CS 573: Algorithms, Fall 2014

Approximation Algorithms

Lecture 7

September 16, 2014

Today's Lecture

Don't give up on **NP-Hard** problems:

(A) Faster exponential time algorithms: $n^{O(n)}$, 3^n, 2^n, etc.

(B) Fixed parameter tractable.

(C) Find an approximate solution.

Part I

Greedy algorithms and approximation algorithms

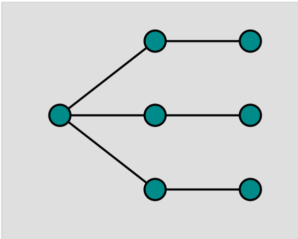
Greedy algorithms

1. ***greedy algorithms***: do locally the right thing...
2. ...and they suck.

VertexCoverMin

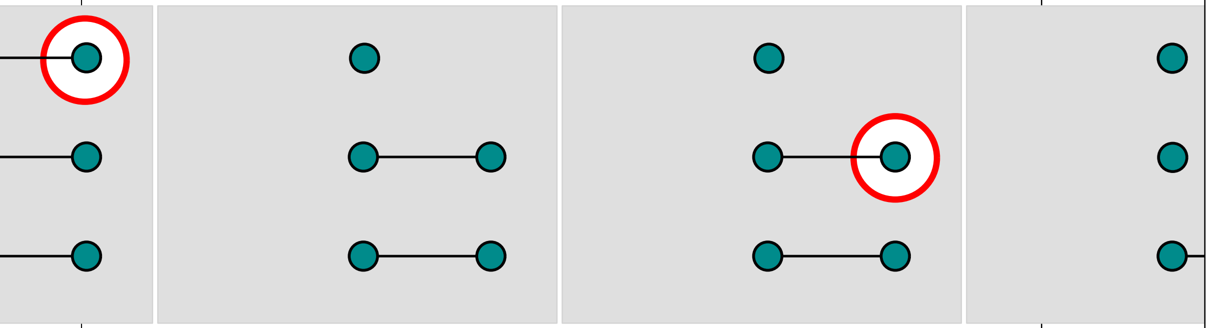
Instance: A graph $\mathbf{G}$.
Question: Return the smallest subset $S \subseteq V(\mathbf{G})$, s.t. S touches all the edges of $\mathbf{G}$.

3. **GreedyVertexCover**: pick vertex with highest degree, remove, repeat.



Greedy algorithms

GreedyVertexCover in action...



Observation

GreedyVertexCover *returns 4 vertices, but opt is 3 vertices.*

Good enough...

Definition

In a ***minimization*** optimization problem, one looks for a valid solution that minimizes a certain target function.

1. VertexCoverMin: $\mathrm{Opt}(\mathbf{G}) = \min_{S \subseteq V(\mathbf{G}), S \text{ cover of } \mathbf{G}} |S|$.
2. $\mathrm{VertexCover}(\mathbf{G})$: set realizing sol.
3. $\mathrm{Opt}(\mathbf{G})$: value of the target function for the optimal solution.

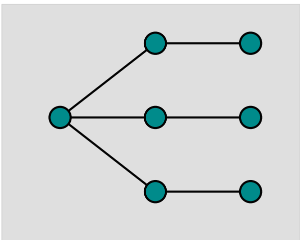
Definition

Alg is ***α-approximation algorithm*** for problem **Min**, achieving an approximation $\alpha \geq 1$, if for all inputs **G**, we have:

$$\frac{\mathbf{Alg}(\mathbf{G})}{\mathrm{Opt}(\mathbf{G})} \leq \alpha.$$

Back to GreedyVertexCover

1. **GreedyVertexCover**: pick vertex with highest degree, remove, repeat.
2. Returns 4, but opt is 3!



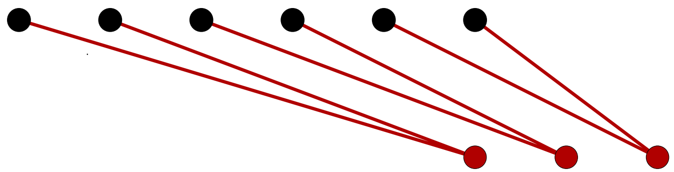
3. Can **not** be better than a $4/3$-approximation algorithm.
4. Actually it is much worse!

How bad is GreedyVertexCover?

Build a bipartite graph.

