NP Completeness

Lecture 3

September 3, 2014

Part I

NP Completeness

Certifiers

Definition

An algorithm $C(\cdot,\cdot)$ is a ***certifier*** for problem X if for every $s \in X$ there is some string t such that $C(s,t) =$ "yes", and conversely, if for some s and t, $C(s,t) =$ "yes" then $s \in X$. The string t is called a certificate or proof for s.

Definition (Efficient Certifier.)

A certifier C is an ***efficient certifier*** for problem X if there is a polynomial $p(\cdot)$ such that for every string s, we have that

- ⋆ $s \in X$ if and only if
- ⋆ there is a string t:
  1. $|t| \leq p(|s|)$,
  2. $C(s,t) =$ "yes",
  3. and C runs in polynomial time.

NP-Complete Problems

Definition

A problem X is said to be **NP-Complete** if

1. $X \in \mathbf{NP}$, and
2. (Hardness) For any $Y \in \mathbf{NP}$, $\mathsf{Y} \leq_P \mathsf{X}$.

Solving NP-Complete Problems

Proposition

Suppose X is **NP-Complete***. Then X can be solved in polynomial time if and only if* $\mathsf{P} = \mathsf{NP}$.

Proof.

$\Rightarrow$ Suppose X can be solved in polynomial time

1. Let $Y \in \mathsf{NP}$. We know $\mathsf{Y} \leq_P \mathsf{X}$.
2. We showed that if $\mathsf{Y} \leq_P \mathsf{X}$ and X can be solved in polynomial time, then Y can be solved in polynomial time.
3. Thus, every problem $Y \in \mathsf{NP}$ is such that $Y \in P$; $NP \subseteq P$.
4. Since $\mathsf{P} \subseteq \mathsf{NP}$, we have $\mathsf{P} = \mathsf{NP}$.

$\Leftarrow$ Since $\mathsf{P} = \mathsf{NP}$, and $X \in \mathsf{NP}$, we have a polynomial time algorithm for X. □

NP-Hard Problems

- Formal definition:

Definition

A problem X is said to be **NP-Hard** if

- (Hardness) For any $Y \in$ **NP**, we have that $\mathbf{Y} \leq_P \mathbf{X}$.

- An **NP-Hard** problem need not be in **NP**!
- Example: Halting problem is **NP-Hard** (why?) but not **NP-Complete**.

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Consequences of proving NP-Completeness

1. If X is **NP-Complete**
   1. Since we believe $\mathsf{P} \neq \mathsf{NP}$,
   2. and solving X implies $\mathsf{P} = \mathsf{NP}$.

   X is unlikely to be efficiently solvable.
2. At the very least, many smart people before you have failed to find an efficient algorithm for X.
3. (This is proof by mob opinion — take with a grain of salt.)

Consequences of proving **NP-Complete**ness

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NP-Complete Problems

Question

Are there any problems that are **NP-Complete**?

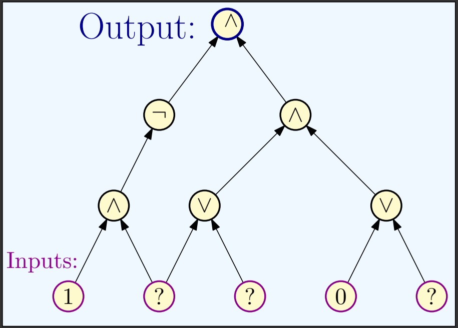
Answer

Yes! Many, many problems are **NP-Complete**.

Circuits

Definition

A circuit is a directed *acyclic* graph with



1. Input vertices (without incoming edges) labelled with 0, 1 or a distinct variable.
2. Every other vertex is labelled $\vee$, $\wedge$ or $\neg$.
3. Single node output vertex with no outgoing edges.

Cook-Levin Theorem

Definition (Circuit Satisfaction (CSAT).)

Given a circuit as input, is there an assignment to the input variables that causes the output to get value 1?

Theorem (Cook-Levin)

CSAT is **NP-Complete***.*

Need to show

1. CSAT is in **NP**.
2. *every* **NP** problem X reduces to CSAT.

CSAT: Circuit Satisfaction

Claim

CSAT is in **NP**.

1. Certificate: Assignment to input variables.
2. Certifier: Evaluate the value of each gate in a topological sort of DAG and check the output gate value.

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CSAT is NP-hard: Idea

- Need to show that *every* **NP** problem X reduces to **CSAT**.
- What does it mean that $X \in$ **NP**?
- $X \in$ **NP** implies that there are polynomials $p()$ and $q()$ and certifier/verifier program C such that for every string s the following is true:
  - If s is a YES instance ($s \in X$) then there is a *proof* t of length $p(|s|)$ such that $C(s, t)$ says YES.
  - If s is a NO instance ($s \notin X$) then for every string t of length at $p(|s|)$, $C(s, t)$ says NO.
  - $C(s, t)$ runs in time $q(|s| + |t|)$ time (hence polynomial time).

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Reducing X to CSAT

1. X is in **NP** means we have access to $p(), q(), C(\cdot,\cdot)$.
2. What is $C(\cdot,\cdot)$? It is a program or equivalently a Turing Machine!
3. How are $p()$ and $q()$ given?
   As numbers.
4. Example: if 3 is given then $p(n) = n^3$.
5. Thus an **NP** problem is essentially a three tuple $\langle p, q, C\rangle$ where C is either a program or a TM.

