Intuitive conclusion

Randomized algorithm are unpredictable in the tactical level, but very predictable in the strategic level.

What is really hiding below the Normal distribution?



Taken from **?**.

Part III

QuickSort with high probability

Show that QuickSort running time is $O(n \log n)$

1. **QuickSort** picks a pivot, splits into two subproblems, and continues recursively.
2. Track single element in input.
3. Game ends, when this element is alone in subproblem.
4. Show every element in input, participates $\leq 32 \ln n$ rounds (with high enough probability).
5. $\mathcal{E}_i$: event ith element participates $> 32 \ln n$ rounds.
6. C_{QS}: number of comparisons performed by **QuickSort**.
7. Running time $O(C_{QS})$.
8. Probability of failure is $\alpha = \Pr\left[C_{QS} \geq 32n \ln n\right] \leq \Pr[\cup_i \mathcal{E}_i] \leq \sum_{i=1}^{n} \Pr\left[\mathcal{E}_i\right]$.
   ... by the union bound.

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Proving that an element...

... participates in small number of rounds.

1. n: number of elements in input for **QuickSort**.
2. x: Arbitrary element x in input.
3. S_1: Input.
4. S_i: input to ith level recursive call that include x.
5. x ***lucky*** in jth iteration, if balanced split...
   $|S_{j+1}| \leq (3/4)|S_j|$ and $|S_j \setminus S_{j+1}| \leq (3/4)|S_j|$
6. $Y_j = 1 \iff x$ lucky in jth iteration.
7. $\Pr\left[Y_j\right] = \frac{1}{2}$.
8. Observation: $Y_1, Y_2, \ldots, Y_m$ are independent variables.
9. x can participate $\leq \rho = \log_{4/3} n \leq 3.5 \ln n$ rounds.
10. ...since $|S_j| \leq n(3/4)^{\#\text{ of lucky iteration in } 1\ldots j}$.
11. If ρ lucky rounds in first k rounds $\implies |S_k| \leq (3/4)^{\rho} n \leq 1$.

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2. Q: How many rounds x participates in $=$ how many coin flips till one gets ρ heads?
3. A: In expectation, 2ρ times.

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In M coin flips: $\Pr[\#\ \textit{heads} \leq M/4] \leq \exp(-M/8)$.

2. Set $M = 32 \ln n \geq 8\rho$.
3. $\Pr[Y_j = 0] = \Pr[Y_j = 1] = 1/2$.
4. $Y_1, Y_2, \ldots, Y_M$ are independent.
5. $\implies$ probability $\leq \rho \leq M/4$ ones in $Y_1, \ldots, Y_M$ is

$$\leq \exp\left(-\frac{M}{8}\right) \leq \exp(-\rho) \leq \frac{1}{n^3}.$$

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1. n input elements. Probability depth of recursion in **QuickSort** $> 32 \ln n$ is $\leq (1/n^3) * n = 1/n^2$.
2. Result:

Theorem

With high probability (i.e., $1 - 1/n^2$) the depth of the recursion of **QuickSort** *is $\leq 32 \ln n$. Thus, with high probability, the running time of* **QuickSort** *is $O(n \log n)$.*

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Alternative proof of high probability of **QuickSort**

1. T: n items to be sorted.
2. $t \in T$: element.
3. X_i: the size of subproblem in ith level of recursion containing t.
4. $X_0 = n$, and $\mathbf{E}\left[X_i \,\middle|\, X_{i-1}\right] \leq \frac{1}{2}\frac{3}{4}X_{i-1} + \frac{1}{2}X_{i-1} \leq \frac{7}{8}X_{i-1}$.
5. $\forall$ random variables $\mathbf{E}\left[X\right] = \mathbf{E}_y\left[\mathbf{E}\left[X \,\middle|\, Y = y\right]\right]$.
6. $\mathbf{E}\left[X_i\right] = \mathbf{E}_y\left[\mathbf{E}\left[X_i \,\middle|\, X_{i-1} = y\right]\right] \leq \mathbf{E}_{X_{i-1}=y}\left[\frac{7}{8}y\right] = \frac{7}{8}\mathbf{E}\left[X_{i-1}\right] \leq \left(\frac{7}{8}\right)^i \mathbf{E}[X_0] = \left(\frac{7}{8}\right)^i n$.