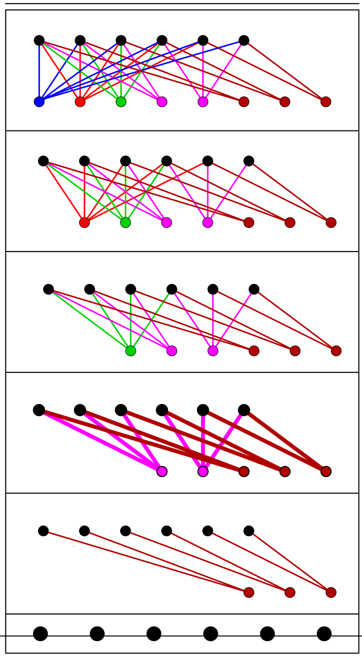
Let the top partite set be of size n.

In the bottom set add $\lfloor n/2 \rfloor$ vertices of degree 2, such that each edge goes to a different vertex above.



Repeatedly add $\lfloor n/i \rfloor$ bottom vertices of degree i, for $i = 2, \ldots, n$.

How bad is GreedyVertexCover?



1. Bottom row taken by Greedy.
2. Top row was a smaller solution.

Lemma

*The algorithm **GreedyVertexCover** is $\Omega(\log n)$ approximation to the optimal solution to VertexCoverMin.*

See notes for details!

Greedy Vertex Cover

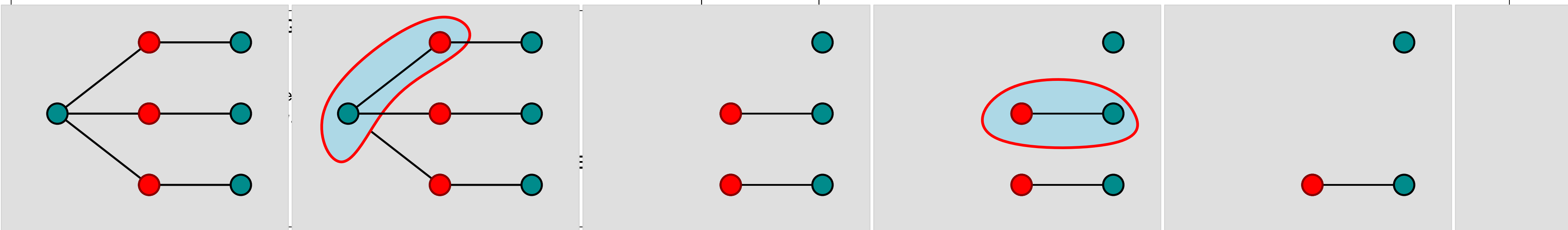
Theorem

*The greedy algorithm for **VertexCover** achieves $\Theta(\log n)$ approximation, where n (resp. m) is the number of vertices (resp., edges) in the graph. Running time is $O(mn^2)$.*

Proof

Lower bound follows from lemma.
Upper bound follows from analysis of greedy algorithm for **Set Cover**, which will be done shortly.
As for the running time, each iteration of the algorithm takes $O(mn)$ time, and there are at most n iterations.

Two for the price of one



Theorem

__ApproxVertexCover__ is a 2-approximation algorithm for VertexCoverMin that runs in $O(n^2)$ time.

Proof...

Two for the price of one - example

Part II

Fixed parameter tractability, approximation, and fast exponential time algorithms (to say nothing of the dog)

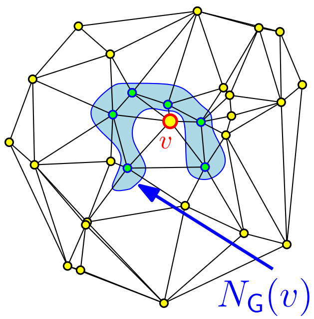
What if the vertex cover is small?

1. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with n vertices
2. $K \leftarrow$ Approximate VertexCoverMin up to a factor of two.
3. Any vertex cover of G is of size $\geq K/2$.
4. Naively compute optimal in $O\left(n^{K+2}\right)$ time.

Induced subgraph

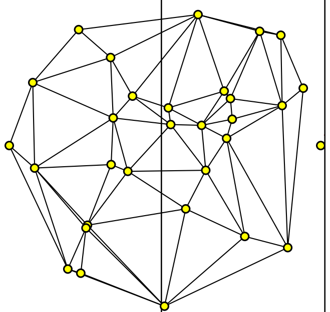
Definition

$N_{\mathsf{G}}(v)$: ***Neighborhood*** of v – set of vertices of $\mathbf{G}$ adjacent to v.



Definition

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a graph. For a subset $S \subseteq \mathbf{V}$, let $\mathbf{G}_S$ be the ***induced subgraph*** over S.



Exact fixed parameter tractable algorithm

Fixed parameter tractable algorithm for **VertexCoverMin**.