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- **NP** problem: a three tuple $\langle p, q, C\rangle$.
  C: program or TM, $p(\cdot)$, $q(\cdot)$: polynomials.
- Problem X: Given string s, is $s \in X$?
- **Equivalent**:
  $\exists$ proof t of length $p\left(|s|\right)$ & $C(s,t)$ returns YES.
  ...$C(s,t)$ runs in $q\left(|s|\right)$ time.
- Reduce from X to **CSAT**...
  Need an algorithm **alg** that
  - takes s (and $\langle p, q, C\rangle$).
    Creates circuit G in poly time in $|s|$.
    ($\langle p, q, C\rangle$ is fixed so $|\langle p, q, C\rangle| = O(1)$.)
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2. Need algorithm **alg** that:
   1. Input: s (and $\langle p, q, C\rangle$).
   2. creates circuit G in poly-time in $|s|$ ($\langle p, q, C\rangle$ fixed).
   3. G satisfiable $\iff$ $\exists$ proof t: $C(s,t)$ returns YES.
3. Simple but Big Idea: Programs are the same as Circuits!
   1. Convert $C(s,t)$ into a circuit G with t as unknown inputs (rest is known including s)
   2. Known: $|t| \leq p(|s|)$ so express boolean string t as $p(|s|)$ variables $t_1, t_2, \ldots, t_k$ where $k = p(|s|)$.
   3. Asking if there is a proof t that makes $C(s,t)$ say YES is same as whether there is an assignment of values to "unknown" variables $t_1, t_2, \ldots, t_k$ that will make G evaluate to true/YES.

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Example: Independent Set

- Formal definition:

Independent Set

> **Instance**: $G = (V, E)$, k
> **Question:** Does $G = (V, E)$ have an **Independent Set** of size $\geq k$

- Certificate: Set $S \subseteq V$.
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Formally why is Independent Set in NP?

1. Input is a "binary" vector:

$$\langle n, y_{1,1}, y_{1,2}, \ldots, y_{1,n}, y_{2,1}, \ldots, y_{2,n}, \ldots, y_{n,1}, \ldots, y_{n,n}, k\rangle$$

   encodes $\langle G, k\rangle$.

   1. n is number of vertices in G
   2. $y_{i,j}$ is a bit which is 1 if edge (i,j) is in G and 0 otherwise (adjacency matrix representation)
   3. k: size of independent set.

2. **Certificate**: $t = t_1 t_2 \ldots t_n$.
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Certifier for **Independent Set**

Certifier $C(s, t)$ for **Independent Set**:

```
if (t_1 + t_2 + ... + t_n < k) then
    return NO
else
    for each (i, j) do
        if (t_i ∧ t_j ∧ y_{i,j}) then
            return NO

return YES
```

Example: Independent Set

A certifier circuit for Independent Set

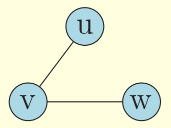
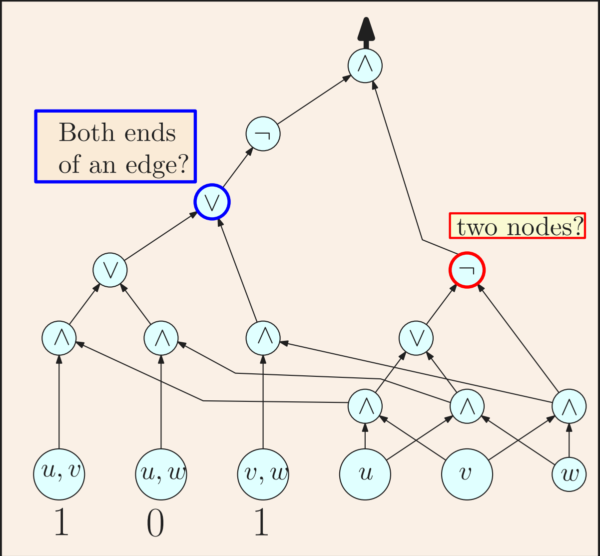


Figure: Graph G with $k = 2$



Programs, Turing Machines and Circuits

1. **alg**: "program" that takes $f(|s|)$ steps on input string s.
2. Questions: What computer is used?
   What does *step* mean?
3. "Real" computers difficult to reason with mathematically:
   1. instruction set is too rich
   2. pointers and control flow jumps in one step
   3. assumption that pointer to code fits in one word
4. Turing Machines:
   1. simpler model of computation to reason with
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   3. all moves are *local* (head moves only one cell)

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Certifiers that at TMs

- Assume $C(\cdot,\cdot)$ is a (deterministic) Turing Machine M
- Problem: Given M, input s, p, q decide if:
  - $\exists$ proof t of length $\leq p(|s|)$
  - M executed on the input s, t halts in $q(|s|)$ time and returns YES.
- **ConvCSAT** reduces above problem to **CSAT**:

1. computes $p(|s|)$ and $q(|s|)$.
2. As such, M:
   - Uses at most $q(|s|)$ memory/tape cells.
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Simulation of Computation via Circuit

- M state at time ℓ: A string $x^\ell = x_1 x_2 \ldots x_k$ where each $x_i \in \{0, 1, B\} \times Q \cup \{q_{-1}\}$.
- Time 0: State of M = input string s, a guess t of $p(|s|)$ "unknowns", and rest $q(|s|)$ blank symbols.
- Time $q(|s|)$? Does M stops in q_{accept} with blank tape.
- Build circuit C_ℓ: Evaluates to YES $\iff$ transition of M from time ℓ to time $\ell + 1$ valid. (Circuit of size $O\Big(q(|s|)\Big)$.
- $\mathcal{C}$: $C_0 \wedge C_1 \wedge \cdots \wedge C_{q(|s|)}$. Polynomial size!
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Key Ideas in reduction:

1. Use TMs as the code for certifier for simplicity
2. Since $p()$ and $q()$ are known to $\mathcal{A}$, it can set up all required memory and time steps in advance
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Note: Above reduction can be done to **SAT** as well. Reduction to **SAT** was the original proof of Steve Cook.

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