Alternative proof of high probability of **QuickSort**

1. $M = 8\log_{8/7} n$: $\mu = \mathbf{E}\left[X_M\right] \leq \left(\frac{7}{8}\right)^M n \leq \frac{1}{n^8} n = \frac{1}{n^7}$.
2. Markov's Inequality: For a non-negative variable X, and $t > 0$, we have:

$$\Pr\left[X \geq t\right] \quad \leq \quad \frac{\mathbf{E}[X]}{t}.$$

3. By Markov's inequality:

$$\Pr\left[\begin{array}{c} t \text{ participates} \\ > M \text{ recursive calls} \end{array}\right] \leq \Pr\left[X_M \geq 1\right] \leq \frac{\mathbf{E}[X_M]}{1} \leq \frac{1}{n^7}.$$

4. Probability any element of input participates $> M$ recursive calls $\leq n(1/n^7) \leq 1/n^6$.

Part IV

Chernoff inequality

Preliminaries

1. X, Y: Random variables are *independent* if $\forall x, y$:

$$\Pr\Big[(X = x) \cap (Y = y)\Big] = \Pr\Big[X = x\Big] \cdot \Pr\Big[Y = y\Big].$$

2. The following is easy to prove:

Claim

If X and Y are independent
$\implies \mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y]$.
$\implies Z = e^X$ *and* $W = e^Y$ *are independent.*

Chernoff inequality

Theorem (Chernoff inequality)

$X_1, \ldots, X_n$: n *independent random variables, such that* $\Pr[X_i = 1] = \Pr[X_i = -1] = \frac{1}{2}$, *for* $i = 1, \ldots, n$. *Let* $Y = \sum_{i=1}^n X_i$. *Then, for any* $\Delta > 0$, *we have*

$$\Pr\Big[Y \geq \Delta\Big] \leq \exp\Big(-\Delta^2/2n\Big).$$

Proof of Chernoff inequality

Fix arbitrary $t > 0$:

$$\mathbf{Pr}\Big[Y \geq \Delta \Big] = \mathbf{Pr}\Big[tY \geq t\Delta \Big]$$

Proof of Chernoff inequality

Fix arbitrary $t > 0$:

$$\mathbf{Pr}\Big[Y \geq \Delta\Big] = \mathbf{Pr}\Big[tY \geq t\Delta\Big] = \mathbf{Pr}\Big[\exp(tY) \geq \exp(t\Delta)\Big]$$

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Fix arbitrary $t > 0$:

$$\mathbf{Pr}\Big[Y \geq \Delta \Big] = \mathbf{Pr}\Big[tY \geq t\Delta \Big] = \mathbf{Pr}\Big[\exp(tY) \geq \exp(t\Delta) \Big]$$
$$\leq \frac{\mathbf{E}\Big[\exp(tY) \Big]}{\exp(t\Delta)},$$

Proof of Chernoff inequality

Continued...

$$\mathbf{E}\left[\exp(tX_i)\right] = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}.$$

Proof of Chernoff inequality

Continued...

$$\mathbf{E}\left[\exp(tX_i)\right] = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \frac{e^t + e^{-t}}{2}.$$

Proof of Chernoff inequality

Continued...

$$\mathbf{E}\Big[\exp(tX_i)\Big] = \frac{e^t + e^{-t}}{2}.$$

Proof of Chernoff inequality

Continued...

$$\mathbf{E}\left[\exp(tX_i)\right] = \frac{e^t + e^{-t}}{2}$$

$$= \frac{1}{2}\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) + \frac{1}{2}\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right).$$

Proof of Chernoff inequality
Continued...

$$\mathbf{E}\Big[\exp(tX_i)\Big] = \frac{1}{2}\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) + \frac{1}{2}\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right).$$

Proof of Chernoff inequality

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$$\mathbf{E}\left[\exp(tX_i)\right] = \frac{1}{2}\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) + \frac{1}{2}\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) = 1 + \frac{t^2}{2!} + + \cdots + \frac{t^{2k}}{(2k)!} + \cdots .$$

Proof of Chernoff inequality

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$$\mathbf{E}\Big[\exp(tX_i)\Big] = 1 + \frac{t^2}{2!} + + \cdots + \frac{t^{2k}}{(2k)!} + \cdots .$$

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$$\mathbf{E}\Big[\exp(tX_i)\Big] = 1 + \frac{t^2}{2!} + + \cdots + \frac{t^{2k}}{(2k)!} + \cdots.$$

However: $(2k)! = k!(k+1)(k+2)\cdots 2k \geq k!2^k$.