Computes minimum vertex cover for the induced graph $\mathbf{G}_X$:

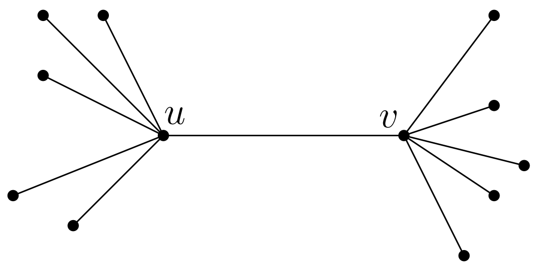
fpVCI (X, β)
```
    // β: size of VC computed so far.
```
if $X = \emptyset$ or $\mathbf{G}_X$ has no edges **then return** β
e $\leftarrow$ any edge uv of $\mathbf{G}_X$.
$\beta_1 =$ **fpVCI** $\left(X \setminus \{u, v\}, \beta + 2\right)$
$\beta_2 =$ **fpVCI** $\left(X \setminus \left(\{u\} \cup N_{\mathsf{G}_X}(v)\right), \beta + \left|N_{\mathsf{G}_X}(v)\right|\right)$
$\beta_3 =$ **fpVCI** $\left(X \setminus \left(\{v\} \cup N_{\mathsf{G}_X}(u)\right), \beta + \left|N_{\mathsf{G}_X}(u)\right|\right)$
return min$(\beta_1, \beta_2, \beta_3)$.

algFPVertexCover $(\mathbf{G} = (\mathbf{V}, \mathbf{E}))$
return fpVCI$(\mathbf{V}, 0)$

Depth of recursion

Lemma
The algorithm **algFPVertexCover** *returns the optimal solution to the given instance of VertexCoverMin.*

Proof...



Depth of recursion II

Lemma
The depth of the recursion of **algFPVertexCover**$(\mathbf{G})$ *is at most* α, *where* α *is the minimum size vertex cover in* $\mathbf{G}$.

Proof.

1. When the algorithm takes both u and v - one of them in opt. Can happen at most α times.
2. Algorithm picks $N_{G_X}(v)$ (i.e., β_2). Conceptually add v to the vertex cover being computed.
3. Do the same thing for the case of β_3.
4. Every such call add one element of the opt to conceptual set cover. Depth of recursion is $\leq \alpha$.

□

Vertex Cover
Exact fixed parameter tractable algorithm

Theorem
$\mathbf{G}$*: graph with* n *vertices. Min vertex cover of size* α. *Then,* **algFPVertexCover** *returns opt. vertex cover.*
Running time is $O\left(3^{\alpha} n^2\right)$.

Proof:

1. By lemma, recursion tree has depth α.
2. Rec-tree contains $\leq 2 \cdot 3^{\alpha}$ nodes.
3. Each node requires $O\left(n^2\right)$ work. ■

Algorithms with running time $O(n^c f(\alpha))$, where α is some parameter that depends on the problem are ***fixed parameter tractable***.

Part III

Traveling Salesperson Problem

TSP

TSP-Min

Instance: $\mathbf{G} = (V, E)$ a complete graph, and $\omega(e)$ a cost function on edges of $\mathbf{G}$.
Question: The cheapest tour that visits all the vertices of $\mathbf{G}$ exactly once.

Solved exactly naively in $\approx n!$ time.
Using DP, solvable in $O(n^2 2^n)$ time.

TSP Hardness

Theorem

*TSP-Min can not be approximated within **any** factor unless* $\mathbf{NP} = \mathbf{P}$.

Proof.

1. Reduction from **Hamiltonian Cycle** into **TSP**.
2. $\mathbf{G} = (\mathbf{V}, \mathbf{E})$: instance of Hamiltonian cycle.
3. $\mathbf{H}$: Complete graph over $\mathbf{V}$.
$$\forall u, v \in \mathbf{V} \quad w_{\mathbf{H}}(uv) = \begin{cases} 1 & uv \in \mathbf{E} \\ 2 & \text{otherwise.} \end{cases}$$
4. $\exists$ tour of price n in $\mathbf{H}$ $\iff$ $\exists$ Hamiltonian cycle in $\mathbf{G}$.
5. No Hamiltonian cycle $\implies$ TSP price at least $n + 1$.
6. But... replace $\mathbf{2}$ by cn, for c an arbitrary number

TSP Hardness - proof continued

Proof.