Proof of Chernoff inequality

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Proof of Chernoff inequality

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$$\mathbf{E}\left[\exp(tX_i)\right] = \sum_{i=0}^{\infty} \frac{t^{2i}}{(2i)!} \leq \sum_{i=0}^{\infty} \frac{t^{2i}}{2^i(i!)}$$

Proof of Chernoff inequality

Continued...

$$\mathbf{E}\left[\exp(tX_i)\right] \leq \sum_{i=0}^{\infty} \frac{t^{2i}}{2^i (i!)}$$

Proof of Chernoff inequality

Continued...

$$\mathbf{E}\Big[\exp(tX_i)\Big] \leq \sum_{i=0}^{\infty} \frac{t^{2i}}{2^i(i!)} = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{t^2}{2}\right)^i$$

Proof of Chernoff inequality

Continued...

$$\mathbf{E}\left[\exp(tX_i)\right] \leq \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{t^2}{2}\right)^i$$

Proof of Chernoff inequality

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Proof of Chernoff inequality

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Proof of Chernoff inequality

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$$\mathbf{E}\Big[\exp(tY)\Big] = \mathbf{E}\left[\exp\left(\sum_i tX_i\right)\right]$$

Proof of Chernoff inequality

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Proof of Chernoff inequality

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Proof of Chernoff inequality

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Proof of Chernoff inequality

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Proof of Chernoff inequality

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Proof of Chernoff inequality

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Proof of Chernoff inequality

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Proof of Chernoff inequality

Continued...

$$\mathbf{E}\Big[\exp(tY)\Big] \leq \exp\left(\frac{nt^2}{2}\right).$$

$$\mathbf{Pr}\Big[\,Y \geq \Delta\Big]$$

Proof of Chernoff inequality

Continued...

$$\mathbf{E}\Big[\exp(tY)\Big] \leq \exp\left(\frac{nt^2}{2}\right).$$

$$\Pr\Big[Y \geq \Delta\Big] \leq \frac{\mathbf{E}\Big[\exp(tY)\Big]}{\exp(t\Delta)}$$

Proof of Chernoff inequality

Continued...

$$\mathbf{E}\Big[\exp(tY)\Big] \leq \exp\left(\frac{nt^2}{2}\right).$$

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Proof of Chernoff inequality

Continued...

$$\mathbf{E}\Big[\exp(tY)\Big] \leq \exp\left(\frac{nt^2}{2}\right).$$

$$\Pr\Big[Y \geq \Delta\Big] \leq \frac{\exp\left(\frac{nt^2}{2}\right)}{\exp(t\Delta)} = \exp\left(\frac{nt^2}{2} - t\Delta\right).$$

Proof of Chernoff inequality

Continued...

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Proof of Chernoff inequality

Continued...

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Set $t = \Delta / n$:

Proof of Chernoff inequality

Continued...

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Proof of Chernoff inequality

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■

Chernoff inequality...

...what it really says

By theorem:

$$\Pr\left[Y \geq \Delta\right] = \sum_{i=\Delta}^{n} \Pr\left[Y = i\right] = \sum_{i=n/2+\Delta/2}^{n} \frac{\binom{n}{i}}{2^n} \leq \exp\left(-\frac{\Delta^2}{2n}\right),$$

Chernoff inequality...

symmetry

Corollary

Let $X_1, \ldots, X_n$ *be* n *independent random variables, such that* $\Pr[X_i = 1] = \Pr[X_i = -1] = \frac{1}{2}$, *for* $i = 1, \ldots, n$. *Let* $Y = \sum_{i=1}^{n} X_i$. *Then, for any* $\Delta > 0$, *we have*

$$\Pr\Big[|Y| \geq \Delta\Big] \leq 2\exp\left(-\frac{\Delta^2}{2n}\right).$$

Chernoff inequality for coin flips

$X_1, \ldots, X_n$ be n independent coin flips, such that $\Pr[X_i = 1] = \Pr[X_i = 0] = \frac{1}{2}$, for $i = 1, \ldots, n$. Let $Y = \sum_{i=1}^{n} X_i$. Then, for any $\Delta > 0$, we have

$$\Pr\left[\frac{n}{2} - Y \geq \Delta\right] \leq \exp\left(-\frac{2\Delta^2}{n}\right) \quad \text{and} \quad \Pr\left[Y - \frac{n}{2} \geq \Delta\right] \leq$$

In particular, we have $\Pr\left[\left|Y - \frac{n}{2}\right| \geq \Delta\right] \leq 2\exp\left(-\frac{2\Delta^2}{n}\right)$.