1. Price of all tours are either:
   (i) n (only if $\exists$ Hamiltonian cycle in $\mathbf{G}$),
   (ii) larger than $cn + 1$ (actually, $\geq cn + (n - 1)$).
2. Suppose you had a poly time c-approximation to TSP-Min.
3. Run it on $\mathbf{H}$:
   (i) If returned value $\geq cn + 1 \implies$ no Ham Cycle since $(cn + 1)/c > n$
   (ii) If returned value $\leq cn \implies$ Ham Cycle since $OPT \leq cn < cn + 1$
4. c-approximation algorithm to TSP $\implies$ poly-time algorithm for **NP-Complete** problem. Possible only if $\mathbf{P} = \mathbf{NP}$.

$\square$

TSP with the triangle inequality

Because it is not that bad after all.

TSP$_{\triangle\neq}$-Min

Instance: $\mathbf{G} = (V, E)$ is a complete graph. There is also a cost function $\omega(\cdot)$ defined over the edges of $\mathbf{G}$, that complies with the triangle inequality.
Question: The cheapest tour that visits all the vertices of $\mathbf{G}$ exactly once.

triangle inequality: $\omega(\cdot)$ if
$$\forall u, v, w \in \mathbf{V}(\mathbf{G}), \qquad \omega(u, v) \leq \omega(u, w) + \omega(w, v).$$

Shortcutting

σ: a path from s to t in $\mathbf{G} \implies \omega(st) \leq \omega(\sigma)$.

TSP with the triangle inequality

Continued...

Definition

Cycle in **G** is ***Eulerian*** if it visits every **edge** of **G** exactly once.

Assume you already seen the following:

Lemma

A graph **G** *has a cycle that visits every edge of* **G** *exactly once (i.e., an Eulerian cycle) if and only if* **G** *is connected, and all the vertices have even degree. Such a cycle can be computed in* $O(n+m)$ *time, where* n *and* m *are the number of vertices and edges of* **G**, *respectively.*

TSP with the triangle inequality

Continued...

1. C_{opt} optimal **TSP** tour in **G**.
2. **Observation**:
   $\omega(C_{\mathrm{opt}}) \geq \mathrm{weight}\Big(\text{cheapest spanning graph of } \mathbf{G}\Big)$.
3. MST: cheapest spanning graph of **G**.
   $\omega(C_{\mathrm{opt}}) \geq \omega(\mathrm{MST}(\mathbf{G}))$
4. $O(n \log n + m) = O(n^2)$: time to compute MST.
   $n = |\mathbf{V}(\mathbf{G})|$, $m = \binom{n}{2}$.

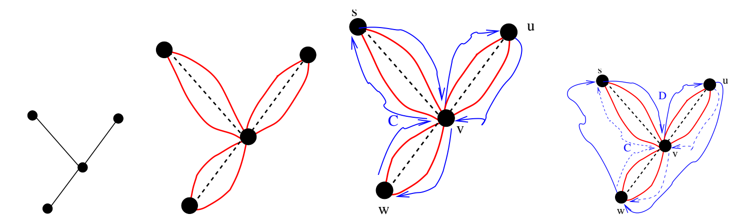
TSP with the triangle inequality

2-approximation

1. $T \leftarrow \mathrm{MST}(\mathbf{G})$
2. $\mathbf{H} \leftarrow$ duplicate very edge of T.
3. H has an Eulerian tour.
4. $\mathbf{C}$: Eulerian cycle in H.
5. $\omega(\mathbf{C}) = \omega(H) = 2\omega(T) = 2\omega(MST(\mathbf{G})) \leq 2\omega(C_{\mathrm{opt}})$.
6. π: Shortcut $\mathbf{C}$ so visit every vertex once.
7. $\omega(\pi) \leq \omega(\mathbf{C})$

TSP with the triangle inequality

2-approximation algorithm in figures



(a) (b) (c) (d)

Euler Tour: vuvwvsv
First occurrences: vuvwvsv
Shortcut String: vuwsv

TSP with the triangle inequality

2-approximation - result

Theorem

$\mathbf{G}$: Instance of TSP$_{\triangle\neq}$-Min.
C_{opt}: min cost TSP tour of $\mathbf{G}$.
$\implies$ Compute a tour of $\mathbf{G}$ of length $\leq 2\omega(C_{\mathrm{opt}})$.
Running time of the algorithm is $O\left(n^2\right)$.

$\mathbf{G}$: n vertices, cost function $\omega(\cdot)$ on the edges that comply with the triangle inequality.