The special case we needed

Lemma

In a sequence of M coin flips, the probability that the number of ones is smaller than $L \leq M/4$ is at most $\exp(-M/8)$.

Proof.

Let $Y = \sum_{i=1}^{m} X_i$ the sum of the M coin flips. By the above corollary, we have:

$$\Pr\Big[Y \leq L\Big] = \Pr\left[\frac{M}{2} - Y \geq \frac{M}{2} - L\right] = \Pr\left[\frac{M}{2} - Y \geq \Delta\right],$$

where $\Delta = M/2 - L \geq M/4$. Using the above Chernoff inequality, we get
$\Pr\Big[Y \leq L\Big] \leq \exp\Big(-\frac{2\Delta^2}{M}\Big) \leq \exp(-M/8)$. □

Part V

The Chernoff Bound — General Case

Problem

Let $X_1, \ldots X_n$ *be* n *independent Bernoulli trials, where*

$$\Pr\Big[X_i = 1\Big] = p_i \qquad \textit{and} \qquad \Pr\Big[X_i = 0\Big] = 1 - p_i,$$

and let denote

$$Y = \sum_i X_i \qquad \mu = \mathbf{E}\big[Y\big].$$

Question: *what is the probability that* $Y \geq (1 + \delta)\mu$.

Theorem (Chernoff inequality)

For any $\delta > 0$,

$$\Pr\Big[Y > (1+\delta)\mu\Big] < \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$

Or in a more simplified form, for any $\delta \leq 2e - 1$,

$$\Pr\Big[Y > (1+\delta)\mu\Big] < \exp\Big(-\mu\delta^2/4\Big),$$

and

$$\Pr\Big[Y > (1+\delta)\mu\Big] < 2^{-\mu(1+\delta)},$$

for $\delta \geq 2e - 1$.

Theorem

Theorem

Under the same assumptions as the theorem above, we have

$$\Pr\Big[Y < (1-\delta)\mu\Big] \leq \exp\left(-\mu\frac{\delta^2}{2}\right).$$

Part VI

Treaps

Balanced binary search trees...

1. Work usually by storing additional information.
2. Idea: For every element x inserted randomly choose ***priority*** $p(x) \in [0, 1]$.
3. $X = \{x_1, \ldots, x_n\}$ priorities: $p(x_1), \ldots, p(x_n)$.
4. x_k: lowest priority in X.
5. Make x_k the root.
6. partition X in the natural way:
   (A) L: set of all the numbers smaller than x_k in X, and
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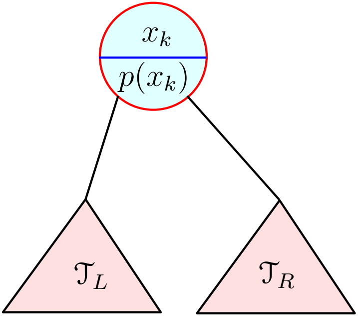
Treaps

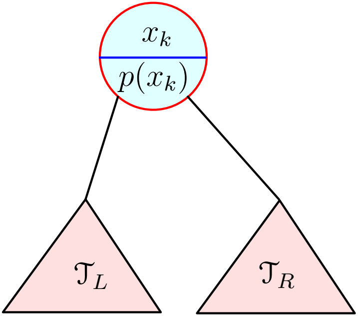


Continuing recursively, we have:

(A) $\boldsymbol{L}$: set of all the numbers smaller than $\boldsymbol{x_k}$ in $\boldsymbol{X}$, and
(B) $\boldsymbol{R}$: set of all the numbers larger than $\boldsymbol{x_k}$ in $\boldsymbol{X}$.

Definition

Resulting tree a ***treap***.
Tree over the elements, and a heap over the priorities; that is, TREAP = TREE + HEAP.



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Treaps continued

Lemma

S: n elements.
Expected depth of treap **T** *for S is $O(\log(n))$.*
Depth of treap **T** *for S is $O(\log(n))$ w.h.p.*

Proof.

QuickSort... □

Treaps continued

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Proof.