TSP with the triangle inequality

3/2-approximation

Definition

$\mathbf{G} = (V, E)$, a subset $M \subseteq E$ is a ***matching*** if no pair of edges of M share endpoints.
A ***perfect matching*** is a matching that covers all the vertices of $\mathbf{G}$.
w: weight function on the edges. ***Min-weight perfect matching***, is the minimum weight matching among all perfect matching, where

$$\omega(M) = \sum_{e \in M} \omega(e).$$

TSP with the triangle inequality

3/2-approximation

The following is known:

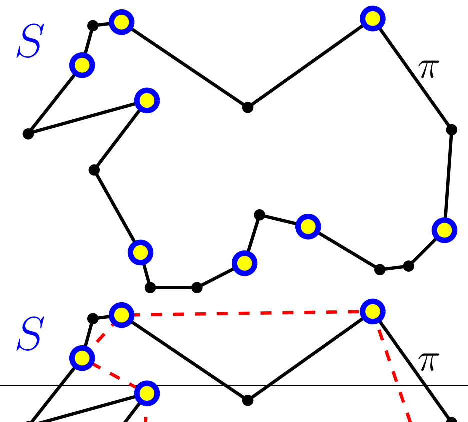
Theorem

Given a graph $\mathbf{G}$ and weights on the edges, one can compute the min-weight perfect matching of $\mathbf{G}$ in polynomial time.

Min weight perfect matching vs. TSP

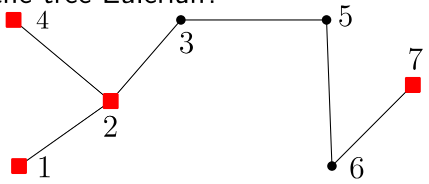
Lemma

$\mathbf{G} = (\mathbf{V}, \mathbf{E})$: complete graph.
$S \subseteq \mathbf{V}$: even size.
$\omega(\cdot)$: a weight function over $\mathbf{E}$.
$\implies$ min-weight perfect matching in $\mathbf{G}_S$ is $\leq \omega(\mathrm{TSP}(\mathbf{G}))/2$.



A more perfect tree?

1. How to make the tree Eulerian?



2. Pesky odd degree vertices must die!
3. Number of odd degree vertices in a graph is even!
4. Compute min-weight matching on odd vertices, and add to MST.
5. $\mathbf{H} = \mathrm{MST} + \mathbf{min} - \mathbf{weight} - \mathbf{matching}$ is Eulerian.
6. Weight of resulting cycle in $\mathbf{H} \leq (3/2)\omega(\mathrm{TSP})$.

Even number of odd degree vertices

Lemma
The number of odd degree vertices in any graph G' is even.

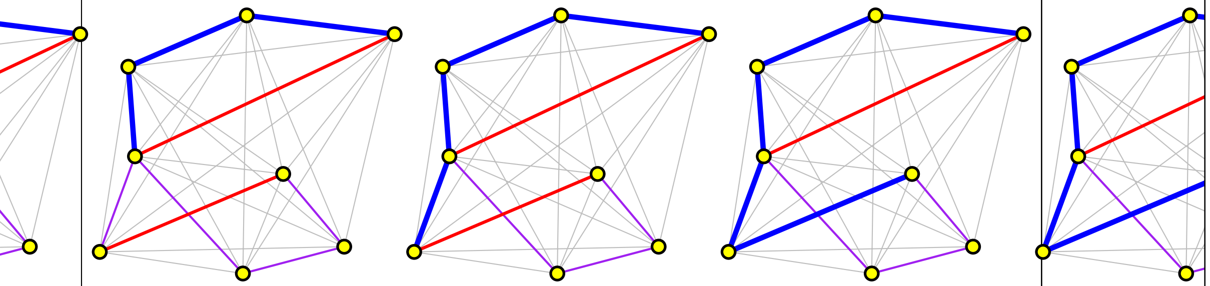
Proof:
$\mu = \sum_{v \in V(G')} d(v) = 2|E(G')|$ and thus even.
$U = \sum_{v \in V(G'), d(v) \text{ is even}} d(v)$ even too.
Thus,

$$\alpha = \sum_{v \in V, d(v) \text{ is odd}} d(v) = \mu - U = \text{even number},$$

since μ and U are both even.
Number of elements in sum of all odd numbers must be even, since the total sum is even.

3/2-approximation algorithm for TSP

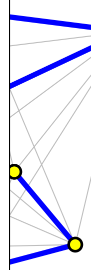
Animated!



3/2-approximation algorithm for TSP

The result

Theorem
Given an instance of TSP with the triangle inequality, one can compute in polynomial time, a $(3/2)$-approximation to the optimal TSP.



Biographical Notes

The **3/2**-approximation for TSP with the triangle inequality is due to **?**.