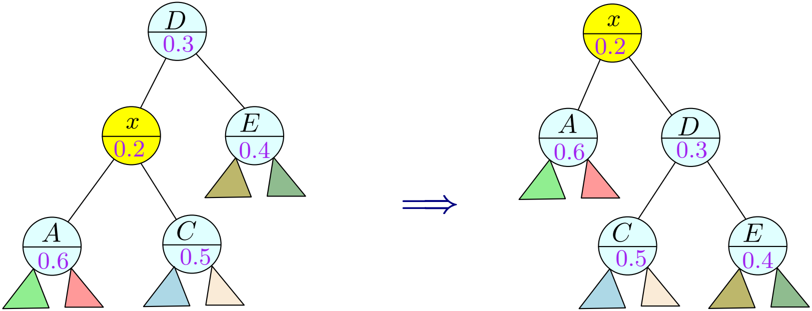
QuickSort... □

Treaps - implementation

Observation

Given n distinct elements, and their (distinct) priorities, the treap storing them is uniquely defined.

Rotate right...



Treaps – insertion

1. x: an element x to insert.
2. Insert it into $\mathbf{T}$ as a regular binary tree.
3. Takes $O(\text{height}(\mathbf{T}))$.
4. x is a leaf in the treap.
5. Pick priority $p(x) \in [0, 1]$.
6. Valid search tree,.. but priority heap is broken at x.
7. Fix priority heap around x.

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Fix treap for a leaf x...

```
RotateUp(x)
   y ← parent(x)
   while p(y) > p(x) do
       if y.left_child = x then
           RotateRight(y)
       else
           RotateLeft(y)
       y ← parent(x)
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Insertion takes $O(\text{height}(\mathsf{T}))$.

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Insertion takes $O(\text{height}(\mathsf{T}))$.

Treaps – deletion

1. Deletion is just an insertion done in reverse.
2. x: element to delete.
3. Set $p(x) \leftarrow +\infty$,
4. rotate x down till its a leaf.
5. Rotate so that child with lower priority becomes new parent.
6. x is now leaf – deleting is easy...

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5. Rotate so that child with lower priority becomes new parent.
6. x is now leaf – deleting is easy...

Treaps – deletion

1. Deletion is just an insertion done in reverse.
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Split

1. x: element stored in treap $\mathbf{T}$.
2. split $\mathbf{T}$ into two treaps – one treap $\mathbf{T}_{\leq x}$ and treap $\mathbf{T}_{>}$ for all the elements larger than x.
3. Set $p(x) \leftarrow -\infty$,
4. fix priorities by rotation.
5. x item is now the root.
6. Splitting is now easy....
7. Restore x to its original priority. Fix by rotations.

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1. x: element stored in treap **T**.
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Meld

1. $\mathbf{T}_L$ and $\mathbf{T}_R$: treaps.
2. all elements in $\mathbf{T}_L$ ¡ all elements in $\mathbf{T}_R$.
3. Want to merge them into a single treap...

Treap – summary

Theorem

Let **T** *be an empty treap, after a sequence of* $m = n^c$ *insertions, where* c *is some constant.*
d*: arbitrary constant.*
The probability depth **T** *ever exceed* $d \log n$ *is* $\leq 1/n^{O(1)}$*.*
A treap can handle insertion/deletion in $O(\log n)$ *time with high probability.*

Proof

Proof.

1. $\mathsf{T}_1, \ldots, \mathsf{T}_m$: sequence of treaps.
2. T_i is treap after ith operation.
3. $\alpha_i = \Pr\Big[\mathrm{depth}(\mathsf{T}_i) > tc' \log n\Big] = \Pr\Big[\mathrm{depth}(\mathsf{T}_i) > c' t\Big(\frac{\log n}{\log|\mathsf{T}_i|}\Big) \cdot \log |\mathsf{T}_i|\Big] \leq \frac{1}{n^{O(1)}}$,
4. Use union bound...

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Bibliographical Notes

1. Chernoff inequality was a rediscovery of Bernstein inequality.
2. ...published in 1924 by Sergei Bernstein.
3. Treaps were invented by Siedel and Aragon **?**.
4. Experimental evidence suggests that Treaps performs reasonably well in practice see **?**.
5. Old implementation of treaps I wrote in `C` is available here: http://valis.cs.uiuc.edu/blog/?p=6060.

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